MPFA Discretization on Quadrilateral Grids

Ivar Aavatsmark, Geir Terje Eigestad, Runhild Aae Klausen, Jan Martin Nordbotten

Center for Integrated Petroleum Research
University of Bergen

IMA Workshop May 13, 2004

Outline

- Motivation
- MPFA: The O-method
- Convergence
- Monotonicity
- Preconditioning of the linear solver

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Reservoir Simulation

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Conservation equations

$$\int_\Omega \frac{\partial}{\partial t}(\phi S_w \rho_w)\, d\tau + \int_{\partial\Omega} \rho_w \boldsymbol{v}_w \cdot \boldsymbol{n}\, d\sigma = \int_\Omega Q_w\, d\sigma$$

$$\int_\Omega \frac{\partial}{\partial t}(\phi(1 - S_w)\rho_o)\, d\tau + \int_{\partial\Omega} \rho_o \boldsymbol{v}_o \cdot \boldsymbol{n}\, d\sigma = \int_\Omega Q_o\, d\sigma$$

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$$\boldsymbol{v}_w = -\frac{k_{rw}}{\mu_w}\boldsymbol{K}(\mathrm{grad}\, p_w - \rho_w g\, \mathrm{grad}\, D)$$

$$\boldsymbol{v}_o = -\frac{k_{ro}}{\mu_o}\boldsymbol{K}(\mathrm{grad}\, p_o - \rho_o g\, \mathrm{grad}\, D)$$

Darcy's law

Equation with elliptic character

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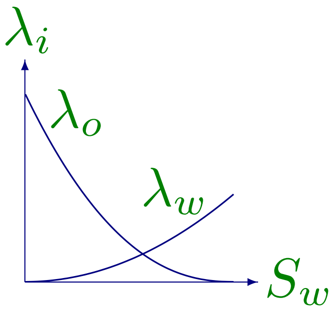

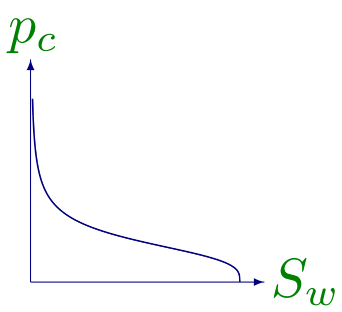

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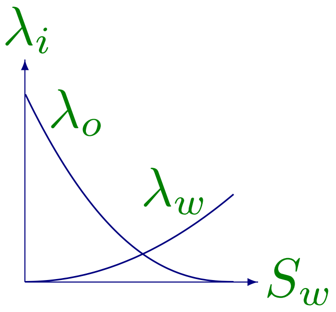

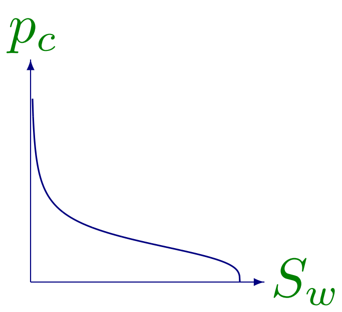

Compressible flow $\Rightarrow$ Parabolic equation with small accumulation term

Equation with hyperbolic character

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$$\phi\frac{\partial S_w}{\partial t} + \boldsymbol{v}\cdot\operatorname{grad} f_w + [\boldsymbol{K}(\rho_w - \rho_o)g\operatorname{grad} D]\cdot\operatorname{grad} h + \operatorname{div}(h\boldsymbol{K}\operatorname{grad} p_c) = (1-f_w)\frac{Q_w}{\rho_w} - f_w\frac{Q_o}{\rho_o}$$

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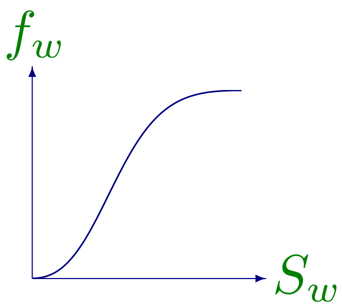

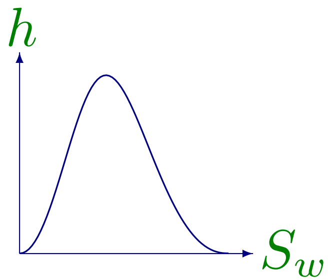

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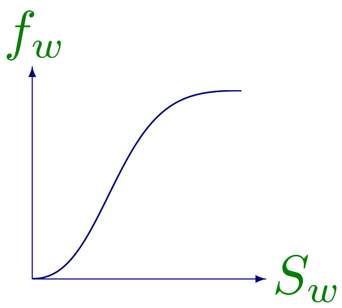

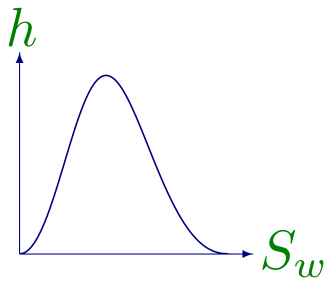

Equation with a strongly nonlinear convection term

Equation properties

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Coupled set of partial differential equations

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Coupled set of partial differential equations

- Almost elliptic part requires implicit discretization
- Almost hyperbolic part with strong nonlinearity requires locally conservative and montone discretization
- A monotone discretization may be achieved by upstream weighting of an explicit flux.

Medium

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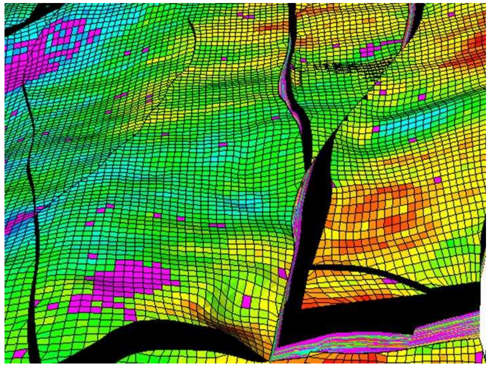

Permeability in lateral grid

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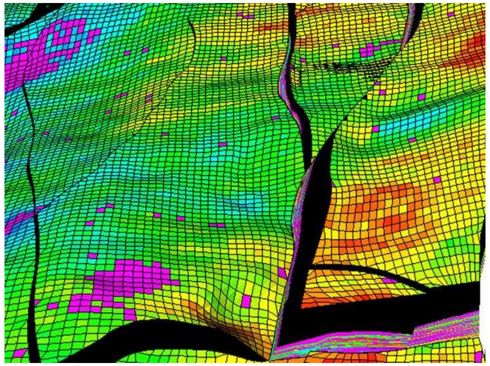

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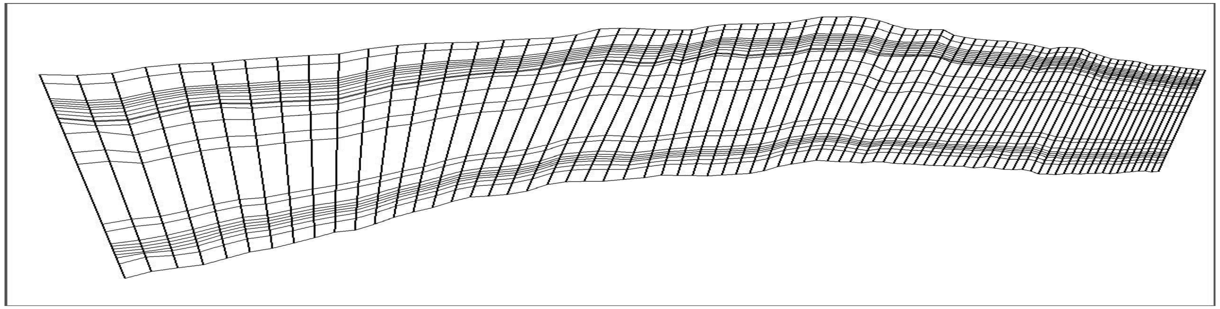

Vertical cross section of a grid

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- The discretization should be valid for strongly heterogenous media

- The discretization should be valid for arbitrary anisotropy of the permeability

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- The discretization must yield convergence for both u and $\boldsymbol{q}$.

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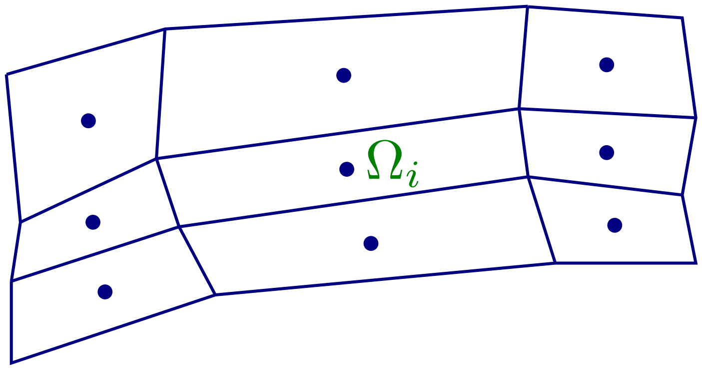
$u(\boldsymbol{x}) = \int_\Omega g(\boldsymbol{x},\boldsymbol{\xi}) f(\boldsymbol{\xi})\, d\tau_{\boldsymbol{\xi}}$ where $g(\boldsymbol{x},\boldsymbol{\xi}) \geq 0$, g Green's function

Control volume formulation

Control volume formulation



$$\int_{\partial\Omega_i} \boldsymbol{q}\cdot\boldsymbol{n}\,d\sigma = \int_{\Omega_i} f\,d\tau.$$

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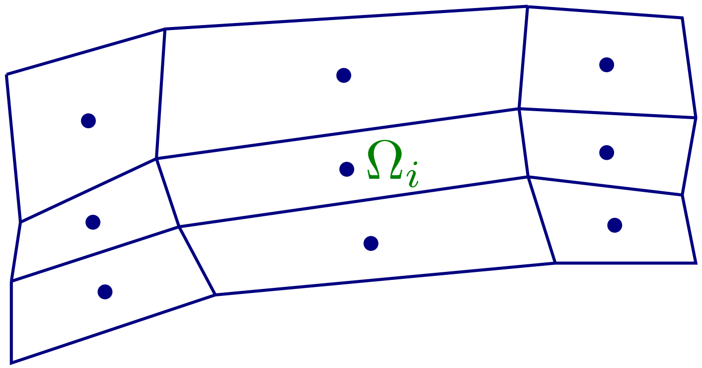

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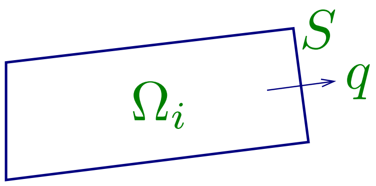

$$q = \int_S \boldsymbol{q}\cdot\boldsymbol{n}\,d\sigma,$$

One-dimensional flow

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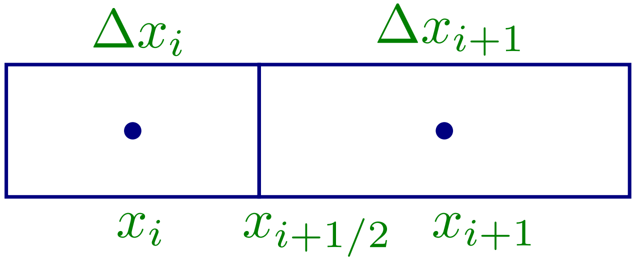

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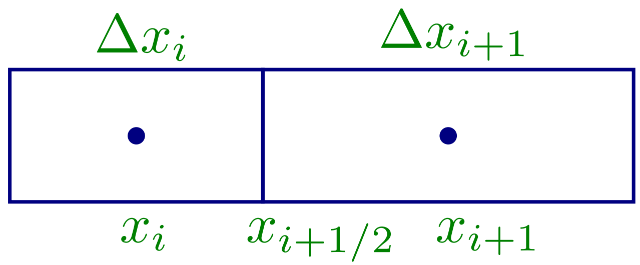

