

Images with Singularities and their Analysis by Polysplines

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Abstract

We present a new approach for analyzing Images with singularities in \mathbb{R}^n .

1 What is an Image?

In the majority of the practical problems (95%) we are rather satisfied with the following

Definition 1 *Image is a piecewise analytic function which is assembled of a finite number of pieces.*

Remark 2 *One may say that 95% of the image is piecewise analytic, there still remains 5% "fractal" stuff. On the other hand the "singularities" in the image are typical for almost every image with bigger or smaller contrast. For that reason we focus on this problem and disregard the fractal parts which are not so typical.*

If G is the rectangle where the Image u is defined then it is assumed by the above definition that we have a partition

$$\overline{G} = \bigcup \overline{D_j}$$

by mutually disjoint domains $D_j \subset G$ and $u = u_j$ is analytic on every D_j . The singularities of the Image are accordingly lying on the set of boundaries

$$\bigcup \partial D_j.$$

2 What is a Polyspline (piecewise polyharmonic spline)?

Let $p \geq 1$ be an integer. We will consider polyharmonic functions in domains D in \mathbb{R}^n or on a manifold. Then polyharmonic function of order p is a solution of the equation

$$\Delta^p u(x) = 0 \quad \text{for } x \in D.$$

We will work with the following notion of Polyspline.

Definition 3 *In a given domain of consideration G a **Polyspline** of order p is defined as a piecewise polyharmonic function which is smooth of order C^{2p-2} , which is assembled of a finite number of pieces.*

This assumes that we have a subdivision

$$G = \bigcup \overline{D_j}$$

by mutually disjoint domains D_j .

3 Approximation of Images by piecewise polyharmonic functions

Since the polyharmonic functions are real-analytic functions it is completely natural that we have the following

Theorem 4 (Sobolev, 1974) *The set of piecewise polyharmonic functions is dense in the set of Images. The convergence will be exponential with p .*

This problem has been considered by *S.L. Sobolev* in "Introduction to the Theory of Cubature Formulas", 1974.

4 Polyharmonic Multiresolution Analysis – a concept of symmetry

In the one-dimensional Images with singularities the wavelets have proved to be an efficient tool for compression. Analogously, we may apply Polysplines to construct Wavelet Analysis in \mathbb{R}^2 and expect good resolution properties for Images having singularities on curves.

In the standard axiomatic of Multiresolution Analysis it is assumed that the scaling spaces V_j are invariant under the groups \mathbb{Z}^n acting as shifts, cf. [9]. So far in order to obtain a richer structure it seems reasonable to assume that the set of symmetries of V_j is much bigger and "massive", e.g. it is a continuous group, or Lie group. It is possible to construct examples of wavelets which satisfy such Multiresolution Analysis by using the so-called Polysplines, cf. [5].

The construction of such Wavelet Analysis is a generalization of the cardinal spline Wavelet Analysis of Chui and Chui–Wang, [2], [3] (see also papers by Aldroubi and Unser who have obtained the same construction in a different way).

5 The simplest example of polyharmonic Wavelet Analysis

The classical today Multiresolution Analysis has been inspired by examples. We provide the examples which generate the **polyharmonic Multiresolution Analysis**. All examples are wavelets constructed by Polysplines. We consider the "cardinal" case as the simplest one.

5.1 Polysplines on strips in \mathbb{R}^n

We assume that for every $\nu \in \mathbb{Z}$ the parallel hyperplanes Γ_ν in \mathbb{R}^n are defined by putting

$$\Gamma_\nu := \mathbb{R}^{n-1} + \nu := \{x = (t, y) \in \mathbb{R}^n : t = \nu \text{ and } y \in \mathbb{R}^{n-1}\}.$$

Let $p \geq 1$ be a fixed integer. **Polyspline** of order p with **break-surfaces** Γ_ν will be called a function u which satisfies the following conditions:

1. It has smoothness C^{2p-2} , i.e.

$$u \in C^{2p-2} \left(\mathbb{R}^n \setminus \bigcup_{\nu} \Gamma_\nu \right).$$

2. It satisfies the following polyharmonic equation between every two spheres, i.e.

$$\Delta^p u(t, y) = 0 \quad \text{for } t \neq \nu \quad \text{for every } \nu \in \mathbb{Z}.$$

3. The following holds

$$u \in L_2(\mathbb{R}^n).$$

4. Two possibilities: In the periodic, we assume that the function $u(t, y)$ is 2π -periodic with respect to every variable y .

