TRANSFORMS WITH CROSSED PRODUCT KERNELS

JULIA VELIKINA

Department of Mathematics

University of Wisconsin – Madison

Advisor: AMOS RON

History

$\{\varphi_n\}$, $\{\psi_n\}$, $n \in \mathbb{Z}_{\geq 0}$ – complete ONS in $L_2([0,1])$

In 1973, **N.Ya.Vilenkin, S.V.Zotikov** considered

$$F : L_1(0,\infty) \to L_\infty(0,\infty) : f \mapsto \int_0^\infty K(x,y) f(x)\,dx,$$

$$K(x,y) := \varphi_{\lfloor x \rfloor}(\{y\}) \psi_{\lfloor y \rfloor}(\{x\}).$$

They proved a number of properties of F,

in particular, when $\{\varphi_n\}$ is Haar system.

Our Setup

Choose

$$\varphi_n(x) = \psi_n(x) := e^{2\pi i n x}$$

Define

$$\mathcal{F}_j[f](y) := 2^j \int_{\mathbb{R}} K(2^j x, 2^j y) f(x) dx, \quad f \in L_2(\mathbb{R}),$$

$$K(x, y) := \varphi_{\lfloor x \rfloor}(\{y\}) \psi_{\lfloor y \rfloor}(\{x\}),$$

Properties

- $\mathcal{F} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is an isometry

- $\mathcal{F}^{-1} = \mathcal{F}$

- $\|\mathcal{F}[f]\|_{L_2(l,l+1)} \leq \sqrt{2}\omega_2(f; \frac{1}{2|l|}), \; l \neq 0$,
  where $\omega_2(f;\delta) := \sup_{|t|\leq\delta} \|f(\cdot + t) - f\|_2$

- If $f \in L_2(\mathbb{R})$ has bounded variation $V(f)$,

$$|\mathcal{F}[f](y)| \leq \frac{1}{|\lfloor y \rfloor|} V(f) \text{ a.e. }, \quad \lfloor y \rfloor \neq 0$$

- $\mathcal{F}_j[f]$ exhibits *self-similarity* propertiy

Self-similarity

For a function $h \in L_2(\mathbb{R})$ define measure of self-similarity

$$\mu_j(h) := \sum_{k=2}^{\infty} \inf_{\alpha_k \in \mathbb{R}} \|h(\cdot + \frac{k-1}{2^j}) - \alpha_k h\|^2_{L_2(\frac{1}{2^j},\frac{2}{2^j})} + \sum_{k=-\infty}^{-2} \inf_{\alpha_k \in \mathbb{R}} \|h(\cdot + \frac{k+1}{2^j}) - \alpha_k h\|^2_{L_2(-\frac{1}{2^j},0)}$$

We say that h is **self-similar** if for some $j \in \mathbb{Z}$, $\mu_j(h) = 0$, that is,

$$h|_{[\frac{k}{2^j},\frac{k+1}{2^j}]} = \alpha_k h|_{[\frac{1}{2^j},\frac{2}{2^j}]}, \ k > 1,$$
$$h|_{[\frac{k}{2^j},\frac{k+1}{2^j}]} = \alpha_k h|_{[-\frac{1}{2^j},0]}, \ k < -1.$$

Self-similarity of the transform

Theorem 1. *Let* $f \in L_2(\mathbb{R})$ *and for some* $m \in \mathbb{Z}$ *and all* $k \in \mathbb{Z}$,

$$f|_{[\frac{k}{2^m},\frac{k+1}{2^m})} \in W_2^3(\frac{k}{2^m},\frac{k+1}{2^m})$$

and

$$\sum_{k\in\mathbb{Z}} \|f^{(3)}\|^2_{L_2(\frac{k}{2^m},\frac{k+1}{2^m})} < \infty.$$

Then for every $\varepsilon > 0$ *there exist* $j \geq m$ *and a self-similar function* $g \in L_2(\mathbb{R})$ *with* $\mu_j(g) = 0$ *such that*

$$\|\mathcal{F}_j[f] - g\|_2 < \varepsilon.$$

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Theorem 2. *For every $\varepsilon > 0$, there exists a sequence of sets $V_j(\varepsilon) \subset L_2(\mathbb{R})$ such that*

(i) $V_j(\varepsilon) \subset V_{j+1}(\varepsilon), \; j \in \mathbb{Z}$;

(ii) $\cup_{j \in \mathbb{Z}} V_j(\varepsilon) = L_2(\mathbb{R})$;

(iii) for each function $f \in V_j(\varepsilon)$ there exists a function $s \in V_j(\varepsilon)$ such that

