Multiresolution Approximation of Curves with Normal Polylines

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Joint work with Ingrid Daubechies and Wim Sweldens.

Background

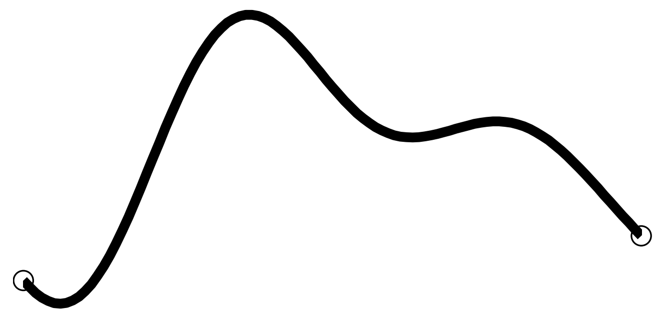
- Huge arbitrary topology meshes for real world 3D data.
- Goals:
  1. Compression,
  2. Progressive reconstruction,
  3. Signal processing, ...
- How mesh is stored/represented affects compression rate. Remeshing to better format improves rate.

Normal Meshes

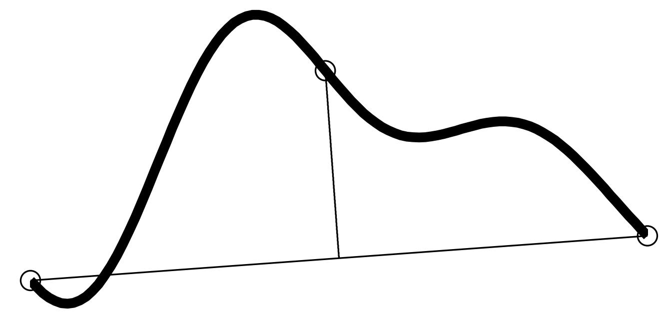
(Guskov, Khodakovsky, Schröder, Sweldens, ...)

- Multiresolution meshes (predict + detail, at each level)
- Purely scalar representation (1 float/vertex),
- No “parameter” info
- Scalar wavelet compression codes can be used.

