Nonlinear Approximation with Schauder Bases

Rémi Gribonval and Morten Nielsen

remi.gribonval@inria.fr, nielsen@math.sc.edu

IMI, Dept of Mathematics, Univ. of South Carolina

Nonlinear approximation in a Banach space X

- Normed Schauder basis $\mathcal{B} = \{g_k\}_{k \geq 1}$

$$\Rightarrow \text{unique representation of } f \in X : f = \sum_{k=1}^{\infty} c_k(f) g_k$$

- Best/Greedy m-term approximation

$$\sigma_m(f) := \inf_{k_1,\ldots,k_m,a_1,\ldots,a_m} \Big\| f - \sum_{1 \leq j \leq m} a_j g_{k_j} \Big\|_X$$

$$\gamma_m(f) := \Big\| f - \sum_{1 \leq k \leq m} c_{\phi(k)}(f) g_{\phi(k)} \Big\|_X$$

with $|c_{\phi(1)}| \geq |c_{\phi(2)}| \geq \ldots \geq |c_{\phi(k)}| \geq \ldots$

Characterization of approximation spaces

- Best/Greedy approximation spaces

$$\mathcal{A}^\alpha := \Big\{ f \in X, \|f\|_{\mathcal{A}^\alpha} := \|f\|_X + \sup_m m^\alpha \sigma_m(f) < \infty \Big\},$$

$$\mathcal{G}^\alpha := \Big\{ f \in X, \|f\|_{\mathcal{G}^\alpha} := \|f\|_X + \sup_m m^\alpha \gamma_m(f) < \infty \Big\}.$$

- Known result [Stechkin, DeVore, Temlyakov] :

$$X = \text{Hilbert space}, \mathcal{B} = \text{ONB} :$$

$$\|f\|_{\mathcal{A}^\alpha} \asymp \|f\|_{\mathcal{G}^\alpha} \asymp \|\{c_k(f)\}\|_{\ell_\tau}, \frac{1}{\tau} = \alpha + \frac{1}{2}.$$

Main elements of the proof :

– $\|f\|_X = \|\{c_k(f)\}\|_{\ell_2}$

– Hardy's inequality.

Theorem 1 (Interpolation theory)

$$\|\{c_k(f)\}\|_{\ell_{q,\infty}} \ll \|f\|_X \ll \|\{c_k(f)\}\|_{\ell_{p,1}}$$

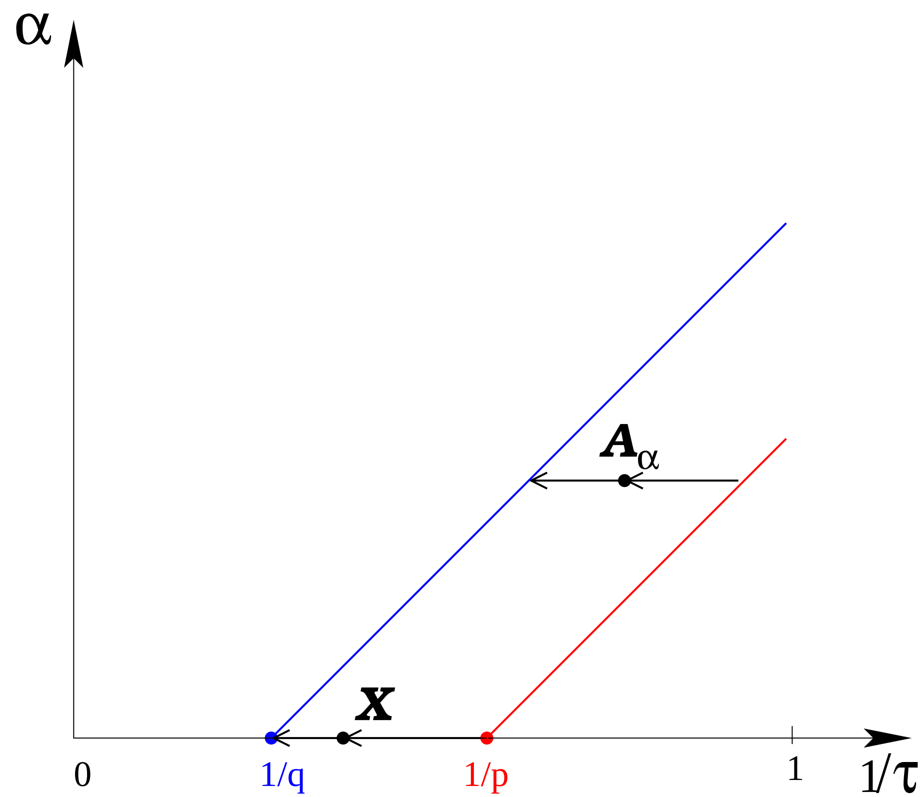
$$\Downarrow$$

$$\forall \alpha > 0, \|\{c_k(f)\}\|_{\ell_{\tau_q}} \ll \|f\|_{\mathcal{A}^\alpha} \ll \|\{c_k(f)\}\|_{\ell_{\tau_p}}$$