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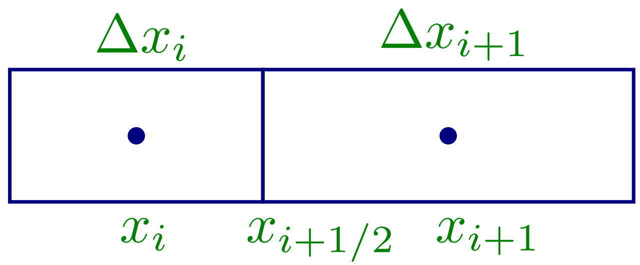

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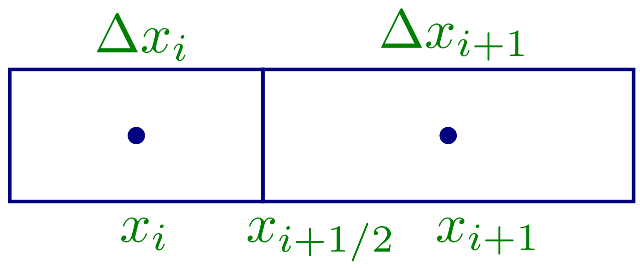

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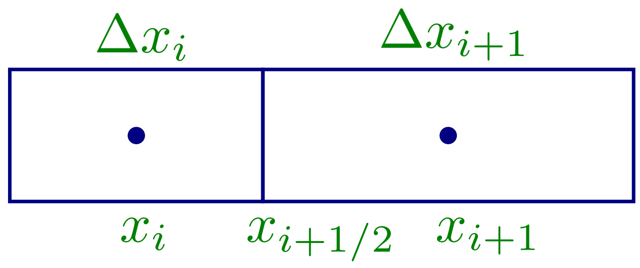
Principles:

One-dimensional flow



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Principles: continuous potential

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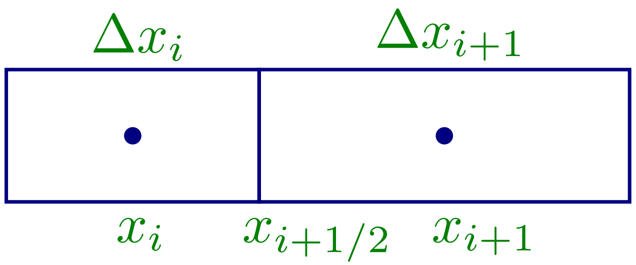

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Principles: continuous potential and continuous flux.

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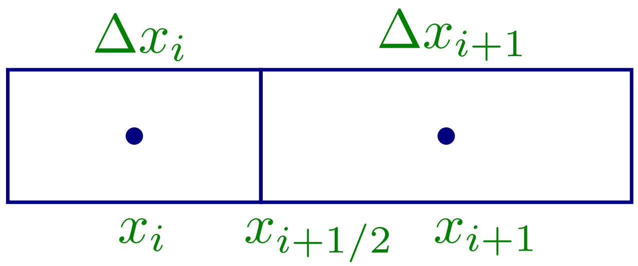

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Principles: continuous potential and continuous flux.
The flux is determined by the interaction of linear potentials in two cells.

Two-dimensional flow

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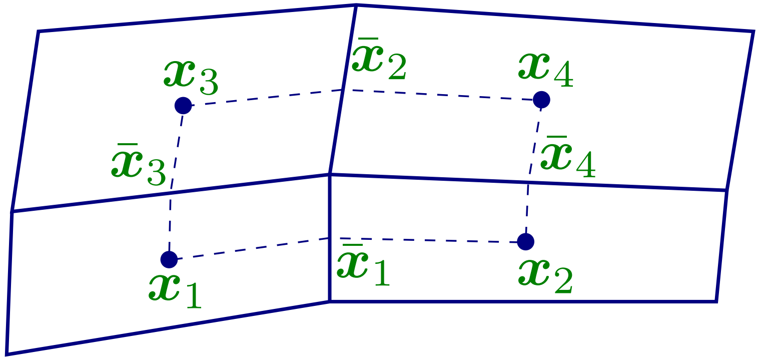
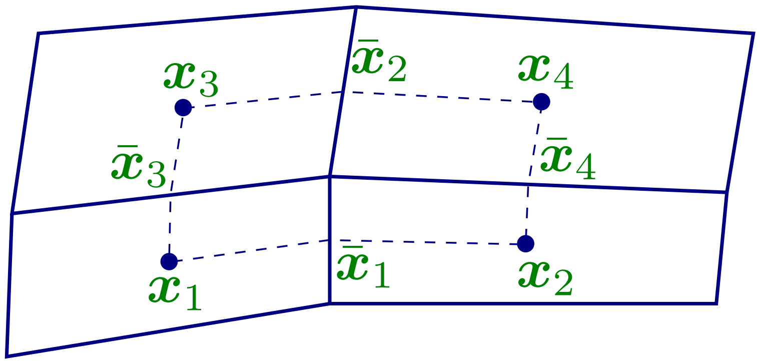


Cells with common corner

Two-dimensional flow



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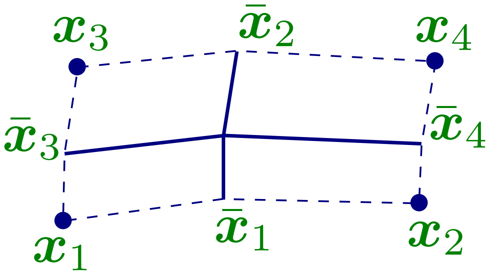
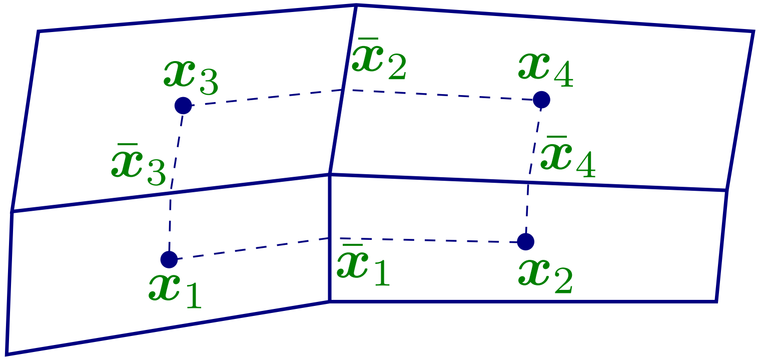


Interaction region

Two-dimensional flow



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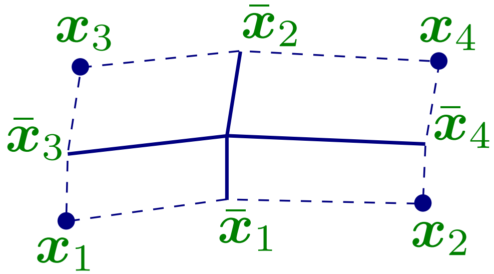
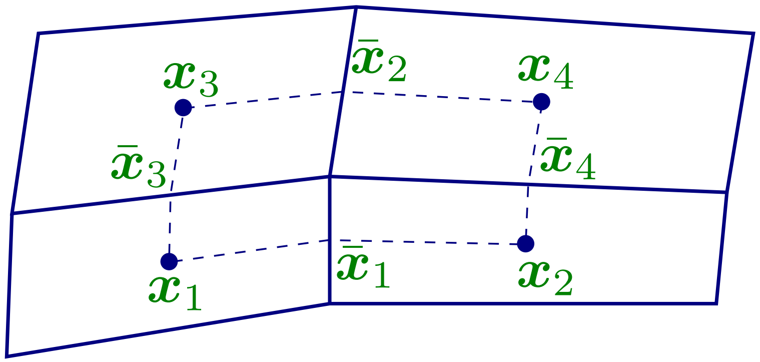


Interaction region

Determine the flux through the half-edges from the interaction

Two-dimensional flow



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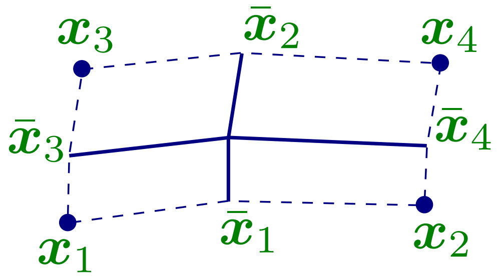
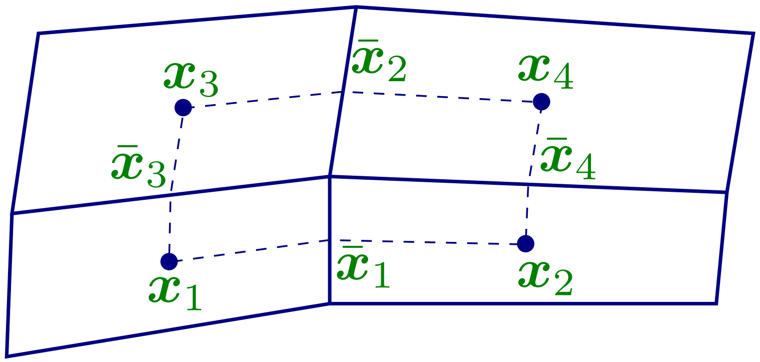


Interaction region

Determine the flux through the half-edges from the interaction of linear potentials in the four cells.

Two-dimensional flow



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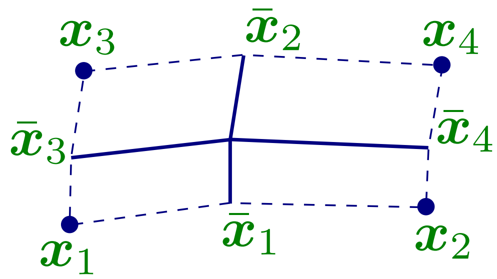
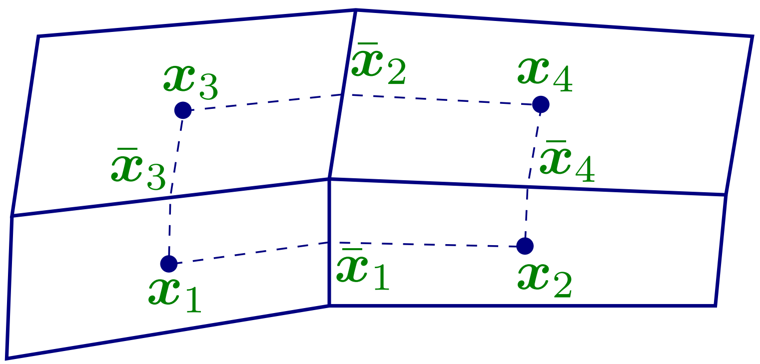


Interaction region

Determine the flux through the half-edges from the interaction of linear potentials in the four cells. Require continuous potential at $\bar{\boldsymbol{x}}_i$

Two-dimensional flow



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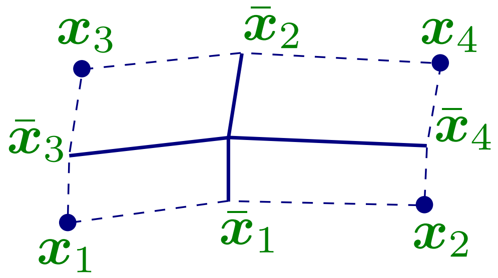

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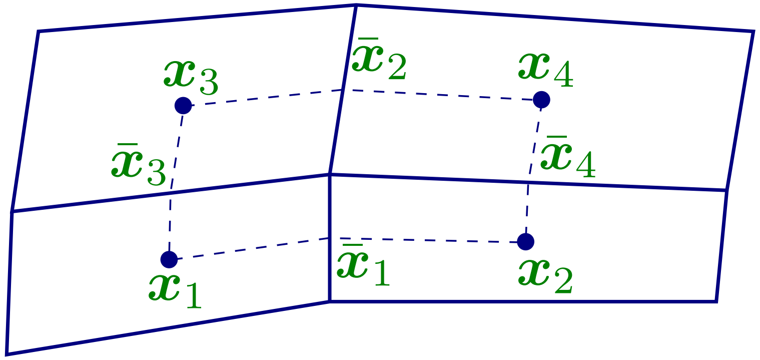

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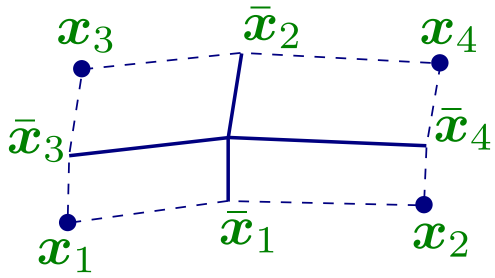
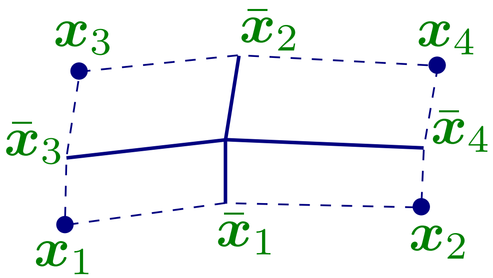


Interaction region

Determine the flux through the half-edges from the interaction of linear potentials in the four cells. Require continuous potential at $\bar{\boldsymbol{x}}_i$ and continuous flux through the half-edges.
This method is called the O-method.