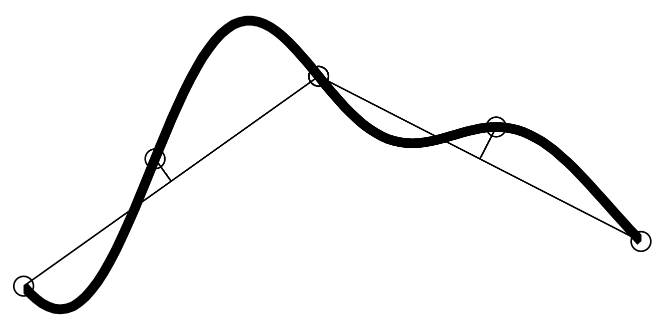
Construction of Normal Polyline



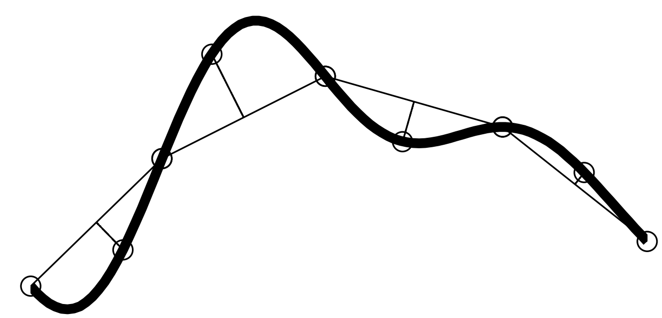
Starting curve $f = (x, y)^T$



Level 1



Level 2