We denote this space of Polysplines by

$$\mathcal{PS}.$$

The set $S = \bigcup_{\nu} \Gamma_\nu$ is **invariant** under the action of the group

$$G \approx \mathbb{R}^{n-1} \times \mathbb{Z},$$

or in the periodic case with respect to the group

$$G \approx \mathbb{T}^{n-1} \times \mathbb{Z},$$

where for $(u, \nu) \in G$ the action is given by

$$(t, y) \mapsto (t - \nu, y - u).$$

It follows that the space of Polysplines \mathcal{PS} is invariant under the same group.

For every $j \in \mathbb{Z}$ the refinement of the set $\mathcal{S} = \bigcup_{\nu} (\mathbb{R}^{n-1} + \nu)$ (in the periodic case $\mathcal{S} = \bigcup_{\nu} (\mathbb{T}^{n-1} + \nu)$) will be defined by

$$\mathcal{S}_j = \bigcup_{\nu} (\mathbb{R}^{n-1} + 2^{-j}\nu),$$

or in the periodic case

$$\mathcal{S}_j = \bigcup_{\nu} (\mathbb{T}^{n-1} + 2^{-j}\nu).$$

5.2 The space of refined Polysplines

Respectively, we have the space of Polysplines with break-surfaces on the set \mathcal{S}_j . Let us denote it by

$$\mathcal{PS}_j.$$

Hence

$$\mathcal{PS}_0 = \mathcal{PS}.$$

Obviously, the space of Polysplines \mathcal{PS}_j is **invariant** under the action of the same group

$$G \approx \mathbb{R}^{n-1} \times \mathbb{Z},$$

or in the periodic case with respect to the group

$$G \approx \mathbb{T}^{n-1} \times \mathbb{Z},$$

where the action of the \mathbb{Z} component is the shift with $\nu 2^{-j}$ for every $\nu \in \mathbb{Z}$.

5.3 The scaling spaces PV_j

Here we suggest for every $j \in \mathbb{Z}$ to define the space PV_j as the closure in $L_2(\mathbb{R}^n)$ of those Polysplines in $L_2(\mathbb{R}^n)$, having as break-surfaces the hyperplanes $\mathbb{R}^{n-1} + 2^{-j}\nu$, i.e.

$$PV_j := \mathcal{PS}_j \cap L_2(\mathbb{R}^n).$$

5.4 The wavelet spaces PW_j

As usual we put for the wavelet spaces

$$PW_j := PV_{j+1} \ominus PV_j.$$

5.5 Representation of polyharmonic functions in a strip

Now we use the decomposition of a polyharmonic function in a strip $\{a < t < b\}$ as

$$u(t, y) = \int_{\mathbb{R}^{n-1}} e^{iy\xi} \hat{u}(t, \xi) d\xi, \quad (1)$$

where for every $\xi \in \mathbb{R}^{n-1}$ the function $\hat{u}(t, \xi)$ is a solution to the ordinary differential equation

$$L_{|\kappa|} \left(\frac{d}{dt} \right) \hat{u}(t, \xi) = 0 \quad \text{for } a < t < b,$$

where

$$L_{|\kappa|} \left(\frac{d}{dt} \right) = \left(\frac{d^2}{dt^2} - k_1^2 - \dots - k_{n-1}^2 \right)^p, \quad (2)$$