O-method

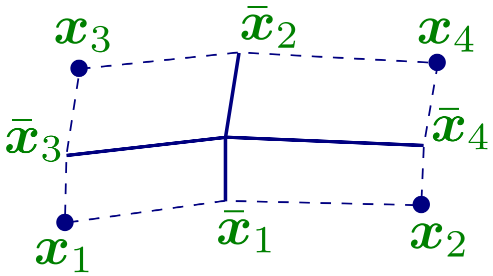
O-method



Interaction region

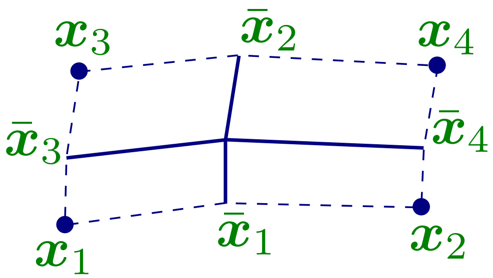
O-method

Linear potential in 4 cells: $4 \cdot 3 = 12$ unknowns



Interaction region

O-method

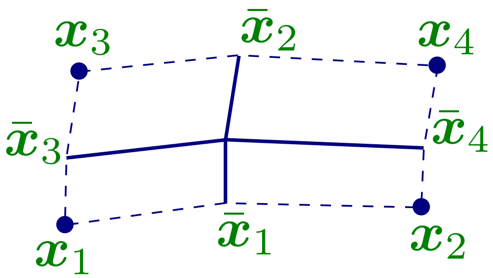


Interaction region

Linear potential in 4 cells: $4 \cdot 3 = 12$ unknowns

Potential continuity at $\bar{\boldsymbol{x}}_i$: 4 conditions

O-method



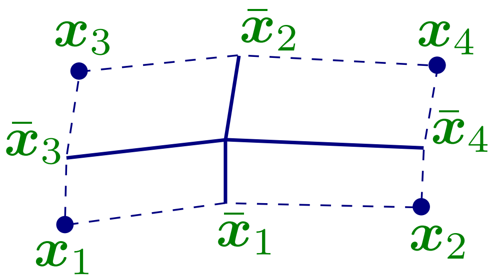
Interaction region

Linear potential in 4 cells: $4 \cdot 3 = 12$ unknowns

Potential continuity at $\bar{\boldsymbol{x}}_i$: 4 conditions

Flux continuity at the edges: 4 conditions

O-method



Interaction region

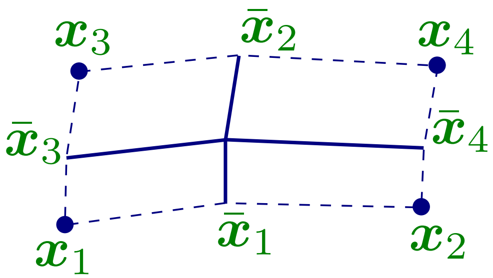
Linear potential in 4 cells: $4 \cdot 3 = 12$ unknowns

Potential continuity at $\bar{\boldsymbol{x}}_i$: 4 conditions

Flux continuity at the edges: 4 conditions

Potential values at cell centers: 4 conditions

O-method



Interaction region

Linear potential in 4 cells: $4 \cdot 3 = 12$ unknowns

Potential continuity at $\bar{\boldsymbol{x}}_i$: 4 conditions

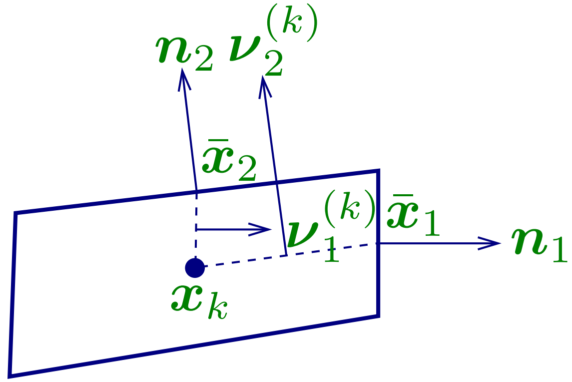
Flux continuity at the edges: 4 conditions

Potential values at cell centers: 4 conditions

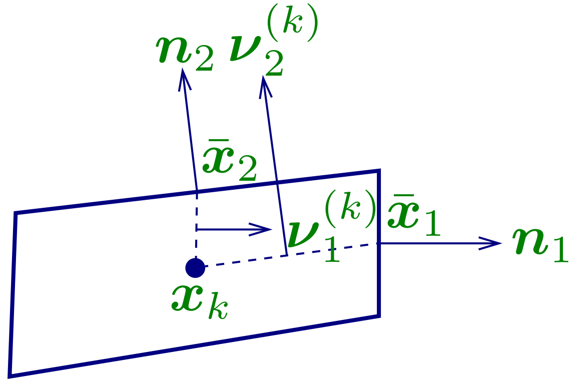
12 equations with 12 unknowns.

Flux expression in each cell

Flux expression in each cell

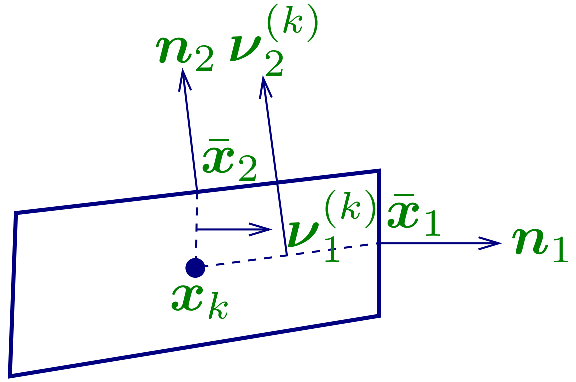


Flux expression in each cell



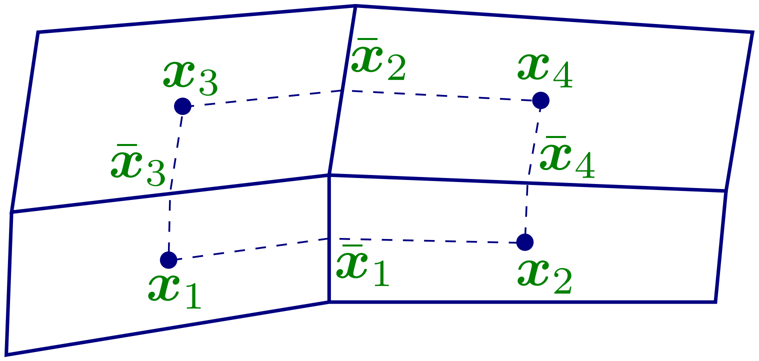
$$\begin{bmatrix} q_1^{(k)} \\ q_2^{(k)} \end{bmatrix} = -\boldsymbol{G}_k \begin{bmatrix} \bar{u}_1 - u_k \\ \bar{u}_2 - u_k \end{bmatrix},$$

Flux expression in each cell



$$\begin{bmatrix} q_1^{(k)} \\ q_2^{(k)} \end{bmatrix} = -\boldsymbol{G}_k \begin{bmatrix} \bar{u}_1 - u_k \\ \bar{u}_2 - u_k \end{bmatrix}, \qquad \boldsymbol{G}_k = \frac{1}{2F_k} \begin{bmatrix} \Gamma_1 \boldsymbol{n}_1^{\mathrm{T}} \boldsymbol{K}_k \boldsymbol{\nu}_1^{(k)} & \Gamma_1 \boldsymbol{n}_1^{\mathrm{T}} \boldsymbol{K}_k \boldsymbol{\nu}_2^{(k)} \\ \Gamma_2 \boldsymbol{n}_2^{\mathrm{T}} \boldsymbol{K}_k \boldsymbol{\nu}_1^{(k)} & \Gamma_2 \boldsymbol{n}_2^{\mathrm{T}} \boldsymbol{K}_k \boldsymbol{\nu}_2^{(k)} \end{bmatrix}$$

Flux equations in an interaction region



Cells with common corner

$$q_1 = q_1^{(1)} = q_1^{(2)}$$

$$q_2 = q_2^{(4)} = q_2^{(3)}$$

$$q_3 = q_3^{(3)} = q_3^{(1)}$$

$$q_4 = q_4^{(2)} = q_4^{(4)}$$

Flux equations in an interaction region

Flux equations in an interaction region

$$q_1 = -g_{1,1}^{(1)}(\bar{u}_1 - u_1) - g_{1,2}^{(1)}(\bar{u}_3 - u_1) = g_{1,1}^{(2)}(\bar{u}_1 - u_2) - g_{1,2}^{(2)}(\bar{u}_4 - u_2)$$

$$q_2 = g_{1,1}^{(4)}(\bar{u}_2 - u_4) + g_{1,2}^{(4)}(\bar{u}_4 - u_4) = -g_{1,1}^{(3)}(\bar{u}_2 - u_3) + g_{1,2}^{(3)}(\bar{u}_3 - u_3)$$

$$q_3 = -g_{2,1}^{(3)}(\bar{u}_2 - u_3) + g_{2,2}^{(3)}(\bar{u}_3 - u_3) = -g_{2,1}^{(1)}(\bar{u}_1 - u_1) - g_{2,2}^{(1)}(\bar{u}_3 - u_1)$$

$$q_4 = g_{2,1}^{(2)}(\bar{u}_1 - u_2) - g_{2,2}^{(2)}(\bar{u}_4 - u_2) = g_{2,1}^{(4)}(\bar{u}_2 - u_4) + g_{2,2}^{(4)}(\bar{u}_4 - u_4)$$

Flux equations in an interaction region

$$q_1 = -g^{(1)}_{1,1}(\bar{u}_1 - u_1) - g^{(1)}_{1,2}(\bar{u}_3 - u_1) = g^{(2)}_{1,1}(\bar{u}_1 - u_2) - g^{(2)}_{1,2}(\bar{u}_4 - u_2)$$

$$q_2 = g^{(4)}_{1,1}(\bar{u}_2 - u_4) + g^{(4)}_{1,2}(\bar{u}_4 - u_4) = -g^{(3)}_{1,1}(\bar{u}_2 - u_3) + g^{(3)}_{1,2}(\bar{u}_3 - u_3)$$

$$q_3 = -g^{(3)}_{2,1}(\bar{u}_2 - u_3) + g^{(3)}_{2,2}(\bar{u}_3 - u_3) = -g^{(1)}_{2,1}(\bar{u}_1 - u_1) - g^{(1)}_{2,2}(\bar{u}_3 - u_1)$$

$$q_4 = g^{(2)}_{2,1}(\bar{u}_1 - u_2) - g^{(2)}_{2,2}(\bar{u}_4 - u_2) = g^{(4)}_{2,1}(\bar{u}_2 - u_4) + g^{(4)}_{2,2}(\bar{u}_4 - u_4)$$

$$\tilde{\boldsymbol{q}} = [q_1,\ q_2,\ q_3,\ q_4]^{\mathrm{T}} \qquad \boldsymbol{u} = [u_1,\ u_2,\ u_3,\ u_4]^{\mathrm{T}} \qquad \boldsymbol{v} = [\bar{u}_1,\ \bar{u}_2,\ \bar{u}_3,\ \bar{u}_4]^{\mathrm{T}}$$

