Gabor Analysis on Amalgam Spaces

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Gabor Frames

- Define $T_a f(t) = f(t-a)$, and $M_b f(t) = e^{2\pi i b t} f(t)$.

- For $\alpha, \beta > 0$, $\{M_{\beta n} T_{\alpha k} g\}_{k,n\in\mathbb{Z}}$ is a Gabor frame for L^2 if $\exists 0 < A, B < \infty$ such that

$$A\|f\|_2^2 \leq \sum_{k,n\in\mathbb{Z}} |\langle f, M_{\beta n} T_{\alpha k} g\rangle|^2 \leq B\|f\|_2^2 \quad \forall f \in L^2(\mathbb{R})$$

- Consequently, $\exists \gamma \in L^2(\mathbb{R})$ such that $\{M_{\beta n} T_{\alpha k}\gamma\}_{k,n\in\mathbb{Z}}$ is also a Gabor frame for L^2 called the canonical dual frame. Moreover, $f = \sum_{k,n\in\mathbb{Z}} \langle f, M_{\beta n} T_{\alpha k} g\rangle M_{\beta n} T_{\alpha k}\gamma$ $= \sum_{k,n\in\mathbb{Z}} \langle f, M_{\beta n} T_{\alpha k}\gamma\rangle M_{\beta n} T_{\alpha k} g \quad \forall f \in L^2$

- Define the following operators:
  $C_g f = \{< f, M_{\beta n} T_{\alpha k} g >\}_{k,n\in\mathbb{Z}}$ (analysis operator),
  $R_g c = \sum_{k,n\in\mathbb{Z}} c_{k,n} M_{\beta n} T_{\alpha k} g$ (synthesis operator).

- For fixed g, γ

$$R_\gamma C_g f = \sum_{k,n\in\mathbb{Z}} < f, M_{\beta n} T_{\alpha k} g > M_{\beta n} T_{\alpha k}\gamma.$$

- $\mathcal{G}(g, \alpha, \beta\ ) = \{M_{\beta n} T_{\alpha k} g\}_{k,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with dual frame $\mathcal{G}(\gamma, \alpha, \beta\ ) = \{M_{\beta n} T_{\alpha k}\gamma\}_{k,n\in\mathbb{Z}}$ is equivalent to
  - $R_\gamma C_g = R_g C_\gamma = I \quad$ on $\quad L^2$
  - $\| \{< f, M_{\beta n} T_{\alpha k} g >\} \|_{l^2} \asymp \| f \|_{L^2}$
  - $\| \{< f, M_{\beta n} T_{\alpha k}\gamma >\} \|_{l^2} \asymp \| f \|_{L^2}$.

Modulation Spaces

- The Short-Time Fourier Transform of f with respect to g is :

$$V_g f(x,\omega) = \int_{\mathbb{R}} f(t)\, \overline{g(t-x)}\, e^{-2\pi i \omega t}\, dt.$$

- Notice that $V_g f(x,\omega) =< f, M_\omega T_x g >$
so we can extend by duality the definition of V_g to $\mathbb{S}^{'}$ for a fixed $g \in \mathbb{S}$.

- Let $1 \leq p,q \leq \infty$ and $g \in \mathbb{S}$. Let m be a positive weight function, e.g., $m(x,y) = (1+ \mid x \mid + \mid y \mid)^s$. The modulation space $\mathbb{M}^{p,q}_m$ is the Banach space of all $f \in \mathbb{S}^{'}$ for which the following norm is finite:
$\|f\|_{\mathbb{M}^{p,q}_m} = \left(\int \left(\int |V_g f(x,y)|^p \, m(x,y)^p \, dx \right)^{q/p} dy \right)^{1/q}$

- The definition is independent of the choice of window g.

- $\mathbb{S}$ is dense in $\mathbb{M}^{p,q}_m$ for $1 \leq p,q < \infty$.

Modulation Spaces (cont.)

- Among the modulation spaces the following classical function spaces occur:
  - $\mathbb{M}^{2,2} = L^2$
  - If $m(x,y) = (1+|x|)^s$ then $\mathbb{M}^{2,2}_m = L^2_s$
  - If $m(x,y) = (1+|y|)^s$ then $\mathbb{M}^{2,2}_m = H^s$
  - $m(x,y) = (1+|x|+|y|)^s$ then $\mathbb{M}^{2,2}_m = L^2_s \cap H^s$

- Feichtinger, Gröchenig:
  If $g, \gamma \in \mathbb{M}^{1,1}$ and $R_\gamma C_g = R_g C_\gamma = I$ on L^2, then
  - $R_\gamma C_g = R_g C_\gamma = I \, on \, \mathbb{M}^{p,q}_m$,
  - $\|f\|_{\mathbb{M}^{p,q}_m} \asymp \|\{< f, T_{\alpha k} M_{\beta n} g >\}\|_{l^{p,q}_{\tilde{m}}}$,
  - $\|f\|_{\mathbb{M}^{p,q}_m} \asymp \|\{< f, T_{\alpha k} M_{\beta n} \gamma >\}\|_{l^{p,q}_{\tilde{m}}}$,

  where $\tilde{m}(k,n) = m(\alpha k, \beta n)$

- Gabor expansions converge unconditionally in the norm of the modulation space.

- $L^p(\mathbb{R})$ for $p \neq 2$ is not a modulation space.