In the case of $u(t, y)$ which is periodic in the variables y we consider variables $y \in \mathbb{T}^{n-1}$ and index $\xi \in \mathbb{Z}^{n-1}$, and we have the representation

$$u(t, y) = \sum_{\mathbb{Z}^{n-1}} e^{iy\xi} \hat{u}(t, \xi). \quad (3)$$

5.6 Recall: L -spline cardinal Wavelet Analysis

The theory of cardinal L -splines has been developed by Ch. Micchelli [10], [11], and by I. Schoenberg [13], in 1976. Let us recall that basics of the Wavelet Analysis for cardinal L -splines with constant coefficients has been developed in the paper **de Boor–DeVore–Ron** [4]. This has been extended and completed in [6]. These results may be applied to the special L -splines for the operators L given by

$$L = L_{|\kappa|} \left(\frac{d}{dv} \right). \quad (4)$$

We will denote the corresponding scaling spaces (of the MRA) by

$$V_j^\kappa.$$

For every fixed κ these are by definition the L -splines for the operator (2) with break-points $2^{-j}\mathbb{Z}$ which are in $L_2(\mathbb{R})$. The wavelet spaces are

$$W_{j-1}^\kappa = V_j^\kappa \ominus V_{j-1}^\kappa.$$

5.7 Decomposition of the sets PV_j

Now thanks to representations (1), (3) we have in the non-periodic case the isomorphisms:

$$PV_j \approx \int_{\mathbb{R}^{n-1}} V_j^\kappa d\kappa,$$

$$PW_j \approx \int_{\mathbb{R}^{n-1}} W_j^\kappa d\kappa.$$

and in the periodic case the isomorphisms

$$PV_j \approx \bigoplus_{\kappa \in \mathbb{Z}^{n-1}} V_j^\kappa,$$

$$PW_j \approx \bigoplus_{\kappa \in \mathbb{Z}^{n-1}} W_j^\kappa.$$

6 Another Examples of polyharmonic Wavelets

We continue with another well studied example – the Polysplines which have break-surfaces on spheres, see [6]. Then the space PV_0 of the Multiresolution Analysis is invariant under the group

$$G = SO(n) \times \mathbb{Z}$$

which is embedded in the group

$$SO(n) \times \mathbb{R}$$

which is a Lie group acting on \mathbb{R}^n . If $(U, \lambda) \in SO(n) \times \mathbb{R}$ and $x \in \mathbb{R}^n$ the action is provided by the relation

$$x \mapsto e^\lambda U(x).$$

Thus the action of the \mathbb{R} component is understood as a multiplication by e^λ for every $\lambda \in \mathbb{R}$. The \mathbb{Z} component is used for refinement.

6.1 Spherical polysplines and wavelets

Let us consider the infinite set of concentric spheres

$$S(0; e^\nu) = e^\nu \mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = e^\nu\} \quad \text{for } \nu \in \mathbb{Z}.$$

The set

$$\mathcal{S} = \bigcup_{\nu} e^\nu \mathbb{S}^{n-1}$$

will play the role of the **grid** in the one-dimensional case. The analogy with the one-dimensional case is based on the log map of the radii:

$$\begin{aligned}\log(e^\nu) &= \nu & \text{for } \nu \in \mathbb{Z}, \text{ or} \\ \log(e^{\mathbb{Z}}) &= \mathbb{Z}.\end{aligned}$$

Let $p \geq 1$ be a fixed integer. **Polyspline** of order p with **break-surfaces** $e^\nu \mathbb{S}^{n-1}$ will be called a function u which satisfies the following conditions:

1. It has smoothness C^{2p-2} , i.e.

$$u \in C^{2p-2}(\mathbb{R}^n \setminus \{0\}).$$

2. It satisfies the polyharmonic equation between every two spheres, i.e.

$$\Delta^p u(x) = 0 \quad \text{for } x \neq e^\nu \quad \text{for every } \nu \in \mathbb{Z}.$$

3. We assume that

$$u \in L_2(\mathbb{R}^n).$$

We denote this space of Polysplines by

$$\mathcal{PS}.$$

The simplest case would be to consider $p = 1$ when we have continuous piecewise harmonic functions. The simplest nontrivial case would be $p = 2$, when we have piecewise biharmonic functions. For simplicity sake, we will restrict ourselves to this case.