Flux equations in an interaction region

$$q_1 = -g_{1,1}^{(1)}(\bar{u}_1 - u_1) - g_{1,2}^{(1)}(\bar{u}_3 - u_1) = g_{1,1}^{(2)}(\bar{u}_1 - u_2) - g_{1,2}^{(2)}(\bar{u}_4 - u_2)$$

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$$q_4 = g_{2,1}^{(2)}(\bar{u}_1 - u_2) - g_{2,2}^{(2)}(\bar{u}_4 - u_2) = g_{2,1}^{(4)}(\bar{u}_2 - u_4) + g_{2,2}^{(4)}(\bar{u}_4 - u_4)$$

$$\tilde{\boldsymbol{q}} = [q_1,\ q_2,\ q_3,\ q_4]^{\mathrm{T}} \qquad \boldsymbol{u} = [u_1,\ u_2,\ u_3,\ u_4]^{\mathrm{T}} \qquad \boldsymbol{v} = [\bar{u}_1,\ \bar{u}_2,\ \bar{u}_3,\ \bar{u}_4]^{\mathrm{T}}$$

$$\tilde{\boldsymbol{q}} = \boldsymbol{C}\boldsymbol{v} + \boldsymbol{F}\boldsymbol{u},$$

Flux equations in an interaction region

$$q_1 = -g^{(1)}_{1,1}(\bar{u}_1 - u_1) - g^{(1)}_{1,2}(\bar{u}_3 - u_1) = g^{(2)}_{1,1}(\bar{u}_1 - u_2) - g^{(2)}_{1,2}(\bar{u}_4 - u_2)$$

$$q_2 = g^{(4)}_{1,1}(\bar{u}_2 - u_4) + g^{(4)}_{1,2}(\bar{u}_4 - u_4) = -g^{(3)}_{1,1}(\bar{u}_2 - u_3) + g^{(3)}_{1,2}(\bar{u}_3 - u_3)$$

$$q_3 = -g^{(3)}_{2,1}(\bar{u}_2 - u_3) + g^{(3)}_{2,2}(\bar{u}_3 - u_3) = -g^{(1)}_{2,1}(\bar{u}_1 - u_1) - g^{(1)}_{2,2}(\bar{u}_3 - u_1)$$

$$q_4 = g^{(2)}_{2,1}(\bar{u}_1 - u_2) - g^{(2)}_{2,2}(\bar{u}_4 - u_2) = g^{(4)}_{2,1}(\bar{u}_2 - u_4) + g^{(4)}_{2,2}(\bar{u}_4 - u_4)$$

$$\tilde{\boldsymbol{q}} = [q_1,\ q_2,\ q_3,\ q_4]^{\mathrm{T}} \qquad \boldsymbol{u} = [u_1,\ u_2,\ u_3,\ u_4]^{\mathrm{T}} \qquad \boldsymbol{v} = [\bar{u}_1,\ \bar{u}_2,\ \bar{u}_3,\ \bar{u}_4]^{\mathrm{T}}$$

$$\tilde{\boldsymbol{q}} = \boldsymbol{C}\boldsymbol{v} + \boldsymbol{F}\boldsymbol{u}, \qquad \boldsymbol{A}\boldsymbol{v} = \boldsymbol{B}\boldsymbol{u}$$

Flux equations in an interaction region

$$q_1 = -g^{(1)}_{1,1}(\bar{u}_1 - u_1) - g^{(1)}_{1,2}(\bar{u}_3 - u_1) = g^{(2)}_{1,1}(\bar{u}_1 - u_2) - g^{(2)}_{1,2}(\bar{u}_4 - u_2)$$

$$q_2 = g^{(4)}_{1,1}(\bar{u}_2 - u_4) + g^{(4)}_{1,2}(\bar{u}_4 - u_4) = -g^{(3)}_{1,1}(\bar{u}_2 - u_3) + g^{(3)}_{1,2}(\bar{u}_3 - u_3)$$

$$q_3 = -g^{(3)}_{2,1}(\bar{u}_2 - u_3) + g^{(3)}_{2,2}(\bar{u}_3 - u_3) = -g^{(1)}_{2,1}(\bar{u}_1 - u_1) - g^{(1)}_{2,2}(\bar{u}_3 - u_1)$$

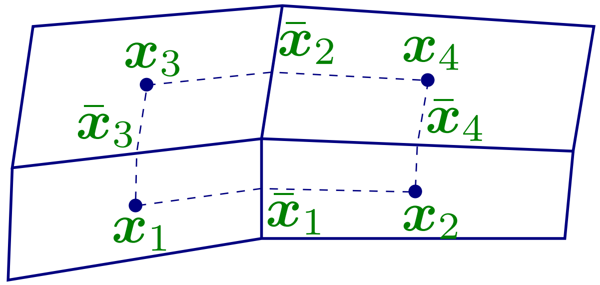
$$q_4 = g^{(2)}_{2,1}(\bar{u}_1 - u_2) - g^{(2)}_{2,2}(\bar{u}_4 - u_2) = g^{(4)}_{2,1}(\bar{u}_2 - u_4) + g^{(4)}_{2,2}(\bar{u}_4 - u_4)$$

$$\tilde{\boldsymbol{q}} = [q_1,\ q_2,\ q_3,\ q_4]^{\mathrm{T}} \qquad \boldsymbol{u} = [u_1,\ u_2,\ u_3,\ u_4]^{\mathrm{T}} \qquad \boldsymbol{v} = [\bar{u}_1,\ \bar{u}_2,\ \bar{u}_3,\ \bar{u}_4]^{\mathrm{T}}$$

$$\tilde{\boldsymbol{q}} = \boldsymbol{C}\boldsymbol{v} + \boldsymbol{F}\boldsymbol{u}, \qquad \boldsymbol{A}\boldsymbol{v} = \boldsymbol{B}\boldsymbol{u} \qquad \Rightarrow \qquad \tilde{\boldsymbol{q}} = \left(\boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B} + \boldsymbol{F}\right)\boldsymbol{u}$$

Flux expression

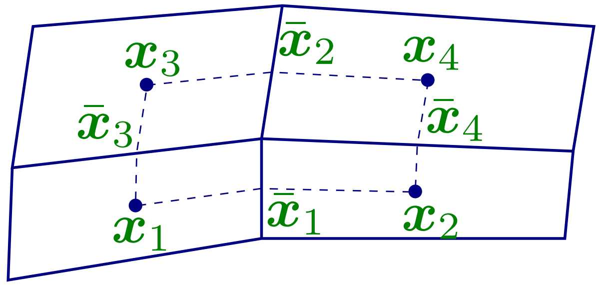
Flux expression



Cells with common corner

$$q_i = \sum_{j=1}^{4} t_{i,j} u_j \quad \text{where} \quad \sum_{j=1}^{4} t_{i,j} = 0$$

Flux expression



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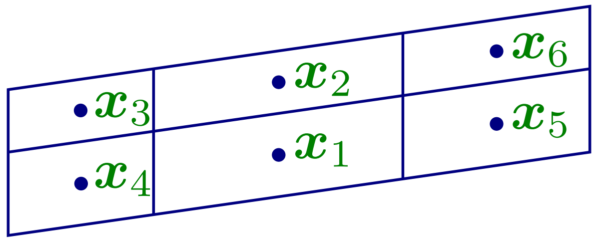
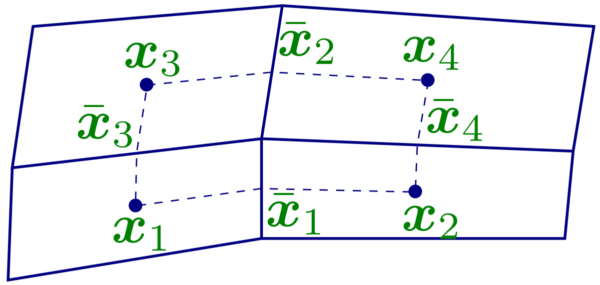


Flux stencil

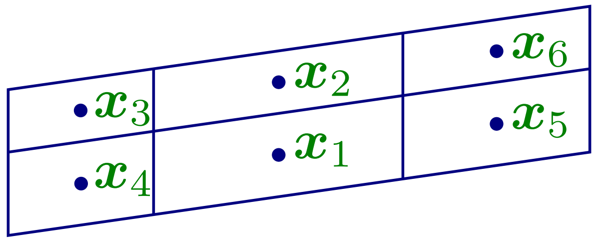
$$q_i = \sum_{j=1}^{6} t_{i,j} u_j \quad \text{where} \quad \sum_{j=1}^{6} t_{i,j} = 0$$

Flux expression



Cells with common corner

$$q_i = \sum_{j=1}^{4} t_{i,j} u_j \quad \text{where} \quad \sum_{j=1}^{4} t_{i,j} = 0$$



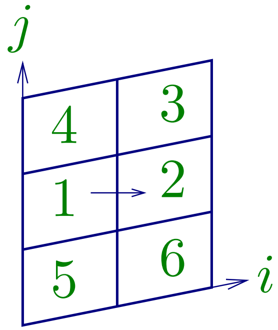
Flux stencil

$$q_i = \sum_{j=1}^{6} t_{i,j} u_j \quad \text{where} \quad \sum_{j=1}^{6} t_{i,j} = 0$$

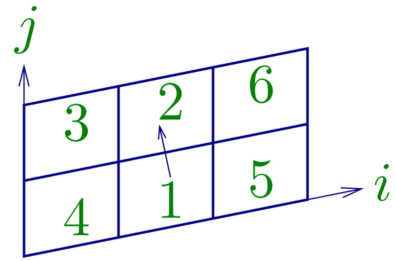
Multipoint flux approximation (MPFA)

