Generalized Multiresolution Analyses,

Wavelets, and

Spectral Multiplicity.

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Generalized Multiresolution Analyses.

Define unitary operators on $L^2(\mathbb{R})$ by

$$Tf(x) = f(x-1); \quad Df(x) = \sqrt{2}f(2x),$$

called the translation and dilation operator, respectively.

Define a *Generalized Multiresolution Analysis* (GMRA) as a sequence of subspaces $\{V_j\}_{j\in\mathbb{Z}}$ with the following five properties:

1) $V_j \subset V_{j+1}$;

2) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$;

3) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;

4) $D(V_j) = V_{j+1}$;

5) V_0 is invariant under T^l for $l \in \mathbb{Z}$.

Remark 1. In the classical MRA case (Mallat), the shift invariance of V_0 is replaced by the more restrictive existence of a scaling function, i.e. a function whose translates forms an orthonormal basis of V_0.

In the Generalized Frame Multiresolution Analysis case (Papadakis, Papar), the shift invariance of V_0 is replaced by the existence of scaling frame functions.

V_0 is called the core space. Since V_0 is shift invariant, we have a unitary representation of the integers on V_0. Our main concern is to analyze this representation to get qualitative information about the GMRA.

For notation, define $\{W_j : j \in \mathbb{Z}\}$ by $V_{j+1} = V_j \oplus W_j$.

Wavelets and GMRAs

A wavelet is a square integrable function such that the collection

$$\{D^nT^l\psi = 2^{\frac{n}{2}}\psi(2^nx - l) : n,\ l \in \mathbb{Z}\}$$

forms an orthonormal basis of $L^2(\mathbb{R})$.

Theorem (BM99). *Let ψ be a wavelet. Define*

$$V_j := \overline{span}\{D^nT^l\psi : n < j,\ l \in \mathbb{Z}\}.$$

Then the V_j's form a GMRA.

This theorem says that to each wavelet ψ we can associate a representation of the integers called the core represenation of ψ. By analyzing the representation as mentioned above, we can get important information about the wavelet.

Two wavelets are core equivalent if their core representations are unitarily equivalent.

The Structure of the Core Representation.

Theorem (Stone). *Let G be a locally compact Abelian group, and let $\pi(g)$ be a unitary representation of G. There exists a projection valued measure p on the (Pontrjagin) dual group $\widehat{G}$ such that*

$$\pi(g) = \int_{\widehat{G}} g(\chi) dp(\chi).$$

Theorem (Stone-Mackey?). *Let p be a projection valued measure on a measurable space $\{\mathcal{S}, \mathcal{B}\}$, with separable Hilbert space H. There exists a finite measure ν and a multiplicity function $m : \mathcal{S} \to \{0, 1, 2, \ldots, \infty\}$ such that:*

1. *ν is equivalent to p,*

2. *there exists a unitary operator*

$$U : H \to \oplus_{j=1}^{\infty} L^2(E_j, \nu|_{E_j}, \mathbb{C}^j) \oplus L^2(E_\infty, \nu|_{E_\infty}, l^2(\mathbb{Z}))$$

such that U intertwines p with the canonical projection valued measure. Here, $E_j = m^{-1}(j)$. Furthermore, the measure ν and multiplicity function m are unique, up to measure class and modulo null sets, respectively.

Two representations of G are unitarily equivalent if and only if their measures are in the same measure class, and their multiplicity functions agree almost everywhere.

We wish to apply these theorems to the core representation of a wavelet. In this setting, the group is the integers; the dual group is the unit circle in the complex plane, which we shall associate to $[-\pi, \pi)$ in the standard way.

The unitary operator described intertwines the p.v.m. p with the canonical p.v.m. (i.e. multiplication by characteristic functions). In other words, the unitary U changes shifts by integers into multiplication by the function $g(\chi)$, a unimodular function.

In general the unitary operator U is not computable. But in our case of wavelets we know this operator quite well; it is the Fourier transform. One consequence of this is that the measure ν is actually the restriction of Lebesgue (Haar) measure to E_1.

As a result, the multiplicity function completely determines the core representation, i.e. two wavelets are core equivalent if and only if their multiplicity functions agree almost everywhere.

Furthermore, since the measure associated to the left regular representation of a group G is Haar (Lebesgue) measure, from our perspective m describes the core representation as a subrepresentation of a multiple of the regular representation.

For example, the regular representation has multiplicity identically 1; the regular representation can also be described as having an orthonormal basis of shifts under the representation. Thus, we have the following:

Theorem (BMM). *ψ is an MRA wavelet iff $m \equiv 1$.*

Since we have established that the structure of the core representation is given by the multiplicity function, it would be nice to be able to compute it. We actually can compute it explicitly.

Wavelet Dimension function (Auscher):

$$D_\psi(\xi) = \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} |\widehat{\psi}(2^j(\xi + 2\pi k))|^2.$$

Theorem 1. *Let* ψ *be a wavelet, and let* $m : [-\pi, \pi] \to \{0, 1, 2, \ldots, \infty\}$ *be its associated multiplicity function. Then*

$$m(\xi) = D_\psi(\xi).$$

Remark 2. It is easy to check that $\int_{-\pi}^{\pi} D_\psi = 2\pi$. As a result, the core representation can't be "too big".