Since the set $\mathcal{S} = \bigcup_\nu e^\nu \mathbb{S}^{n-1}$ is **invariant** under the action of the group

$$G \approx SO(n) \times \mathbb{Z},$$

where the action of the \mathbb{Z} component is understood as multiplication by e^ν for every $\nu \in \mathbb{Z}$, it follows that the space of Polysplines \mathcal{PS} is invariant under the same group.

6.2 The refinement of the "grid" $\mathcal{S} = \bigcup_\nu e^\nu \mathbb{S}^{n-1}$

We will consider the refinement of the set $\mathcal{S} = \bigcup_\nu e^\nu \mathbb{S}^{n-1}$ given for every $j \in \mathbb{Z}$ by

$$\mathcal{S}_j = \bigcup_\nu e^{2^{-j}\nu} \mathbb{S}^{n-1}.$$

Obviously, if we consider the log of the radii $e^{2^{-j}\nu}$ we see that this refinement corresponds to the one-dimensional refinement

$$2^{-j}\mathbb{Z}.$$

The set \mathcal{S}_j is invariant under the group

$$G \approx SO(n) \times \mathbb{Z},$$

where the action of the \mathbb{Z} -component is the multiplication by $e^{\nu 2^{-j}}$ for every $\nu \in \mathbb{Z}$. Under the log map we see that this corresponds to the 2^{-j} -shift:

$$\log(2^{-j}) = 2^{-j},$$

i.e. this corresponds to the 2^{-j} -shift invariance of the refined grid $2^{-j}\mathbb{Z}$.

6.3 The space of refined Polysplines

Respectively, we have the space of Polysplines with break-surfaces on the set \mathcal{S}_j . Let us denote it by

$$\mathcal{PS}_j.$$

Hence

$$\mathcal{PS}_0 = \mathcal{PS}.$$

Obviously, the space of Polysplines \mathcal{PS}_j is **invariant** under the action of the same group

$$G \approx SO(n) \times \mathbb{Z}.$$

where the action of the \mathbb{Z} component is the multiplication by $e^{\nu 2^{-j}}$ for every $\nu \in \mathbb{Z}$.

6.4 The scaling spaces PV_j

Here we suggest for every $j \in \mathbb{Z}$ to define the space PV_j as the closure in $L_2(\mathbb{R}^n)$ of those Polysplines in $L_2(\mathbb{R}^n)$, having as break-surfaces the spheres with radii $e^{\nu \cdot 2^{-j}}$, i.e.

$$PV_j := \mathcal{PS}_j \cap L_2(\mathbb{R}^n).$$

6.5 The wavelet spaces PW_j

As usual we put for the wavelet spaces

$$PW_j := PV_{j+1} \ominus PV_j.$$

6.6 Representation of Polysplines in annulus

The main reason for our interest in this special set of break-surfaces $\mathcal{S} = \bigcup e^\nu \mathbb{S}^{n-1}$ is the possibility to decompose the Polysplines in the space $\mathcal{PS} \cap L_2(\mathbb{R}^n)$ in infinitely many one-dimensional L -splines!

Vekua (1949, for \mathbb{R}^2) and Sobolev (1966, for \mathbb{R}^n) have generalized the Almansi representation of a polyharmonic function for the case of the annulus:

Theorem 5 Every function h polyharmonic of order p in the annulus $A_{a,b} = \{a < r < b\}$ is representable in the form

$$h(x) = \sum_{k=0}^{p-1} |x|^{2k} h_k(x) + R(x)$$

where h_k are harmonic functions in the annulus $A_{a,b}$ and the function R is a canonical polyharmonic function which is a linear combination of the fundamental solutions for the operators Δ^j for $j = 1, 2, \dots, p$, which is of order p and has singularities inside $A_{a,b}$.