Stencils

Stencils

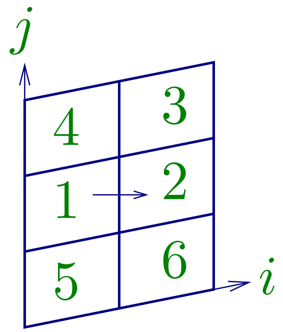


i-flux stencil

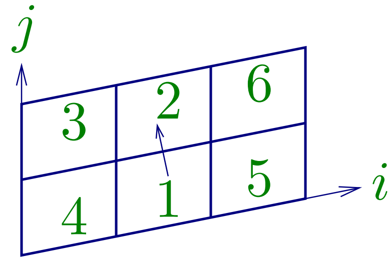


j-flux stencil

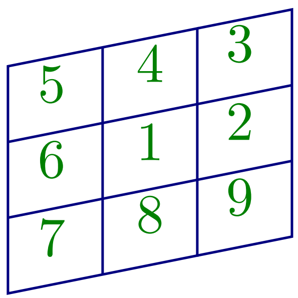
Stencils



i-flux stencil



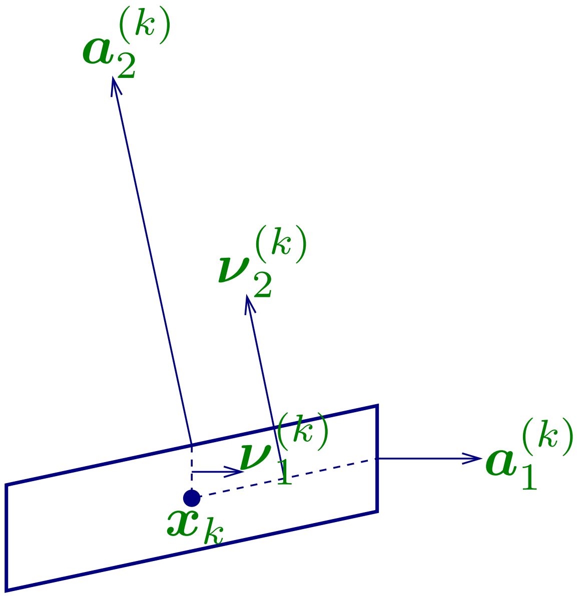
j-flux stencil



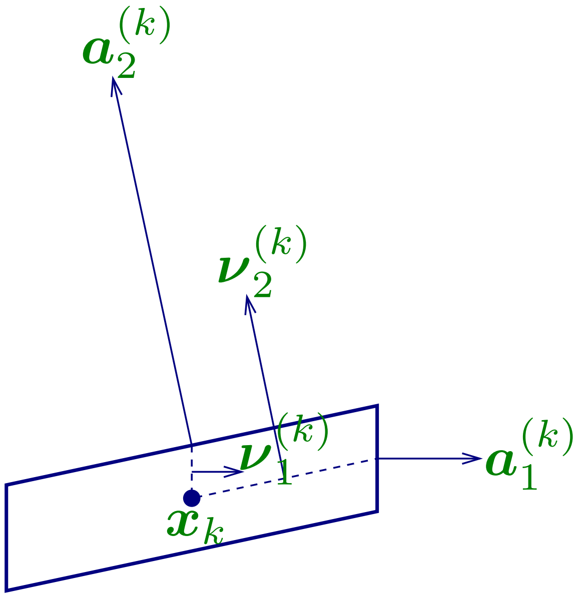
Cell stencil

Parallelogram cells

Parallelogram cells



Parallelogram cells

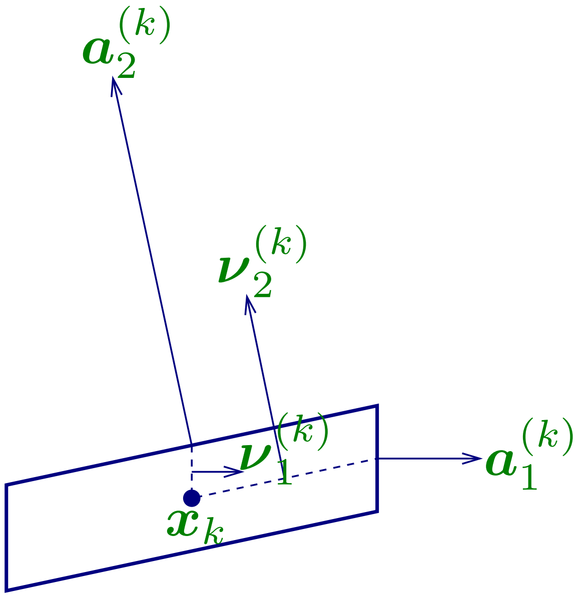


$$G_k = \frac{1}{V_k}\left[a_1^{(k)} \quad a_2^{(k)}\right]^{\mathrm{T}} K_k \left[a_1^{(k)} \quad a_2^{(k)}\right]$$

$$G_k = \frac{1}{|\det J_k|} J_k^{\mathrm{T}} K_k J_k$$

Congruence transformation
$\Rightarrow G_k$ positive definite

Parallelogram cells



$$G_k = \frac{1}{V_k} \begin{bmatrix} a_1^{(k)} & a_2^{(k)} \end{bmatrix}^{\mathrm{T}} K_k \begin{bmatrix} a_1^{(k)} & a_2^{(k)} \end{bmatrix}$$

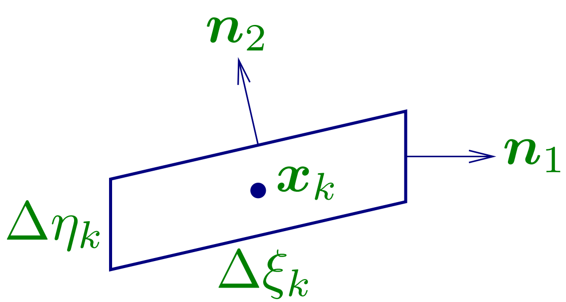
$$G_k = \frac{1}{|\det J_k|} J_k^{\mathrm{T}} K_k J_k$$

Congruence transformation
$\Rightarrow G_k$ positive definite

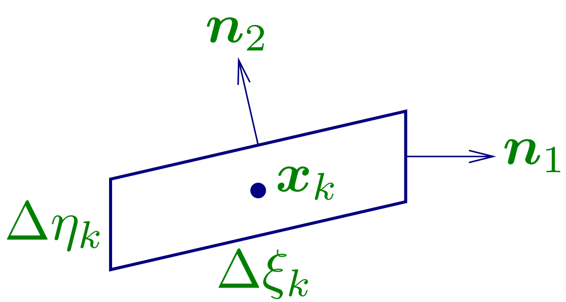
Symmetry and positive definiteness of G_k is crucial for proving that the coefficient matrix in the system of equations is symmetric and positive definite.

Parallelogram cells

Parallelogram cells

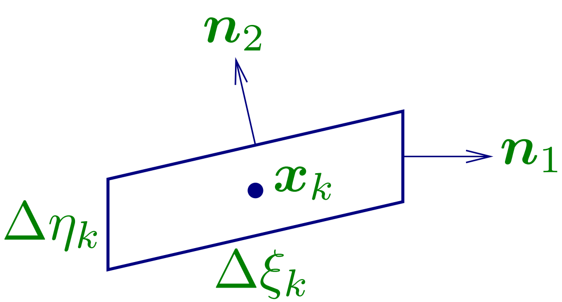


Parallelogram cells



$$\boldsymbol{G}_k = \frac{1}{\Delta\xi_k \Delta\eta_k} \boldsymbol{D}_k \boldsymbol{H}_k \boldsymbol{D}_k,$$

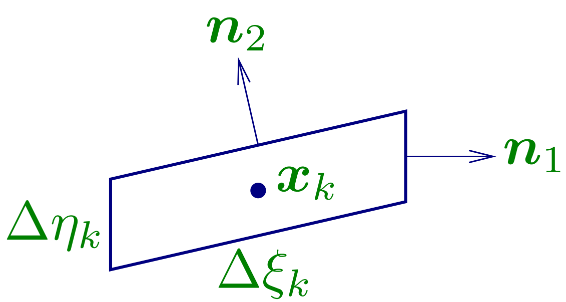
Parallelogram cells



$$\boldsymbol{G}_k = \frac{1}{\Delta\xi_k \Delta\eta_k} \boldsymbol{D}_k \boldsymbol{H}_k \boldsymbol{D}_k,$$

$$\boldsymbol{H}_k = \frac{1}{\det[\boldsymbol{n}_1,\ \boldsymbol{n}_2]} \begin{bmatrix} \boldsymbol{n}_1^{\mathrm{T}} \boldsymbol{K}_k \boldsymbol{n}_1 & \boldsymbol{n}_1^{\mathrm{T}} \boldsymbol{K}_k \boldsymbol{n}_2 \\ \boldsymbol{n}_2^{\mathrm{T}} \boldsymbol{K}_k \boldsymbol{n}_1 & \boldsymbol{n}_2^{\mathrm{T}} \boldsymbol{K}_k \boldsymbol{n}_2 \end{bmatrix}, \qquad \boldsymbol{D}_k = \mathrm{diag}(\Delta\eta_k,\ \Delta\xi_k).$$

Parallelogram cells

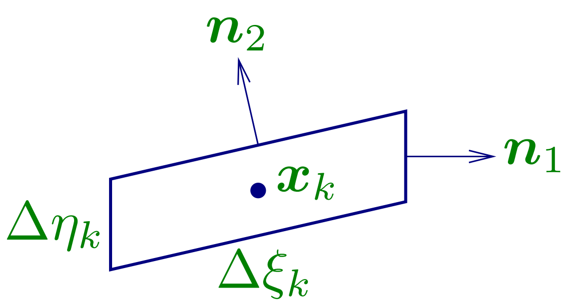


$$\boldsymbol{G}_k = \frac{1}{\Delta\xi_k \Delta\eta_k} \boldsymbol{D}_k \boldsymbol{H}_k \boldsymbol{D}_k,$$

$$\boldsymbol{H}_k = \frac{1}{\det[\boldsymbol{n}_1,\ \boldsymbol{n}_2]} \begin{bmatrix} \boldsymbol{n}_1^{\mathrm{T}} \boldsymbol{K}_k \boldsymbol{n}_1 & \boldsymbol{n}_1^{\mathrm{T}} \boldsymbol{K}_k \boldsymbol{n}_2 \\ \boldsymbol{n}_2^{\mathrm{T}} \boldsymbol{K}_k \boldsymbol{n}_1 & \boldsymbol{n}_2^{\mathrm{T}} \boldsymbol{K}_k \boldsymbol{n}_2 \end{bmatrix}, \qquad \boldsymbol{D}_k = \mathrm{diag}(\Delta\eta_k,\ \Delta\xi_k).$$

$\boldsymbol{H}_k$ contains anisotropy and grid skrewness. $\boldsymbol{D}_k$ contains cell dimensions.

Parallelogram cells



$$\boldsymbol{G}_k = \frac{1}{\Delta\xi_k \Delta\eta_k} \boldsymbol{D}_k \boldsymbol{H}_k \boldsymbol{D}_k,$$

$$\boldsymbol{H}_k = \frac{1}{\det[\boldsymbol{n}_1,\ \boldsymbol{n}_2]} \begin{bmatrix} \boldsymbol{n}_1^{\mathrm{T}} \boldsymbol{K}_k \boldsymbol{n}_1 & \boldsymbol{n}_1^{\mathrm{T}} \boldsymbol{K}_k \boldsymbol{n}_2 \\ \boldsymbol{n}_2^{\mathrm{T}} \boldsymbol{K}_k \boldsymbol{n}_1 & \boldsymbol{n}_2^{\mathrm{T}} \boldsymbol{K}_k \boldsymbol{n}_2 \end{bmatrix}, \qquad \boldsymbol{D}_k = \mathrm{diag}(\Delta\eta_k,\ \Delta\xi_k).$$

$\boldsymbol{H}_k$ contains anisotropy and grid skrewness. $\boldsymbol{D}_k$ contains cell dimensions.

If $\boldsymbol{n}_i^{\mathrm{T}} \boldsymbol{K} \boldsymbol{n}_j = 0$, $i \neq j$, then $\boldsymbol{H}_k$ and $\boldsymbol{G}_k$ are diagonal, and the multipoint flux reduces to a two-point flux. Such a grid is called $\boldsymbol{K}$-orthogonal.

O-method

O-method

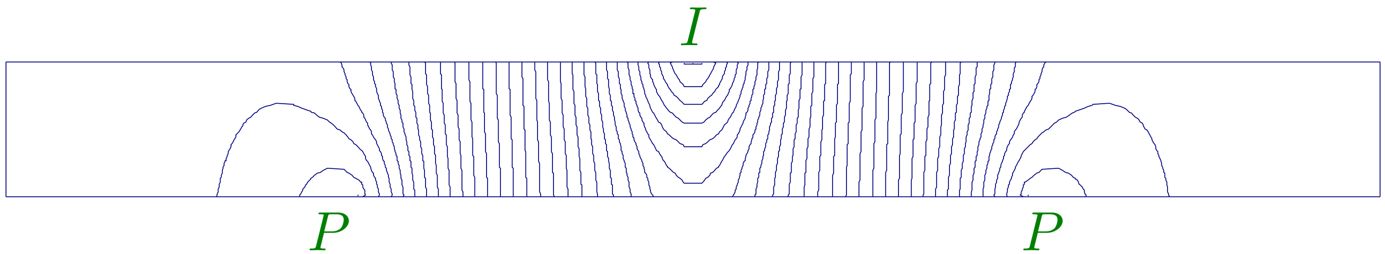
- L^2-stability can be proved for the O-method on parallelogram grids.