Amalgam Spaces

- Let $1 \leq p, q \leq \infty$ $\alpha, \beta > 0$ and ν be a weight function subordinate to a polynomial weight ω on $\mathbb{R}$.
  The weighted amalgam space $W\left(L^p, l^q_{\tilde{\nu}}\right)$, is the Banach space of measurable functions on $\mathbb{R}$ with norm:

$$\|f\|_{W\left(L^p, l^q_{\tilde{\nu}}\right)} = \left(\sum_{k\in\mathbb{Z}} \|f.T_{\alpha k}\chi_{[0,\alpha)}\|_p^q \, \tilde{\nu}(k)^q\right)^{1/q}.$$

- $W\left(L^p, l^p\right) = L^p$.

- For $p, q \neq 2$, $W\left(L^p, l^q_{\tilde{\nu}}\right)$ is not a modulation space.

- Define $S^{p,q}_{\tilde{\nu}} = l^q_{\tilde{\nu}}\left(\mathcal{F}L^p\left[0, 1/\beta\right)\right)$ to be the space of all sequences $\{c_{k\,n}\}_{k,n\in\mathbb{Z}\times\mathbb{Z}}$ such that:

$$\|c\| = \left(\sum_{k\in\mathbb{Z}} \left\|\sum_{n\in\mathbb{Z}} c_{k,n} e^{2\pi i \beta k \cdot}\right\|_p^q \tilde{\nu}(k)^q\right)^{1/q} < \infty$$

(with adjustment if p=1)

Gabor Frame Expansions in $W\,(L^p, l^q_{\tilde{\nu}}\,)$

- Proposition 1: Let $\alpha, \beta > 0$, $1 \le p, q \le \infty$ and $g \in W\,(L^\infty, l^1_{\tilde{\omega}}\,)$.Then there exists $C = C(\alpha, \beta, \omega) > 0$ such that if $\{m_k\}$ is any sequence of $1/\beta$-periodic functions in $L^p(Q_{1/\beta})$. Then:

  $$\|\sum_{k\in\mathbb{Z}} m_k T_{\alpha k} g\|_{W\,(L^p, l^q_{\tilde{\nu}})} \le C\|g\|_{W\,(L^\infty, l^1_{\tilde{\omega}})}\big(\sum_{k\in\mathbb{Z}} \|m_k\|_p^q\,\tilde{\nu}(k)^q\big)^{1/q}$$

  (weak convergence for endpoint p, q).

- The synthesis operator has now the following form: $R_g c = \sum_{k\in\mathbb{Z}} m_k T_{\alpha k} g$ where m_k is the unique sequence in $L^p[0, 1/\beta)$ such that $\widehat{m}_k(n) = c_{kn}$. (Note that this definition agrees with the usual one for $1 < p < \infty$)

- Proposition 2: Let $\alpha, \beta > 0$, $1 \le p, q \le \infty$ and $g \in W\,(L^\infty, l^1_{\tilde{\omega}}\,)$.Then, $\|C_g f\|_{S^{p,q}_{\tilde{\nu}}} \le C\|g\|_{W\,(L^\infty, l^1_{\tilde{\omega}})}\|f\|_{W\,(L^p, l^q_{\tilde{\nu}})}$.

- Theorem 1. Let $g, \gamma \in W\,(L^\infty, l^1_{\tilde{\omega}}\,)$ be such that $\mathcal{G}(g, \alpha, \beta\,)$ is a frame for L^2 with dual frame $\mathcal{G}(\gamma, \alpha, \beta\,)$. Then given $f \in W\,(L^1, l^\infty_{1/\tilde{\omega}}\,)$, the following statements are equivalent.

  - $f \in W\,(L^p, l^q_{\tilde{\nu}}\,)$
  - $C_g f = \{< f, M_{\beta n} T_{\alpha k} g >\} \in\ l^q_{\tilde{\nu}}\,(\mathcal{F}L^p\,[0, 1/\beta))$
  - $C_\gamma f = \{< f, M_{\beta n} T_{\alpha k} \gamma >\} \in\ l^q_{\tilde{\nu}}\,(\mathcal{F}L^p\,[0, 1/\beta))$
  - $\|f\| \asymp \|C_g f\| \asymp \|C_\gamma f\|$

  (In this case, $f = R_\gamma C_g f = R_g C_\gamma f$, and these series converges conditionally.)

Necessary conditions on the windows

- Proposition 4: Let g be a measurable function, $\alpha, \beta > 0$. Assume that C_g is bounded from $W(L^p, l^\infty)$ to $S^{p,\infty}$. Then $g \in W(L^\infty, l^p)$

- A function $f : \mathbb{R} \to \mathbb{C}$ has persistency length a if there is $\delta > 0$ and a compact set K congruent to $[0, a[$ modulo a such that for every $x \in K, |f(x)| \geq \delta$.

- Theorem 2 Let g, γ be measurable functions on $\mathbb{R}$ and $\alpha, \beta > 0$ be given. Suppose the following:
  - The frame operators are bounded on $W(L^p, l^\infty)$,
  - γ has persistency length $1/\beta$,
  - $\forall f \in W(L^p, l^\infty) \sum_{k,n} \langle f, M_{\beta n} T_{\alpha k} g \rangle M_{\beta n} T_{\alpha k} \gamma$ converges

  unconditionally in L^p_{loc}.
  Then $g \in W(L^\infty, l^p)$.