We have the following representation of a function polyharmonic in an annulus:

Theorem 6 (OK, 2001) If the function $h(x)$ is **biharmonic** in the annulus $A = \{x : a < |x| < b\}$ then it permits the following representation

$$h(x) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d_k} h_{k,\ell}(\log r) Y_{k,\ell}(\theta).$$

Here $Y_{k,\ell}$, $k = 0, 1, 2, \dots$; $\ell = 1, 2, \dots, d_k$, form a basis of the space of spherical harmonics. For every k, ℓ the function $h_{k,\ell}$ is a solution to the following equation

$$M_{k,2}(z) = \prod_{j=1}^4 (z - \lambda_j),$$

where $\Lambda_k := [\lambda_1, \lambda_2, \lambda_3, \lambda_4]$, with

$$\lambda_1 = -n - k + 2, \quad \lambda_2 = -n - k + 4, \quad \lambda_3 = k, \quad \lambda_4 = k + 2. \quad (5)$$

Respectively for the Polysplines which are in L_2 we have the following representation

Theorem 7 (OK, 2001) Let $h \in \mathcal{PS} \cap L_2(\mathbb{R}^n)$. Then in the above representation the functions $h_{k,\ell}(v) e^{\frac{n}{2}v}$ are L -splines for the operator

$$\prod_{j=1}^4 \left(z - \lambda_j - \frac{n}{2} \right),$$

and

$$h_{k,\ell}(v) e^{\frac{n}{2}v} \in L_2(\mathbb{R}).$$

6.7 Recall: L -spline cardinal Wavelet Analysis

The theory of cardinal L -splines has been developed by Ch. Micchelli [10], [11], and by I. Schoenberg [13], in 1976. Let us recall that basics of the Wavelet Analysis for cardinal L -splines with constant coefficients has been developed in the paper de Boor–DeVore–Ron [4]. This has been extended and completed in [6]. These results may be applied to the special L -splines for the operators L given by

$$L^{(k)}\left(\frac{d}{dv}\right) = \prod_{j=1}^4 \left(\frac{d}{dv} - \lambda_j - \frac{n}{2}\right). \quad (6)$$

We will denote the corresponding scaling spaces (of the MRA) by

$$V_j^{(k)}.$$

For every fixed k these are by definition the L -splines for the operator (6) with break-points $2^{-j}\mathbb{Z}$ which are in $L_2(\mathbb{R})$.

6.8 Isomorphism for the spaces PV_j

Using the above representation we obtain the following isomorphism.

Theorem 8 (OK, 2001) *The following isomorphism holds:*

$$PV_0 \approx \left[\bigoplus_{k=0}^{\infty} \bigoplus_{\ell=1}^{d_k} \right]' V_0^{(k)},$$

where in $\bigoplus_{\ell=1}^{d_k} V_0^{(k)}$ we have taken in fact d_k copies of the space $V_0^{(k)}$. More generally, for every $j \in \mathbb{Z}$,

$$PV_j \approx \left[\bigoplus_{k=0}^{\infty} \right]' \bigoplus_{\ell=1}^{d_k} V_j^{(k)}.$$

Here the $'$ means that we take only elements which are in $L_2(\mathbb{R}^n)$, i.e. if

$$g^{k,\ell}(v) \in V_j^{(k)}$$

then

$$\sum_{k=0}^{\infty} \sum_{\ell=1}^{d_k} \int_{-\infty}^{\infty} |g^{k,\ell}(v)| e^{nv} dv < \infty.$$

We obtain the corresponding $f(x) \in PV_j$ by putting

$$f^{k,\ell}(v) = e^{-\frac{n}{2}v} g^{k,\ell}(v)$$

and

$$f(x) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d_k} f^{k,\ell}(\log r) Y_{k,\ell}(\theta).$$

6.9 Isomorphism for the spaces PW_j

Now if we have

$$PW_{j-1}^{(k)} = PV_j^{(k)} \ominus PV_{j-1}^{(k)},$$

then we have a similar decomposition for the polyharmonic wavelet spaces. We may prove the following

Theorem 9 (OK, 2001) *For every $j \in \mathbb{Z}$ if PW_j is the wavelet space defined by*

$$PW_{j-1} = PV_j \ominus PV_{j-1}$$

then we have the decomposition isomorphism

$$PW_j \approx \left[\bigoplus_{k=0}^{\infty} \bigoplus_{\ell=1}^{d_k} \right]' W_j^{(k)},$$

where we have denoted by $W_j^{(k)}$ the space of all polyharmonic wavelets ψ for the operator $\mathcal{L} [\Lambda_k + \frac{n}{2}]$ on the mesh $2^{-j}\mathbb{Z}$.¹

The above isomorphism is realized through decomposition and reconstruction relations which are also component-wise.