O-method

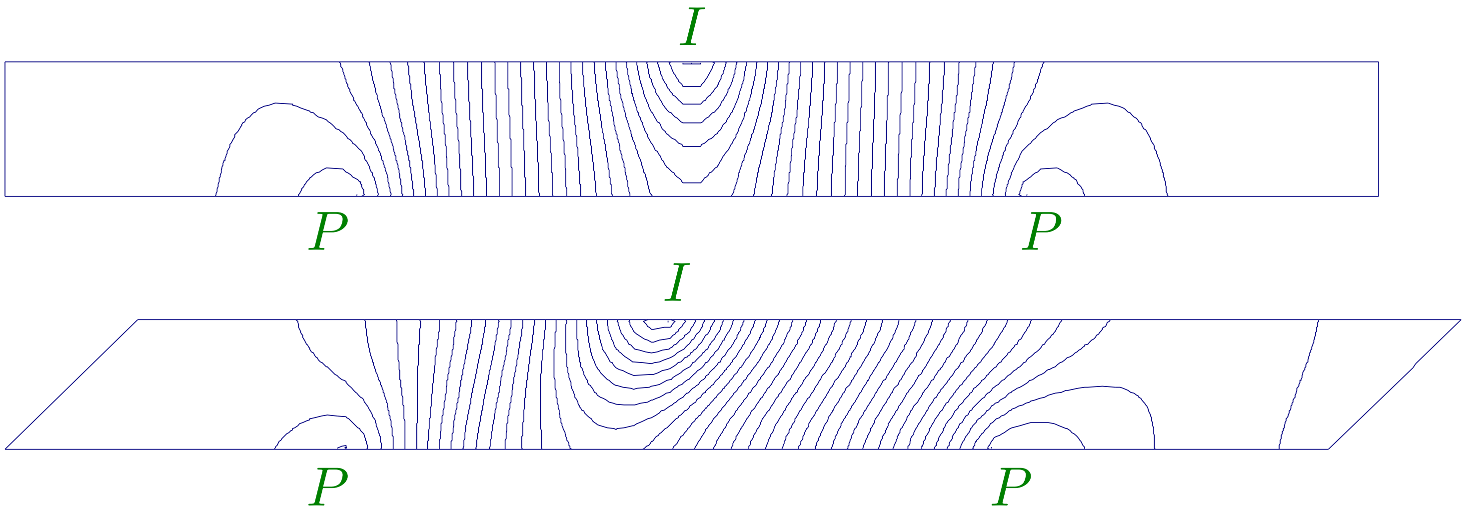
- L^2-stability can be proved for the O-method on parallelogram grids.
- The O-method extends to three dimensions. The interaction region contains 8 cells. The flux stencil has 18 cells, and the cell stencil has 27 cells.

Results — pressure

Results — pressure

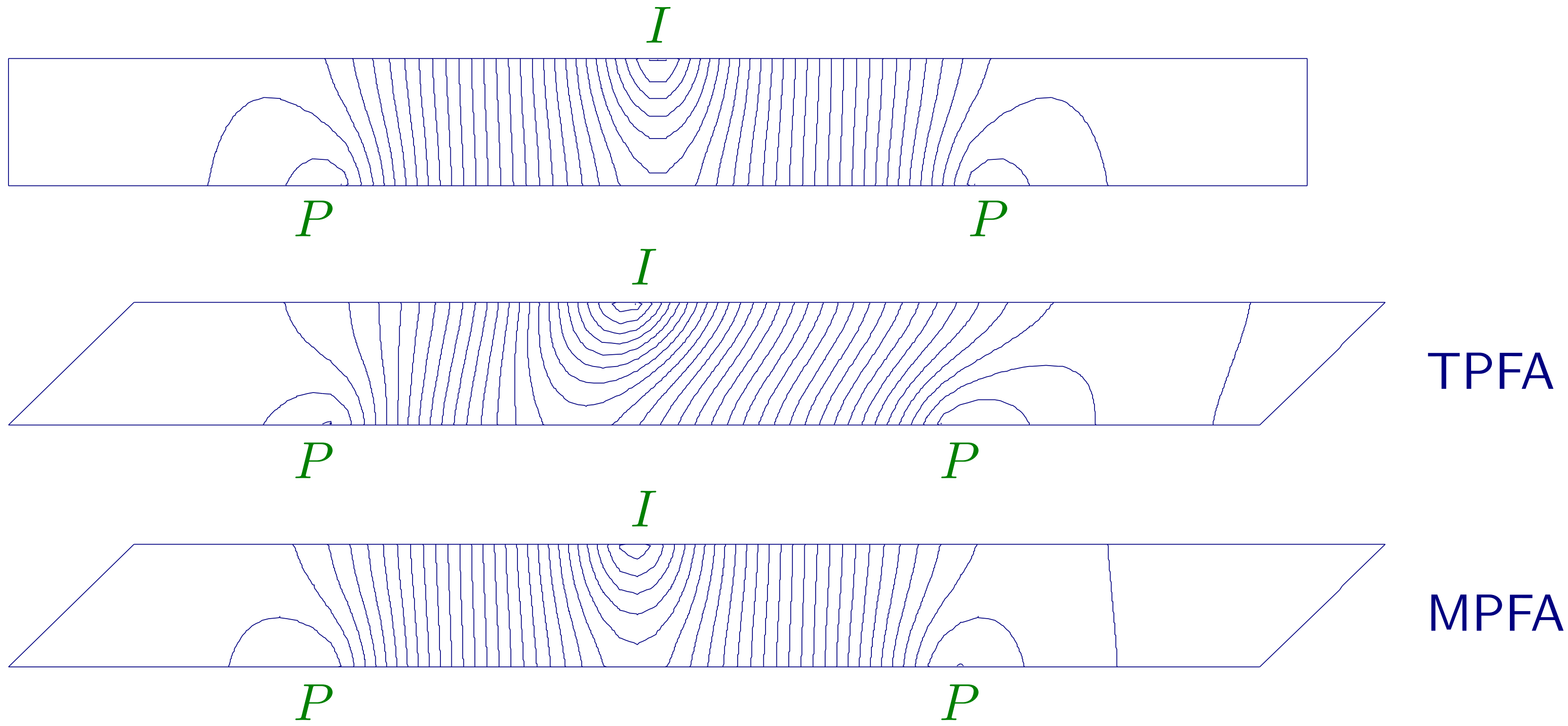


Results — pressure



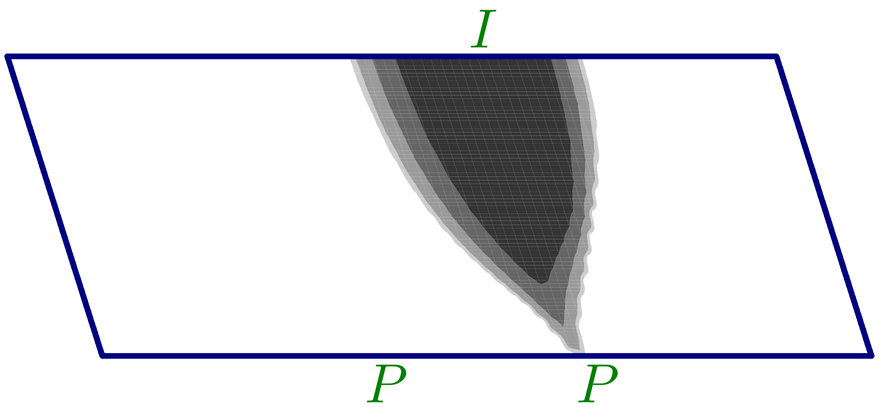
TPFA

Results — pressure



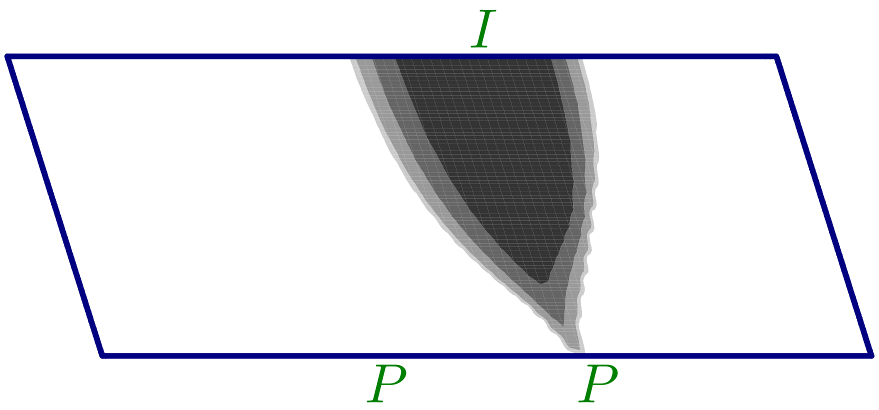
Results — saturation

Results — saturation

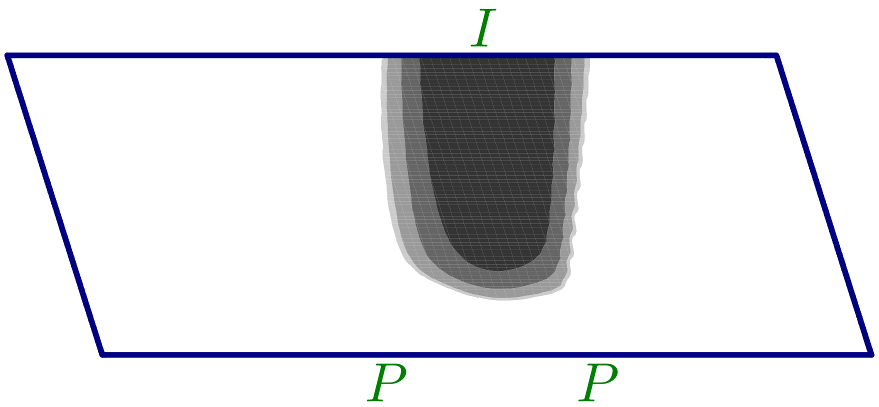


TPFA

Results — saturation



TPFA



MPFA

Related methods — U-method

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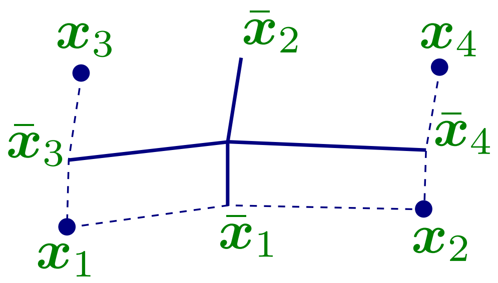
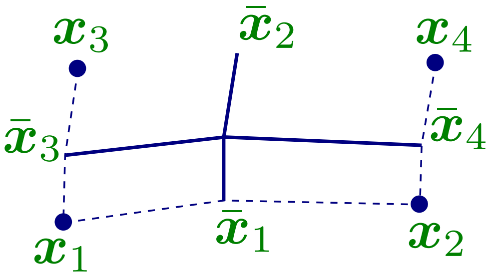


Interaction region

Related methods — U-method

Linear potential in 4 cells: $4 \cdot 3 = 12$ unknowns



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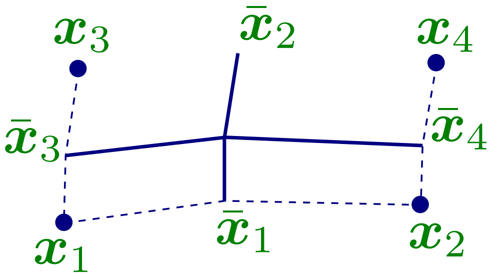

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Linear potential in 4 cells: $4 \cdot 3 = 12$ unknowns

Weak potential continuity at central edge: 1 condition

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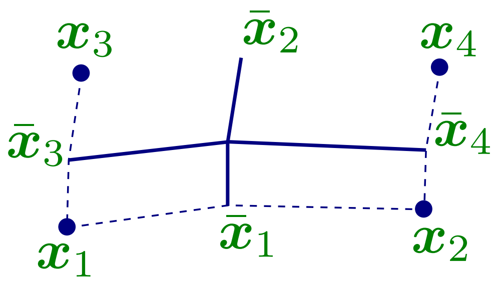

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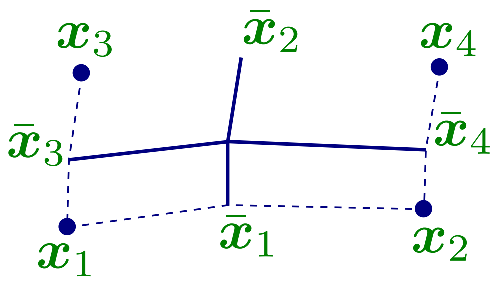

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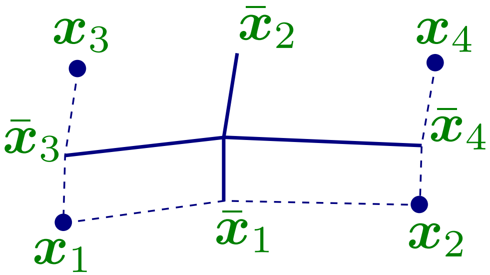

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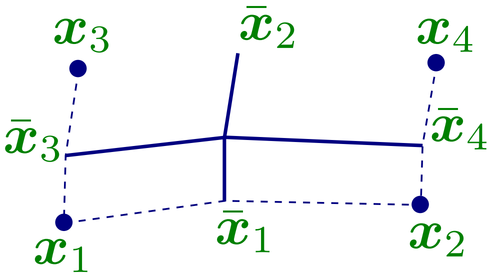

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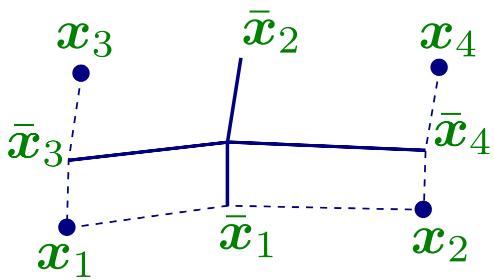

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Non-symmetric matrix of coefficients.
Looser hinging might be advantageous by strong heterogeneity.