$$\frac{1}{\tau_q} = \alpha + \frac{1}{q} \qquad \frac{1}{\tau_p} = \alpha + \frac{1}{p}$$

Examples

- ⋆ $p = 1, q = \infty$ in any Banach space
- ⋆ $p > 1, q < \infty$ if X is uniformly convex and uniformly smooth. [Gurariĭ and Gurariĭ 1971]



<u>Definition</u> : $1 \leq P_X(\mathcal{B}) \leq Q_X(\mathcal{B}) \leq \infty = \sup/\inf$ of p, q such that

$$\|\{c_k(f)\}\|_{\ell_{q,\infty}} \ll \|f\|_X \ll \|\{c_k(f)\}\|_{\ell_{p,1}}$$

Converse results

- Sharpness of left-side line if $\mathcal{B}$ is quasi-greedy.

$$\begin{array}{rcl} \exists\alpha > 0, \|\{c_k(f)\}\|_{\ell_{\tau_q}} & \ll & \|f\|_{\mathcal{A}^\alpha} \\ & \Downarrow & \\ \|\{c_k(f)\}\|_{\ell_{q,\infty}} & \ll & \|f\|_X \end{array}$$

- Sharpness of right-side line if $\mathcal{B}$ is greedy.

$$\begin{array}{rcl} \exists\alpha > 0, \|f\|_{\mathcal{A}^\alpha} & \ll & \|\{c_k(f)\}\|_{\ell_{\tau_p}} \\ & \Downarrow & \\ \|f\|_X & \ll & \|\{c_k(f)\}\|_{\ell_{p,1}} \end{array}$$

Structured bases

- $\mathcal{B}$ is quasi-greedy iff $\gamma_m(f) \to 0$
- $\mathcal{B}$ is greedy iff $\gamma_m(f) \le C\sigma_m(f)$

Theorem [Konyagin,Temlyakov] :
greedy $\Leftrightarrow$ unconditional and democratic.

- $\mathcal{B}$ is democratic iff

$$\|\sum_{k\in\Lambda_n} g_k\|_X \asymp \|\sum_{k\in\Lambda'_n} g_k\|_X$$

whenever $|\Lambda_n| = |\Lambda'_n| = n$.

- For $\mathcal{B}$ quasi-greedy and democratic we can define $w(\mathcal{B}) := \{w_n\}$ with

$$w_n \asymp \|\textstyle\sum_{k\in\Lambda_n} \pm g_k\|_X$$

Theorem 2 (Greedy approximation)

Assume $\mathcal{B}$ is quasi-greedy. Then the following are equivalent

1. $\mathcal{B}$ is democratic
2. For all $\alpha > 0$

$$\|f\|_{\mathcal{G}^\alpha} \asymp \|\{c_k(f)\}\|_{\ell_{1/\alpha}(w)}$$

where $\|\{c_k\}\|_{\ell_{\tau,\infty}(w)} := \sup_m w_m m^{1/\tau} |c_{\phi(m)}|$.

Corollaries

- $\mathcal{B}$ greedy :

$$\|f\|_{\mathcal{A}^\alpha} \asymp \|f\|_{\mathcal{G}^\alpha} \asymp \|\{c_k(f)\}\|_{\ell_{1/\alpha}(w)} .$$

- $\mathcal{B}$ quasi-greedy and $\|\{c_k(f)\}\|_{\ell_{p,\infty}} \ll \|f\|_X \ll \|\{c_k(f)\}\|_{\ell_{p,1}}$,

$$\|f\|_{\mathcal{A}^\alpha} \asymp \|f\|_{\mathcal{G}^\alpha} \asymp \|\{c_k(f)\}\|_{\ell_{\tau_p}} .$$

- $\mathcal{B}$ quasi-greedy in $X =$ Hilbert space

$$\|f\|_{\mathcal{A}^\alpha} \asymp \|f\|_{\mathcal{G}^\alpha} \asymp \|\{c_k(f)\}\|_{\ell_{\tau_2}} .$$