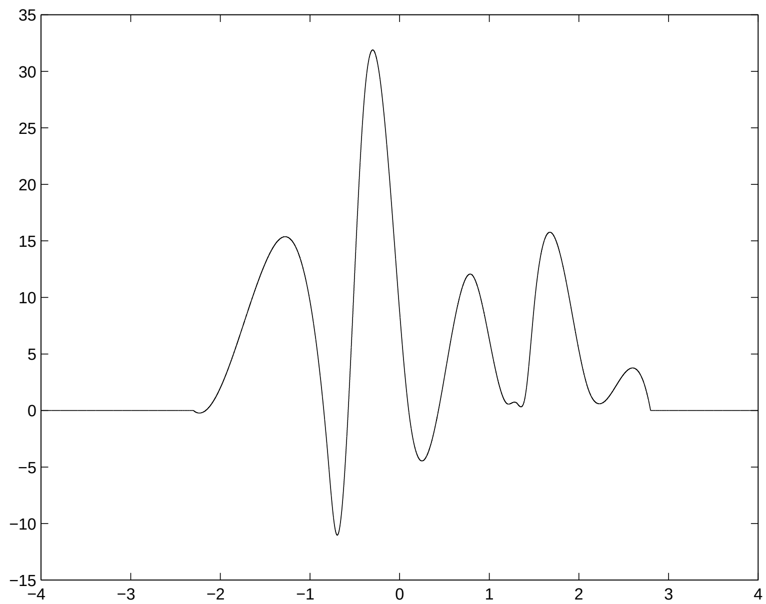
$$\|f - s\|_2 = \|\mathcal{F}_j[f] - \mathcal{F}_j[s]\|_2 < \varepsilon$$

and $\mathcal{F}_j[s]$ is self-similar, that is,

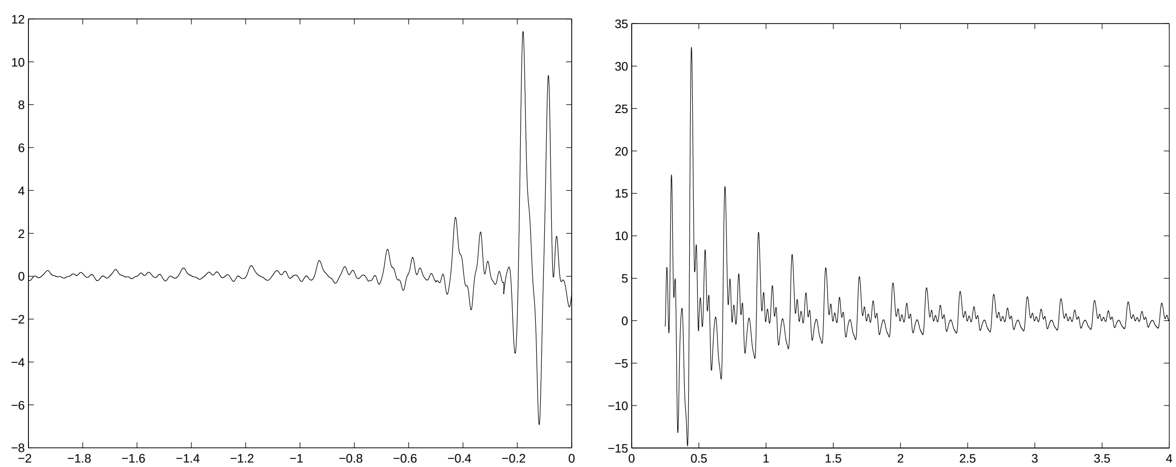
$$\mu_j(\mathcal{F}_j[s]) = 0.$$

Example

We start with a function f

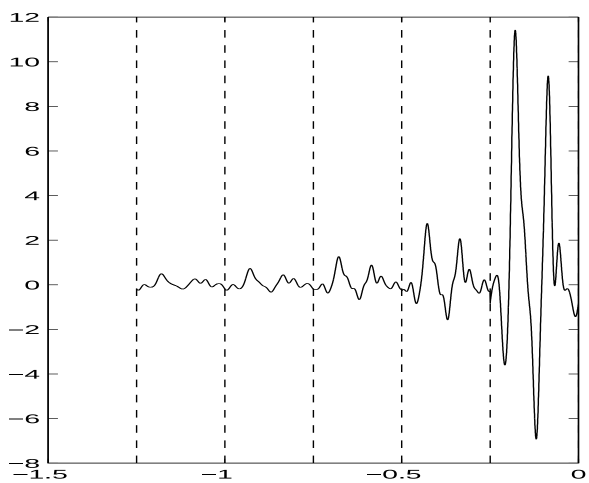


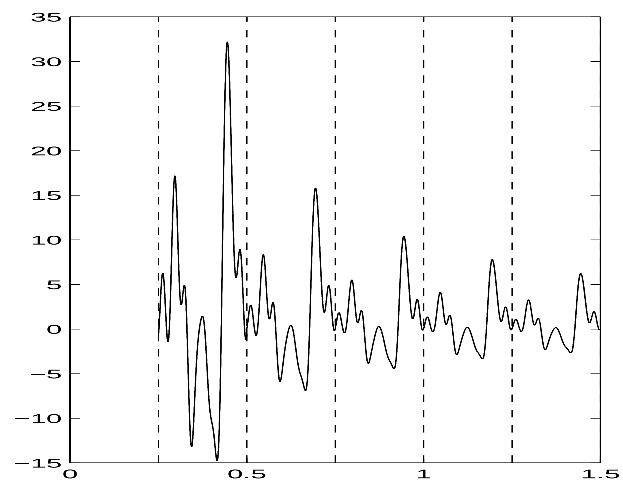
Its transform $\mathcal{F}_2[f]$ exhibits good self-similarity



Example

We construct a self-similar approximant h for $\mathcal{F}_2[f]$ with $\mu_2(h) = 0$.





Set $\tilde{f} := \mathcal{F}_2^{-1}[h]$, $\|f - \tilde{f}\|_2 / \|f\|_2 = 0.0176$