Related methods — Edwards & Rogers

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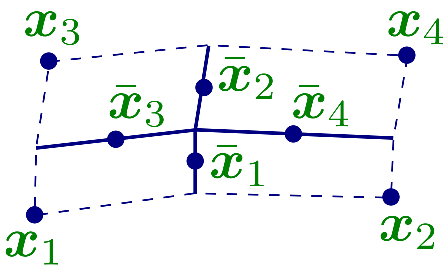

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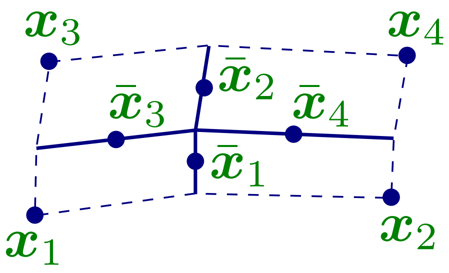

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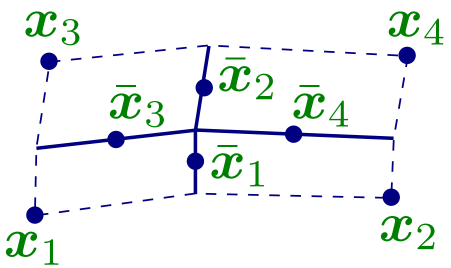

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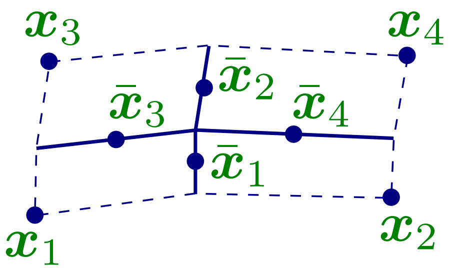

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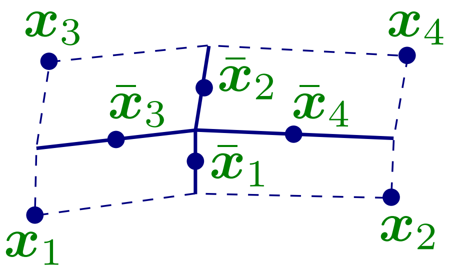

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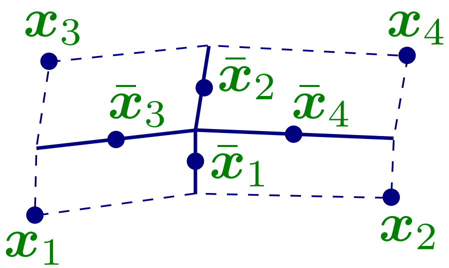

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Does not reduce to five-point stencil for $\boldsymbol{K}$-orthogonal grids.

Expanded mixed finite element method

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$$\boldsymbol{q} = -\boldsymbol{K}\boldsymbol{g}, \qquad \boldsymbol{g} = \operatorname{grad} u, \qquad \operatorname{div} \boldsymbol{q} = f$$

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- Apply RT_0 functions. $\boldsymbol{g}$ must be constant on each edge, but is not continuous.
- Klausen proved convergence with trapezoidal quadrature: If $u \in H^2(\Omega)$ and $\boldsymbol{q} \in (H^2(\Omega))^d$, then

$$\|u_h - u\|_{L^2(\Omega)} + \|\boldsymbol{q}_h - \boldsymbol{q}\|_{H(\mathrm{div},\Omega)} \leq Ch$$

Convergence for O-method on parallelogram grids

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Recall the flux expressions

$$\begin{bmatrix} q_1^{(k)} \\ q_2^{(k)} \end{bmatrix} = -\boldsymbol{G}_k \begin{bmatrix} \bar{u}_1 - u_k \\ \bar{u}_2 - u_k \end{bmatrix}, \qquad \boldsymbol{G}_k = \frac{1}{\Delta\xi_k \Delta\eta_k} \boldsymbol{D}_k \boldsymbol{H}_k \boldsymbol{D}_k,$$

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Klausen showed that these fluxes can be expressed by the fluxes from the Expanded Mixed Finite Element Method with trapezoidal quadrature.

Relation between EMFEM and MPFA O-method

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$$\boldsymbol{Q} = \frac{1}{2\det\boldsymbol{H}}\begin{bmatrix} \alpha & \alpha & \delta & \delta & \beta & -\beta & \beta & -\beta \\ \delta & \delta & \alpha & \alpha & -\beta & \beta & -\beta & \beta \\ -\gamma & \gamma & -\gamma & \gamma & \alpha & \alpha & \delta & \delta \\ \gamma & -\gamma & \gamma & -\gamma & \delta & \delta & \alpha & \alpha \end{bmatrix}$$

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$$\alpha = 2h_{11}h_{22} - h_{12}^2, \qquad \beta = h_{11}h_{12}$$

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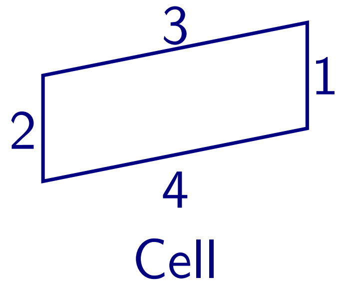

Convergence

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Hence, if $u \in H^2(\Omega)$ and $\boldsymbol{q} \in (H^2(\Omega))^d$, then for the O-method on parallelogram grids:

$$\|u_h - u\|_{L^2(\Omega)} + \|\boldsymbol{q}_h - \boldsymbol{q}\|_{H(\text{div},\Omega)} \leq Ch$$

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The proof can be extended to arbitrary quadrilaterals using the Piola mapping.

Convergence of nonregular solutions

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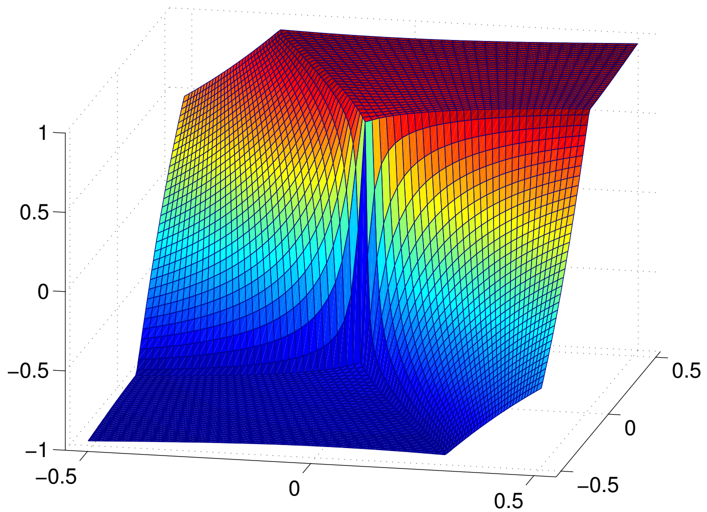
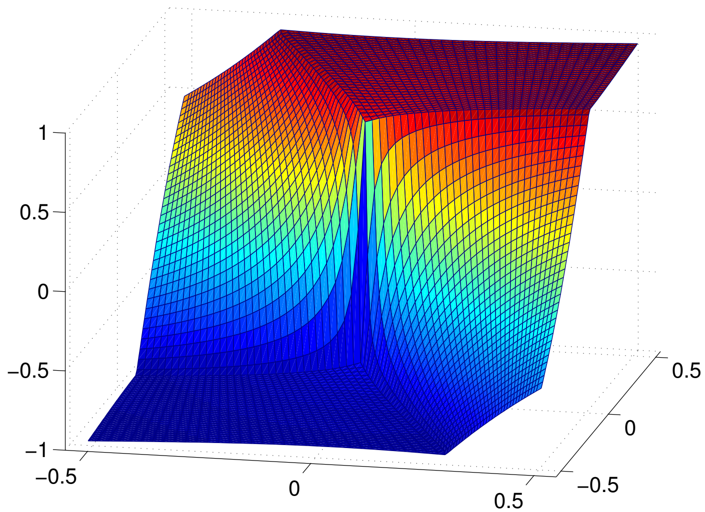


Potential $u \sim r^{\alpha}$

Convergence of nonregular solutions



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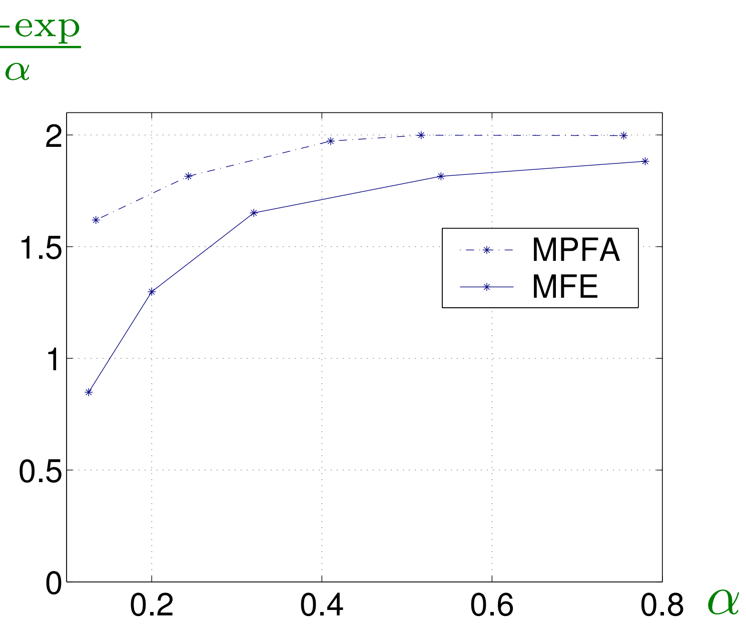
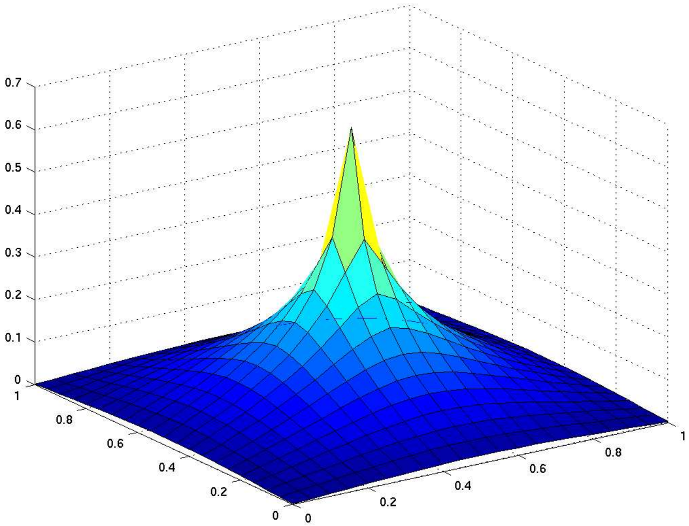