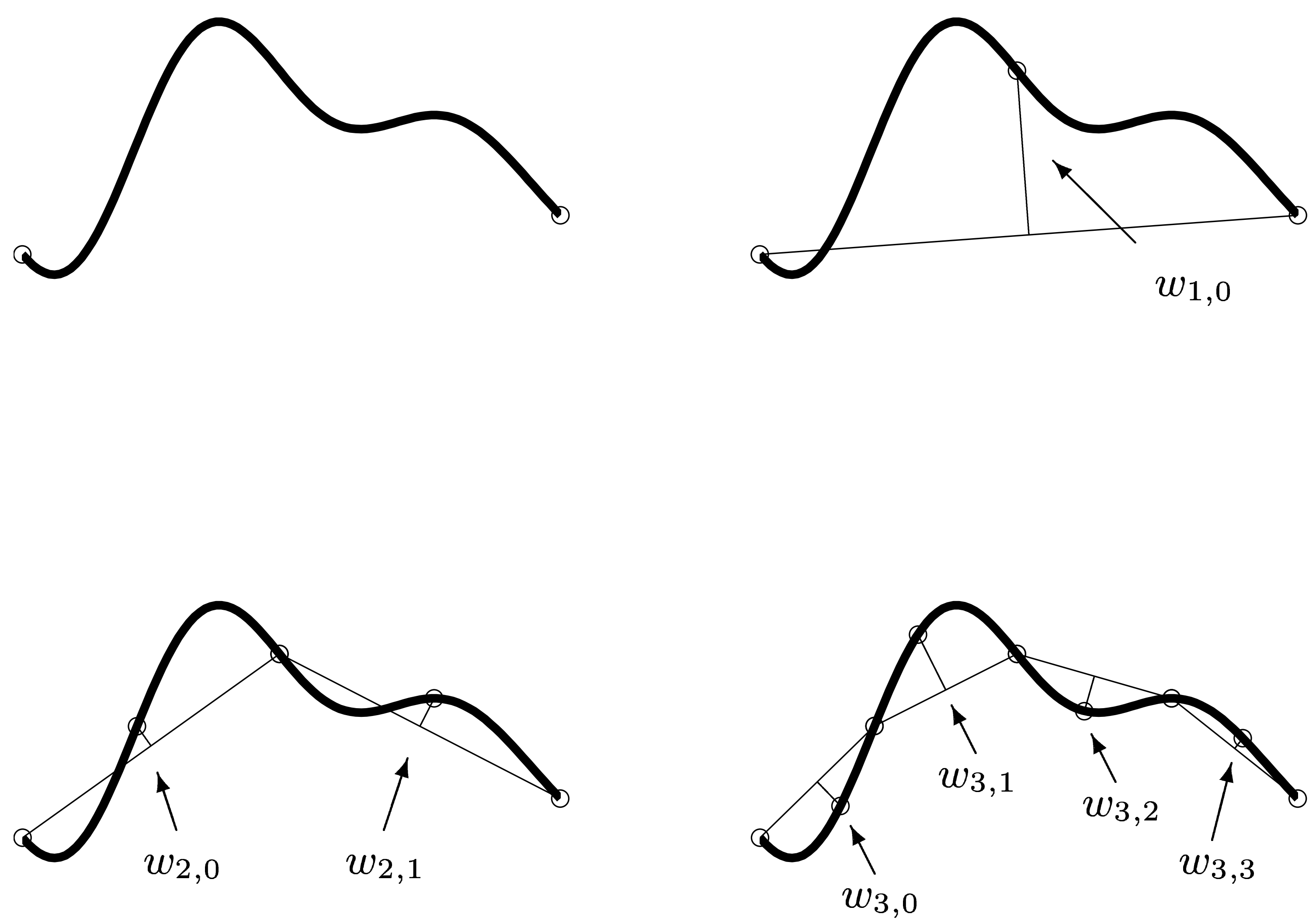
Level 3

One-dimensional analysis

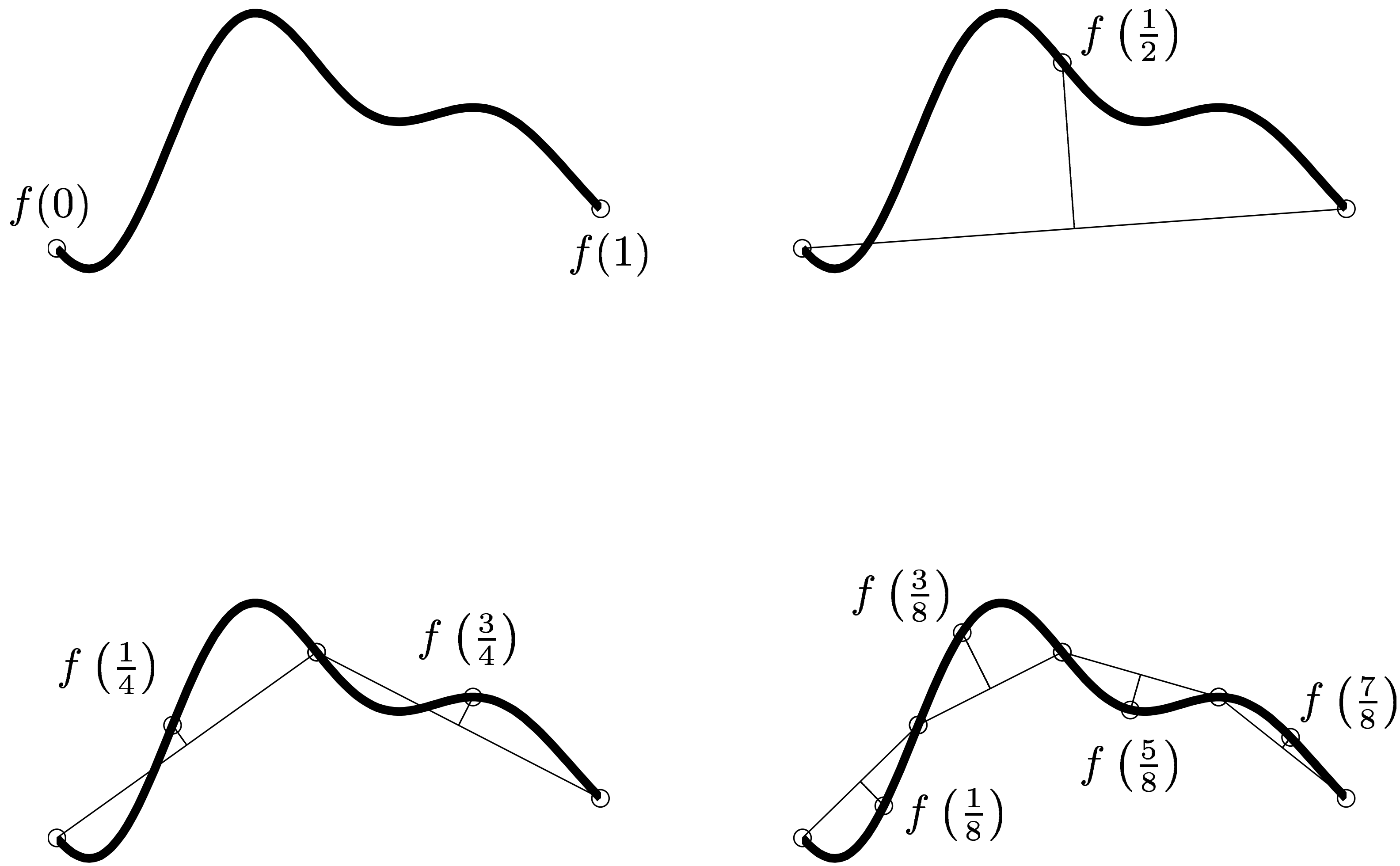
Issues:

- Decay of normal offsets (“wavelet coefficients”),
- Regularity of parameterization.