6.10 Polyharmonic Multiresolution Analysis

Theorem 10 *The sequence of spaces PV_j , $j \in \mathbb{Z}$, has the following properties:*

1. *It is a strictly increasing sequence, i.e.*

$$\dots \subset PV_{-1} \subset PV_0 \subset PV_1 \subset PV_2 \subset \dots$$

and approximates $L_2(\mathbb{R}^n)$, i.e.

$$\bigcap_{j=-\infty}^{\infty} PV_j = \{0\}, \quad \bigcup_{j=-\infty}^{\infty} PV_j \quad \text{is dense in } L_2(\mathbb{R}^n);$$

2. **"Shift invariance"**: *For every $f \in L_2(\mathbb{R}^n)$ and every $\ell \in \mathbb{Z}$ holds*

$$f(x) \text{ belongs to } PV_0 \iff f(x \cdot e^\ell) \text{ belongs to } PV_0;$$

3. **Rotational invariance**: *Let U be an orthogonal transform of \mathbb{R}^n . Then*

$$f(x) \text{ belongs to } PV_0 \iff f(Ux) \text{ belongs to } PV_0;$$

4. **Father wavelet**: *There is an element $\phi_1^{k,\ell}(v, \theta)$ such that for every sequence $\{c_\rho^{k,\ell}\} \in \ell_2^{(3)}$ the function*

$$f(x) = \sum_{\rho=-\infty}^{\infty} \sum_{k=0}^{\infty} \sum_{\ell=1}^{d_k} c_\rho^{k,\ell} \phi_1^{k,\ell}(v - \rho, \theta)$$

¹The sum $\bigoplus_{\ell=1}^{d_k} W_j^{(k)}$ is understood as taking d_k identic copies of the space $W_j^{(k)}$.

is in PV_0 .

5. **Refinement operator:** There is a canonical linear "scaling" operator which provides a way to express every $f \in PV_j$ through elements of the space PV_{j+1} , i.e.

$$f(x) = \sum_{\nu=-\infty}^{\infty} M_{\nu} g(re^{-\frac{\nu}{2}h}\theta).$$

Here $g \in PV_{j+1}$ and the operator M_{ν} defined in a canonical way, while the sum on ν is finite.

6. There is a **Riesz type inequality** for the elements $f(x)$ against the coefficients $c_p^{k,\ell}$.

In a similar sense there exists a unique **mother wavelet**.

6.11 Zero moments of polyharmonic wavelets

In the one-dimensional case the wavelets are orthogonal to the polynomials of order p . One of the advantages of the large symmetry group G is that the polyharmonic wavelets enjoy "enormous cancellation properties":

The *polyharmonic wavelets* are orthogonal to the functions polyharmonic of order p .

Theorem 11 Let f be a polynomial polyharmonic of order p , i.e. $\Delta^p f(x) = 0$. For every $j \in \mathbb{Z}$ if some element $\psi \in PW_j$ has a compact support then the integral

$$\int_{\mathbb{R}^n} f(x) \psi(x) dx = 0.$$

Note that the space of the polynomials polyharmonic of order p is infinite dimensional. Due to the compactness of the support of ψ we see that it is not necessary to consider global solutions of $\Delta^p f = 0$ but only some of local character, e.g. in an annulus.

This should be compared with the standard paradigm of MRA where the wavelet functions are orthogonal only to finite dimensional subspaces.

7 Further examples

7.1 The case of the ray hyperplanes

Let us consider N hyperplanes in \mathbb{R}^n such that their intersection pairwise is the same. For simplicity we consider the case \mathbb{R}^3 .