Error $\|u_h - u\|_{L^2}$

Monotonicity

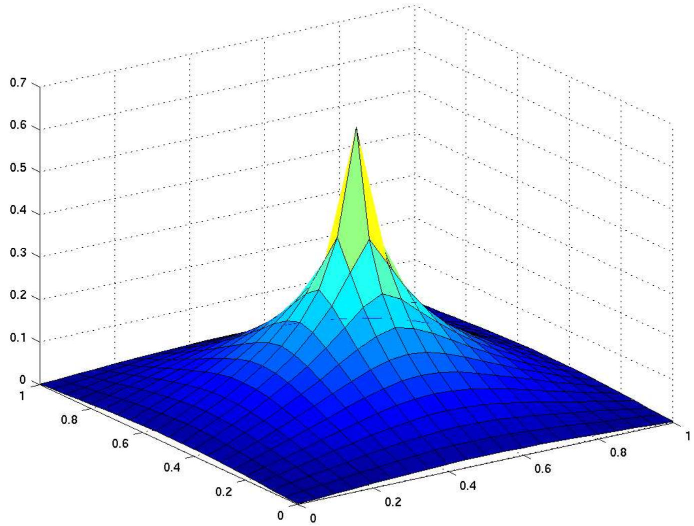
Monotonicity

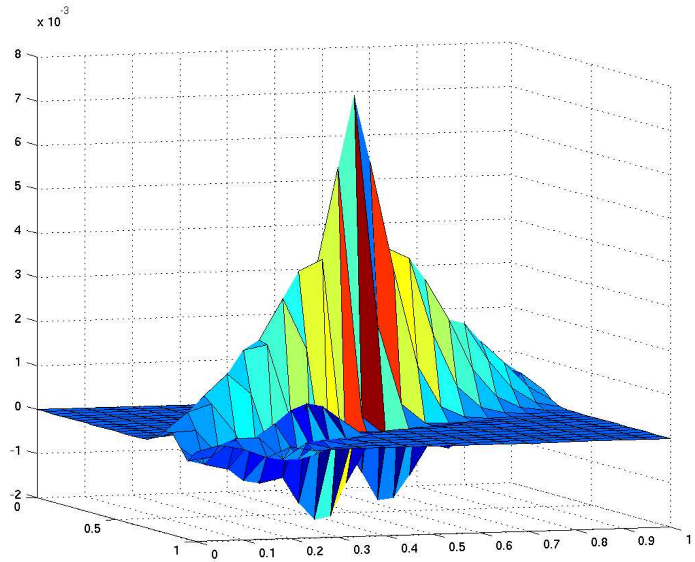


Isotropic

$\boldsymbol{A}^{-1} \geq \boldsymbol{O}$

Monotonicity





Isotropic

$\boldsymbol{A}^{-1} \geq \boldsymbol{O}$

Anisotropy ratio $1:1000$, $\theta = \pi/6$.

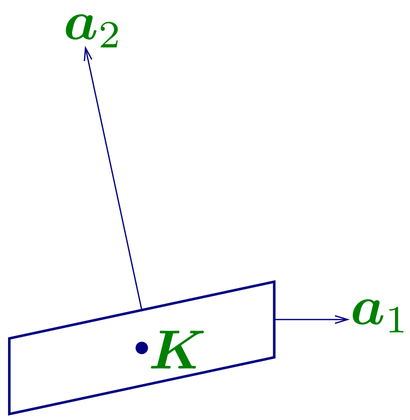
$\boldsymbol{A}^{-1} \not\geq \boldsymbol{O}$

Monotonicity

Monotonicity

$$\boldsymbol{A}\boldsymbol{u} = \boldsymbol{b}$$

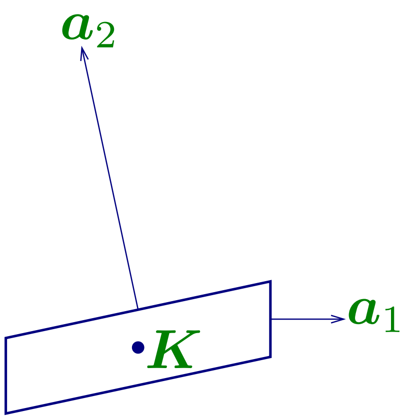
$$\boldsymbol{A}^{-1} \geq \boldsymbol{O}$$

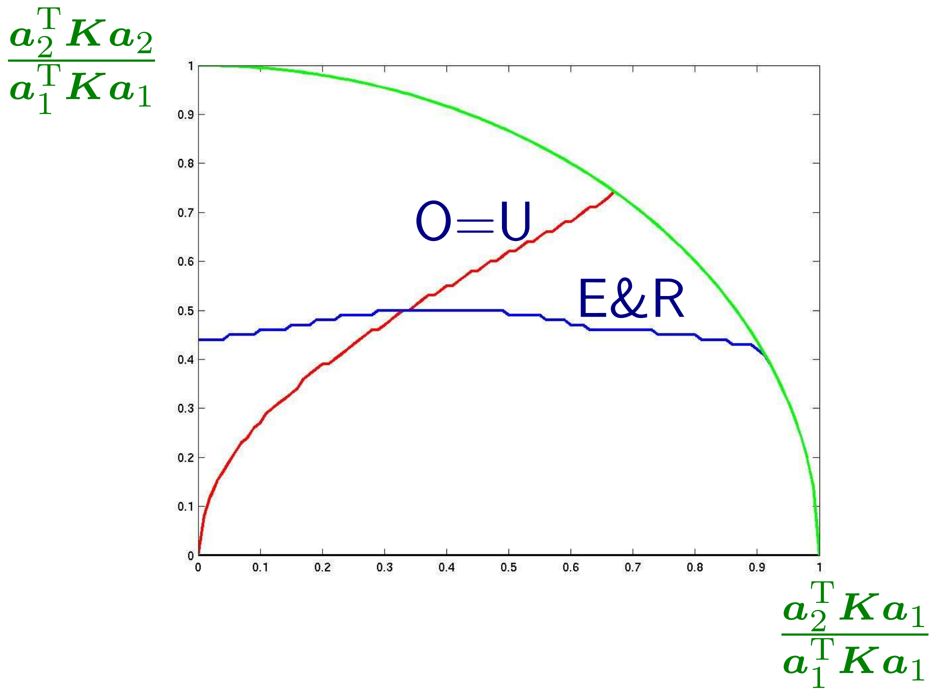


Monotonicity

$$Au = b$$

$$A^{-1} \geq O$$





Homogeneous medium
Uniform grid

Enhanced monotonicity

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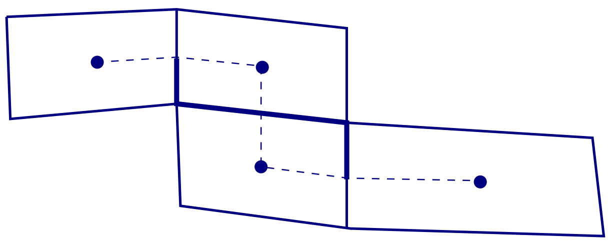
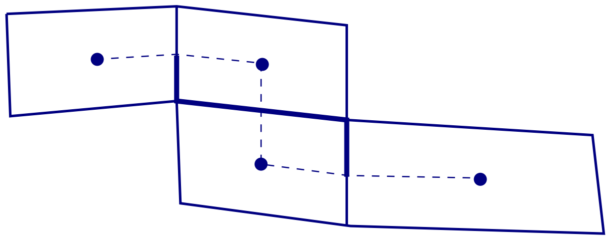


Z-stencil $\sim$ U-stencil

Enhanced monotonicity



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Flux continuity on 3 edges

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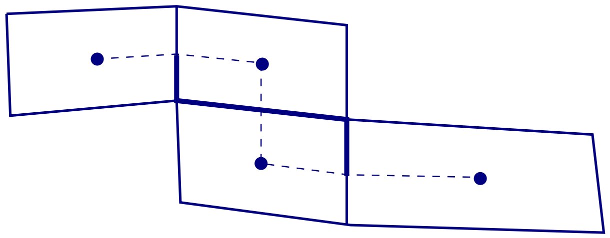

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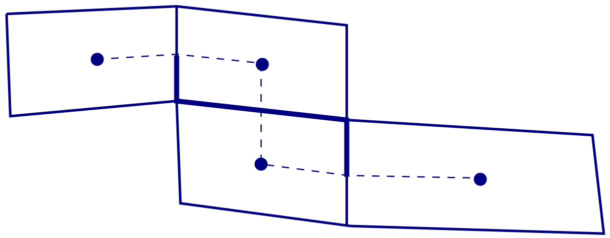

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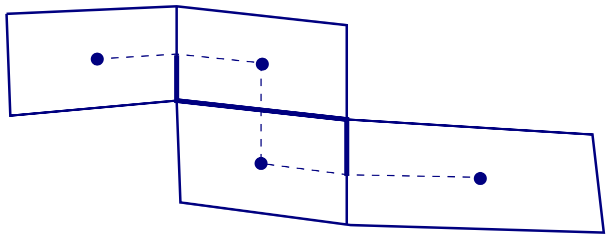

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Full potential continuity at two edges

O- and Z-methods may be combined based on proximity to a corner.

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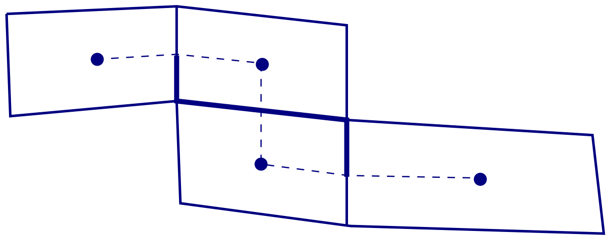

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Combined O&Z increased the monotone cases by 18% compared to the O-method in a large set of grids in a layered medium.

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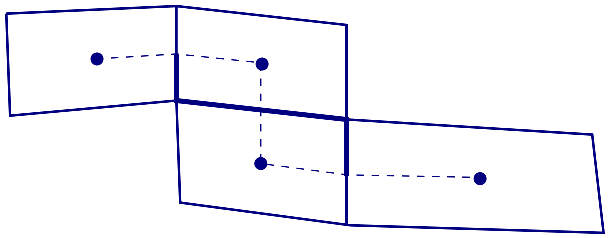

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Proximity is important for monotonicity.

Iterative solution — Edwards splitting

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$$\boldsymbol{A}\boldsymbol{u} = \boldsymbol{b}$$

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Iterative solution — Edwards splitting

$$
\begin{aligned}
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\rho(\boldsymbol{I} - \boldsymbol{B}^{-1}\boldsymbol{A}) &< 1
\end{aligned}
$$

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$\boldsymbol{B}$ is obtained by removing the off-diagonal elements in $\boldsymbol{G}_k$, i.e., discretizing as if the grid was $\boldsymbol{K}$-ortogonal.

Iterative solution — single-phase flow

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Spectral radius ρ depends on

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- orientation of the principal axes of $\boldsymbol{K}$

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Spectral radius ρ depends only weakly on

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- heterogeneity

Iterative solution — single-phase flow

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- orientation of the principal axes of $\boldsymbol{K}$
- grid skewness
- anisotropy ratio

Spectral radius ρ depends only weakly on

- grid aspect ratio
- heterogeneity

For moderate skewness angles ($< 20^\circ$) and moderate anisotropy ratio (< 100) the spectral radius satisfies $\rho \lessapprox 0.6$. This ensures fast convergence in the iteration.

Conclusions

- MPFA methods are well suited for reservoir simulation on non-$\boldsymbol{K}$-orthogonal grids.
- The O-method gives convergence of order h in potential and flow density.
- The O-method is conditionally monotone. The monotonicity range may be enhanced by combination of MPFA methods based on proximity.
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