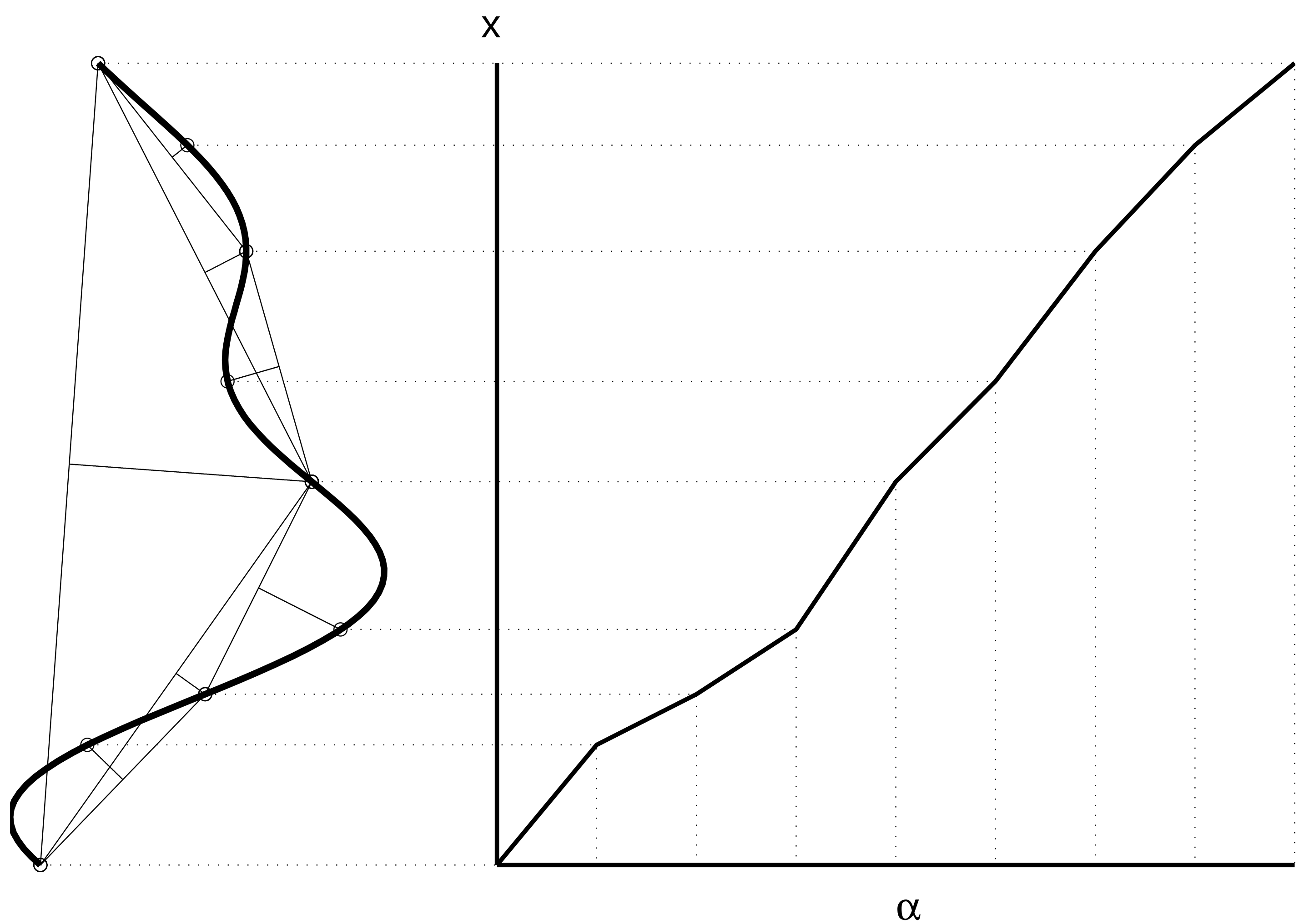
Normal Offsets $\boldsymbol{w}_j = \{w_{j,k}\}$



Normal Parameterization $f(\alpha) = (x(\alpha), y(\alpha))^T$



Normal Parameterization, $x(\alpha)$



One-dimensional analysis, cont.