Let us denote by $(x_1, x_2) = re^{i\theta}$ the polar coordinates in \mathbb{R}^2 . Let the hyperplanes be

$$T_{\nu} := e^{i\nu\frac{2\pi}{N}}T_0 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \theta = \theta_{\nu}\}, \quad \nu = 1, \dots, N$$

where we put

$$\theta_\nu = \frac{2\pi}{N}.$$

We have the set

$$\mathcal{S} = \bigcup_{\nu} e^{i\nu \frac{2\pi}{N}} T_0.$$

Let u_ν be defined for $\theta_{\nu-1} \leq \theta \leq \theta_\nu$, $\nu = 1, \dots, N$. For simplicity we put $u_{N+1} = u_1$.

Let $p \geq 1$ be a fixed integer. **Polyspline** of order p with **break-surfaces** T_ν will be called a function u which satisfies the following conditions:

1. It has smoothness C^{2p-2} , i.e.

$$u \in C^{2p-2} \left(\mathbb{R}^n \setminus \bigcup_{\nu} T_\nu \setminus \{0\} \right).$$

2. It satisfies the following polyharmonic equation between every two planes, i.e.

$$\Delta^p u(x) = 0 \quad \text{for } \theta_{\nu-1} \leq \theta \leq \theta_\nu, \quad \text{for every } \nu = 1, 2, \dots, N.$$

3. We assume that

$$u \in L_2(\mathbb{R}^3).$$

We denote this space of Polysplines by

$$\mathcal{PS}.$$

Since the set $\mathcal{S} = \bigcup_{\nu} T_\nu$ is **invariant** under the action of the group

$$G \approx \mathbb{R} \times \mathbb{R} \times C_N,$$

where for $(u, v, \nu) \in G$ and $(r, \theta, x_3) \in \mathbb{R}^3$ the action is given by

$$(r, \theta, x_3) \longmapsto \left(r e^u, \theta + \nu \frac{2\pi}{N}, x_3 + v \right).$$

It follows that the space of Polysplines \mathcal{PS} is invariant under the same group.

For every $j \in \mathbb{Z}$ the refinement of the set $\mathcal{S} = \bigcup_{\nu} T_\nu$ will be defined by

$$\mathcal{S} = \bigcup_{\nu} e^{i2^{-j} \nu \frac{2\pi}{N}} T_0.$$

This set is invariant under the group

$$G \approx \mathbb{R} \times \mathbb{R} \times C_{2N},$$

We introduce as above the spaces PV_j and the wavelet spaces $PW_{j-1} = PV_j \ominus PV_{j-1}$.

We obtain the same results as in the previous Section for decomposition in one-dimensional Wavelet Analysis. For every fixed $\lambda, \xi \in \mathbb{R}^2$ the ordinary differential operator

$$L = \left(\frac{\partial^2}{\partial \theta^2} - \lambda^2 \right) \left(\frac{\partial^2}{\partial \theta^2} - (\lambda - 2i)^2 \right),$$

generates Multiresolution Analysis $V_j^{\lambda, \xi}$ and wavelet spaces $W_{j-1}^{\lambda, \xi} = V_j^{\lambda, \xi} \ominus V_{j-1}^{\lambda, \xi}$. We have again the isomorphism of the spaces

$$PV_j \approx \int_{\mathbb{R}^2} V_j^{\lambda, \xi} d\lambda d\xi,$$

$$PW_j \approx \int_{\mathbb{R}^2} W_j^{\lambda, \xi} d\lambda d\xi.$$

8 Other examples and their symmetry groups

Other examples are generated by the above considering products of the groups. The refinement is always with respect to a component \mathbb{Z} or \mathbb{S} . On the other hand as usual, other groups are obtained from the above by applying diffeomorphisms.

9 David Donoho's ridgelets and curvelets and the Polyspline approach

D. Donoho has insistently pointed out that the standard wavelet paradigm is not efficient for analyzing images having $(n - 1)$ -dimensional singularities, see e.g. [1]. For that reason he has created with coauthors the ridgelets and the curvelets which are more effective. The above polyharmonic wavelets are an alternative approach to the same problem since the singularities of the Polysplines lie by definition on $(n - 1)$ -dimensional surfaces.

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