Introduction

- Many physical systems involve processes occuring at more than one scale. E. g. flow, transport in porous, mechanics of multiscale composite materials, . . .
- It is of interest to analyze and characterize these processes at large (coarse) length scales.
- Detailed computational analysis resolving all scales prohibitively expensive. Ignoring fine scales not good since they affect coarse scale.
- Need for methods which systematically take into account fine scale information to produce accurate equations for coarse scale.
- Keywords: “coarsening,” “upscaling,” “homogenization.”

Background

Consider a non-linear ODE,

$$\frac{d}{dx}g(x,u(x)) = f(x,u(x)), \qquad g(0,u(0)) = 0,$$

or, written as an integral equation,

$$g(x,u) = \int_0^x f(t,u(t))dt \equiv Kf(x,u). \tag{1}$$

Decompose u into coarse/fine scales given by projections P/Q,

$$u = Pu + Qu.$$

We want to reduce (1) to an equation for just the coarse part Pu.

Decompose (1),

$$Pg(x, Pu + Qu) = PKf(x, Pu + Qu) \tag{2}$$

$$Qg(x, Pu + Qu) = QKf(x, Pu + Qu). \tag{3}$$

The second equation, (3), gives an implicit relationship between Qu and Pu. We assume there exists a well-defined mapping we call h, such that

$$Qu(x) = h(x, Pu). \tag{4}$$

(Note h is in general not of the form $h(x, Pu(x))$, but may depend globally on Pu.)

Substituting (4) into (2), we get

$$Pg(x, Pu + h(x, Pu)) = PKf(x, Pu + h(x, Pu)),$$

which is the reduced equation we were looking for. Symbolically

$$\bar{g}(x, Pu) = \bar{K}\bar{f}(x, Pu).$$

Wavelet-based Reduction

(Beylkin, Brewster, Gilbert)

Discretize (1) in a fine Haar wavelet space, V_{j+1} and let P/Q project on V_j and W_j respectively. (Coarse space is V_j ∼local averages.)

$$\boldsymbol{g} = K_{j+1}\boldsymbol{f},$$

$$\boldsymbol{g} = \{g(x_k, u_k)\}, \quad \boldsymbol{f} = \{f(x_k, u_k)\}, \quad \boldsymbol{x} = \{x_k\}, \quad \boldsymbol{u} = \{u_k\} \in \mathbb{R}^{2^{j+1}}$$

$$K_{j+1} = \delta_{j+1} \begin{pmatrix} \frac{1}{2} & 0 & \dots & 0 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 1 & \dots & 1 & \frac{1}{2} \end{pmatrix} \in \mathbb{R}^{2^{j+1} \times 2^{j+1}}, \qquad \delta_{j+1} = 2^{-(j+1)},$$

- QK_{j+1} is *diagonal* and (3) *decouples* into 2^{j+1} scalar equations. Hence, h is local, $h = h(x, Pu(x))$, or

$$(Q\boldsymbol{u})_k = h(x_k, (P\boldsymbol{u})_k), \qquad k = 0, \ldots, 2^j.$$

- If $Q\boldsymbol{u} \ll 1$ we can Taylor expand g, f around $P\boldsymbol{u}$ and h becomes very simple. It also implies that if

$$\begin{aligned} g(x_k, u_k) &= a(x_k, u_k) + \delta_{j+1}^2 b(x_k, u_k), \\ f(x_k, u_k) &= c(x_k, u_k) + \delta_{j+1}^2 d(x_k, u_k), \end{aligned}$$

there are $\bar{a}, \bar{b}, \bar{c}, \bar{d}$ such that

$$\begin{aligned} \bar{g}(x_k, u_k) &= \bar{a}(x_k, u_k) + \delta_j^2 \bar{b}(x_k, u_k), \\ \bar{f}(x_k, u_k) &= \bar{c}(x_k, u_k) + \delta_j^2 \bar{d}(x_k, u_k), \end{aligned}$$

and

$$\bar{\boldsymbol{g}} = K_j \bar{\boldsymbol{f}} + \mathcal{O}(\delta_j^4).$$

Apply reduction recursively

$$\boldsymbol{g}_{j+1}, \boldsymbol{f}_{j+1} \;\to\; \boldsymbol{g}_{j}, \boldsymbol{f}_{j} \;\to\; \boldsymbol{g}_{j-1}, \boldsymbol{f}_{j-1} \;\to\; \cdots$$

$$\boldsymbol{a}_{j+1}, \boldsymbol{b}_{j+1}, \boldsymbol{c}_{j+1}, \boldsymbol{d}_{j+1} \;\to\; \boldsymbol{a}_{j}, \boldsymbol{b}_{j}, \boldsymbol{c}_{j}, \boldsymbol{d}_{j} \;\to\; \boldsymbol{a}_{j-1}, \boldsymbol{b}_{j-1}, \boldsymbol{c}_{j-1}, \boldsymbol{d}_{j-1} \;\to$$

There are explicit recursion relations which allow us to obtain a, b, c, d at level $j-1$ from those at level j, etc.

At scale 0 we get an equation for $u_0 \in V_0$,

$$a_0(u_0) + L^2 b_0(u_0) = \frac{L}{2}\left(c_0(u_0) + L^2 d_0(u_0)\right)$$

where L is the size of the domain.

The nonlinear functions a_0, b_0, c_0 and d_0 are obtained from the reduction symbolically, without explicit use of solution. Also: asymptotic case when j (starting level) $\to \infty$.