Parameterization typically *not* smooth even if curve is:

In linear case, let α be normal parameterization $f(\alpha) = (x(\alpha), y(\alpha))^T$. Then

$$\Big(f(\alpha+h) - f(\alpha-h)\Big) \cdot \Big(f(\alpha+h) - 2f(\alpha) + f(\alpha-h)\Big) = 0,$$

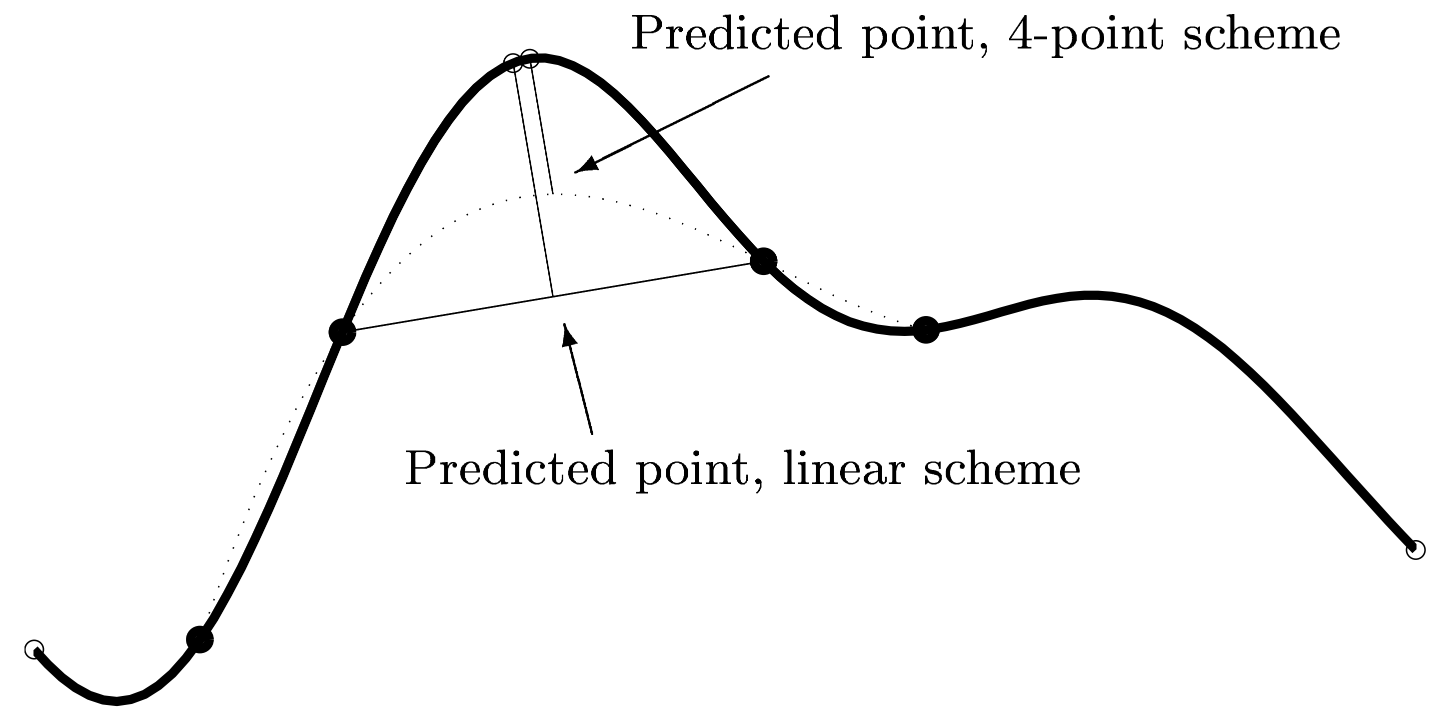
at odd dyadic points ($\alpha = (2k+1)2^{-j}$, $h = 2^{-j}$).