Counting parts of the regular representation.

We would now like to use the multiplicity function to tell us information about whether the representation admits frames or Riesz bases of shifts of some collection of vectors.

Theorem (HL00). *Suppose a representation* π *of* G *has a collection of vectors* $\{x_n\}$ *such that the collection* $\{\pi(G)x_n\}$ *is a frame for the representation space* H*. Then* π *is a subrepresentation of some multiple* ($\leq n$) *of the left regular representation.*

From the perspective of the Stone-Mackey theorem, this theorem says that a representation of an Abelian group G has frame generators if and only if the measure is the restriction of Haar measure to a subset of the dual group.

Since we know this is the case for the core representation, we use the multiplicity function to tell us which parts of the regular representation we have. In particular, we can ask whether the core representation has any vectors, or more specifically, a collection of generators, with linearly independent shifts.

The second part of the question is equivalent to the multiplicity function agreeing with a constant function almost everywhere. Moreover, that value tells the number of generators required.

The Stone-Mackey theorem can be used to show that the GMRA corresponding to a wavelet has a collection of functions such that the translates applied to this collection forms a normalized tight frame for V_0.

Papadakis (P00) also showed this fact, and in particular, explicitly calculated these scaling frame functions. In other words, every GMRA of $L^2(\mathbb{R})$ is a GFMRA.

In the case of wavelets on $L^2(\mathbb{R})$, we have information about the size of the multiplicity function since its integral is 2π. Thus, for example, we know that given a wavelet, its core space contains a function with linearly independent shifts if and only if the wavelet is actually MRA.

Strongly Disjoint Frames of Translates.

In the spirit of measuring the parts of the regular representation as above, we consider the characterization of strongly disjoint frames.

Definition 1 (HL00). Suppose $\{x_n\}$ is a tight frame for H and $\{y_n\}$ is a tight frame for K. We say that $\{x_n\}$ and $\{y_n\}$ are strongly disjoint if $\{x_n \oplus y_n\}$ is a tight frame for $H \oplus K$.

If π is a representation of a locally compact Abelian group on a Hilbert space H, then π admits a frame vector if there is a $v \in H$ such that $\{\pi(G)v\}$ is a frame.

Theorem 2. *Suppose π_H and π_K are representations of G on H and K, with v, w frame vectors for H and K respectively. Then the corresponding frames are strongly disjoint if and only if the multiplicity functions for each representation are orthogonal, i.e. they have disjoint support.*

Multiplicity Theory of Refinable Spaces.

Suppose V is a closed linear subspace of $L^2(\mathbb{R})$ which is invariant under shifts by integers. Suppose further that $V \subset DV$. Then we have the following result of Baggett and Merrill.

Theorem (BM99). *If V is as above, and $m(\xi)$ is the multiplicity function of V, then $m(\xi)$ satisfies the inequality*

$$m(\xi) \leq m(\frac{\xi}{2}) + m(\frac{\xi}{2} + \pi).$$

Furthermore, if we add the assumption that $m(\xi)$ is finite almost everywhere, we have the equality

$$m(\xi) + m_1(\xi) = m(\frac{\xi}{2}) + m(\frac{\xi}{2} + \pi).$$

If we define the closed shift invariant subspace W to be the orthogonal complement of V in DV, i.e.

$$V \oplus W = DV,$$

then the function $m_1(\xi)$ describes the structure of W as above.

An Application to Biorthogonal Riesz Wavelets.

Suppose ψ is a biorthogonal Riesz wavelet, with dual $\tilde{\psi}$. It is an easy exercise to show that ψ generates a GMRA in the same way an orthonormal wavelet does. Here we shall outline the proof of the following fact:
Theorem 3. *Suppose ψ is a biorthogonal Riesz wavelet with GMRA $\{V_j\}$. Then there exists an orthonormal wavelet η which generates the same GMRA.*

If $\psi, \tilde{\psi}$ are biorthogonal wavelets, then we have the following dimension function of Auscher:

$$D_{\psi,\tilde{\psi}}(\xi) = \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} \hat{\psi}(2^j(\xi + 2\pi k))\hat{\tilde{\psi}}(2^j(\xi + 2\pi k)).$$

It is not surprising that the multiplicity function of the core space for ψ agrees with this dimension function. Indeed, one can extend the proof of the orthonormal case. This fact is also proven in a preprint of Ron and Shen (RS01).

The significance for us here is that the dimension function, which is the multiplicity function, satisfies the consistency equation

$$D(\xi) + 1 = D(\frac{\xi}{2}) + D(\frac{\xi}{2} + \pi).$$

Therefore, the multiplicity function $m_1(\xi)$ of

$$W_0 = V_1 \ominus V_0$$

is identically 1. Thus, by the above discussion, there exists a vector $\eta \in W_0$ whose translates form an orthonormal basis. This η is in fact an orthonormal wavelet which clearly generates the GMRA we started with.

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