Smooth parameterization $\Rightarrow$

$$\frac{d}{d\alpha}\left|\frac{df(\alpha)}{d\alpha}\right|^2 + \frac{h^2}{12}\left(\frac{d^3}{d\alpha^3}\left|\frac{df(\alpha)}{d\alpha}\right|^2 - \frac{d}{d\alpha}\left|\frac{d^2 f(\alpha)}{d\alpha^2}\right|^2\right) = \mathcal{O}(h^4).$$

$h \to 0 \;\Rightarrow\;$ parameterization $\sim$ arclength and curve is line or circle.

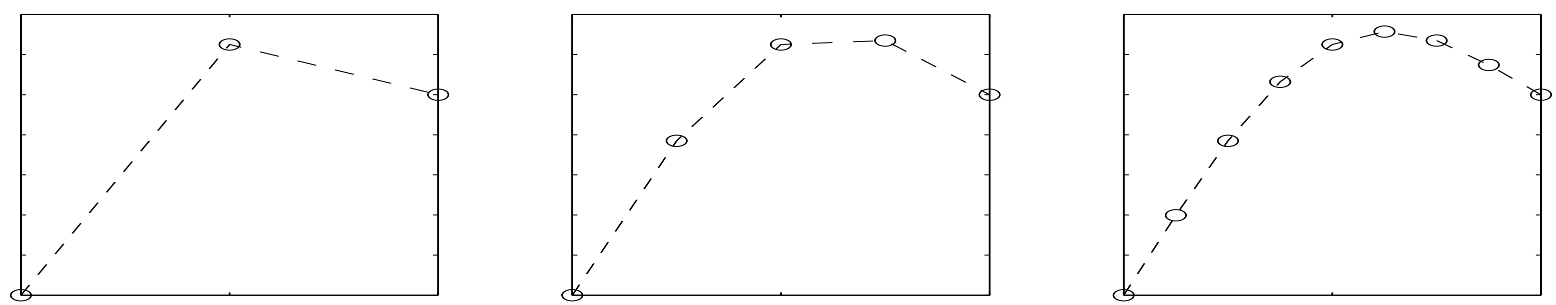
Higher Order Subdivision Scheme as Predictor



Subdivision

Procedure to generate successively finer mesh from a coarse mesh.

Ex. (1D, $\boldsymbol{y}_j = \{y_{j,k}\}$),



$\boldsymbol{y}_0$ $\boldsymbol{y}_1$ $\boldsymbol{y}_2$

Let S be a *subdivision operator* such that $\boldsymbol{y}_{j+1} = S\boldsymbol{y}_j$ and

$$y_{j+1,k} = \sum_{\ell \in I_k} S_{k,\ell} y_{j,\ell}, \qquad I_k = [k-B, k+B].$$

Here S is linear, stationary (indep. of j) and local ($B < \infty$).

Subdivision, cont.

Limit: $\lim_{j\to\infty} \tilde{\boldsymbol{y}}_j(t) \to \phi(t) \in C^?$

Order of S is p if S preserves q-degree polynomials for $0 \le q < p$.

Derived subdivision schemes, $S^{[q]}$, act on divided differences of $\boldsymbol{y}_j$,

$$S^{[q]}\boldsymbol{y}_j^{[q]} = \boldsymbol{y}_{j+1}^{[q]}, \qquad \boldsymbol{y}^{[q]} = D_j^q \boldsymbol{y}_j.$$

(Also, $S^{[q]} = D_{j+1}^q S D_j^{-q}$.) $S^{[q]}$ well-defined for $q \le$ order of S.

Examples, interpolating schemes $(y_{j+1,2k} = y_{j,k})$:

- "2-point" (linear), $S \sim [\frac{1}{2},\ \frac{1}{2}]$, $\phi \in C^{1-\varepsilon}$
- "4-point" (cubic), $S \sim \frac{1}{16}[-1,\ 9,\ 9,\ -1]$, $\phi \in C^{2-\varepsilon}$.
- "6-point", $S \sim \frac{1}{256}[3,\ -25,\ 150,\ 150,\ -25,\ 3]$, $\phi \in C^{2.83}$.

One-dimensional analysis, results

Theorem. Let S be predictor subdivision and $y(x)$ the curve. If

- S is of order $\geq p$ and, for some m, $\left|S^{[p]^j}\right|_\infty \leq C m^j$,
- $y \in C^{p+1}$,
- $|\Delta \boldsymbol{x}_j|_\infty \leq C\delta^j$, for some $\delta < 1$.

Then

$$\tilde{\boldsymbol{x}}_j(\alpha) \to x(\alpha) \in C^{p-\log_2 m-\varepsilon}.$$

and the normal offsets $\boldsymbol{w}_j$ satisfy the estimate

$$|\boldsymbol{w}_j|_\infty \leq C 2^{-j(p-\log_2 m+1-\varepsilon)}.$$

Examples:

- "2-point", $S^{[1]} = I$, so $p = 1$, $m = 1$ and $x(\alpha) \in C^{1-\varepsilon}$,
- "4-point", $\left|S^{[3]}\right|_\infty = 2$, so $p = 3$, $m = 2$ and $x(\alpha) \in C^{2-\varepsilon}$.

Sketch of proof

1. Consider nonlinear subdivision schemes T_j,

$$\boldsymbol{x}_{j+1} = T_j \boldsymbol{x}_j, \qquad |T_j \boldsymbol{x}_j - S\boldsymbol{x}_j|_\infty \le 2^{-nj}.$$

   (a) Estimate $\boldsymbol{x}_j^{[q]}$, given $\left|{S^{[p]}}^j\right|_\infty \le Cm^j$,

$$\left|\boldsymbol{x}_j^{[q]}\right|_\infty \le 1 + \begin{cases} (m2^{q-p})^j, & n > p - \log_2 m, \\ (2^{q-n})^j, & n < p - \log_2 m. \end{cases}$$

   (b) $\Rightarrow$ regularity of limit,

$$\tilde{x}(t) \to x(t) \in \begin{cases} C^{p-\log_2 m - \varepsilon}, & n > p - \log_2 m, \\ C^{n-\varepsilon}, & n < p - \log_2 m. \end{cases}$$

2. Show parameterization is given by some T_j with n large enough.

(a) Estimate (x-component of) offset,

$$|N_j \boldsymbol{x}_j - S\boldsymbol{x}_j|_\infty \le |Sy(\boldsymbol{x}_j) - y(S\boldsymbol{x}_j)|_\infty .$$

(b) Approximate commutation with nonlinear functions,

$$\left|\boldsymbol{x}^{[p]}\right|_\infty \le \text{const} \quad \Rightarrow \quad |Sy(\boldsymbol{x}) - y(S\boldsymbol{x})|_\infty \le 2^{-j(p+1)}.$$

(c) Result follows by induction, starting from

$$\begin{aligned} & & \left|\boldsymbol{x}^{[1]}\right|_\infty &\le \text{const} \\ (2a, 2b) & \Rightarrow & |N_j \boldsymbol{x}_j - S\boldsymbol{x}_j|_\infty &\le |Sy(\boldsymbol{x}_j) - y(S\boldsymbol{x}_j)|_\infty \le 2^{-2j}, \\ (1a) & \Rightarrow & \left|\boldsymbol{x}^{[2]}\right|_\infty &\le \text{const} \end{aligned}$$

etc. till $|N_j \boldsymbol{x}_j - S\boldsymbol{x}_j|_\infty \le 2^{-nj}$ for n large enough.