Digital Ridgelet Transform based on True Ridge Functions

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Ridgelets on the Continuum

Goal: Construct systems that behave very well at representing functions with linear singularities.

- Ridge functions: $\rho(x, y) = r(\cos(\theta)x + \sin(\theta)y)$.
- Ridgelet function $\rho_{a,b,\theta}(x, y) = \psi((\cos(\theta)x + \sin(\theta)y - b)/a)/a^{3/2}$ where $\psi(t)$ is a wavelet.
  - Continuous transform in L^2. Tight Frame of the space of compact supported functions of L^2.
- Orthonormal Ridgelet functions
  - Not ridge functions, but obey certain localizations properties in a radial x angular frequency domain.
  - Orthonormal set of L^2.

Discretization of Ridgelets

- Problem: translating the notion of ridgelets, (if possible) from continuum concepts, useful in theoretical discussion, to algorithmic concepts capable of widespread application.
- The theory of ridgelets is closely related to the theories of Radon transformation, and of rotation and scaling of images, all of which seem natural and simple on the continuum, and for which it is widely believed that there is no simple, inevitable definition for digital data.

Our definition offers:

- Analysis and synthesis by true ridge functions.
- exact reconstruction. The analysis operator is invertible on its range; the appropriately preconditioned operator has a tightly controlled spread of singular values.
- a near-parseval relationship.
- a fast algorithm requiring only $O(N \log(N))$ flops for data in and n by n grid, where $N = n^2$ is the total number of data.
- strong analogy between the relationship of ridgelets, polar fourier transforms, and radon transforms, and between digital ridgelets, pseudopolar transforms, and a notion of Radon transform for digital data called Fast Slant Stack (FSS).
- cartesian data structures

Digital Ridgelets

We consider images I as n by n arrays indexed by coordinates (u, v) ranging in the square $-n/2 \leq u, v < n/2$ centered at $(0, 0)$. Let $\theta^s_{\ell;n}$ be defined so that

$$\tan(\theta^1_{\ell;n}) = 2\ell/n, \qquad \text{cotan}(\theta^2_{\ell;n}) = 2\ell/n, \quad -n/2 \leq \ell < n/2.$$

The lines $v = \tan(\theta^1_{\ell;n})u + t$ we speak of as ‘basically horizontal lines’ and the lines $u = \text{cotan}(\theta^2_{\ell;n})v + t$ we speak of as ‘basically vertical lines’. Each family of lines is equispaced in slope, rather than angle.

Let n be given. A normalized digital ridgelet $\rho_{j,k,s,\ell}$ is an n by n array built as ridge functions from fractionally-differenciated Meyer wavelets by the formula

$$\rho_{j,k,s,\ell}(u, v) = \psi_{j,k}(u + \tan(\theta^s_\ell)v), \qquad s = 1,$$

and

$$\rho_{j,k,s,\ell}(u, v) = \psi_{j,k}(u + \text{cotan}(\theta^s_\ell)v), \qquad s = 2,$$

where the parameter m underlying the definition obeys $m = 2n$.

Digital Ridgelet Analysis operator

The normalized digital ridgelet analysis operator applied to an $n \times n$ image $(I(u,v) : -n/2 \le u, v < n/2)$ is the array with $4n^2$ entries

$$RI = (\langle I, \tilde{\rho}_\lambda \rangle : \lambda \in \Lambda)$$

The next 'result' in really a distillation of computational experience.

Empirical Fact. *The normalized transforms R, and its adjoint $R^\star$ have their nonzero singular values within about 10% of each other. The generalized inverse $R^\dagger$ can be computed to seven digits accuracy in 4 iterations or less of a conjugate gradient solver.*

As a corollary of this empirical result, we have that *the system ρ_λ makes a frame, with the ratio of frame bounds empirically smaller than 1.10* . Thus the system (ρ_λ) behaves nearly as well as would a tight frame or ortho basis.

Digital Ortho-Ridgelets

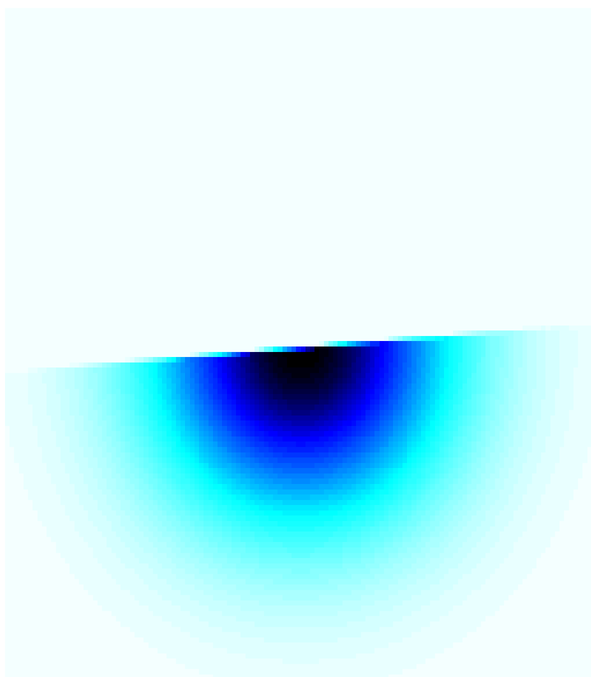
Let $W_{i,\ell}(u)$ denote a discrete orthonormal Cohen-Daubechies-Feauveau--Jawerth boundary adjusted wavelet for the discrete interval $-n/2 \leq u < n/2$. There are n of these wavelets, with indices $0 \leq i < \log_2(n)$ and $0 \leq l < 2^i$. For real sequences (W_u) and (V_u) indexed by $-n/2 \leq u < n/2$, let $\{W, V\}$ denote the inner product $\sum_l W_u V_u$.

Given the normalized discrete Ridgelet transform array RI, we define the Digital Ortho-Ridgelet transform array UI by taking the wavelet transform along the angular variable ℓ of RI:

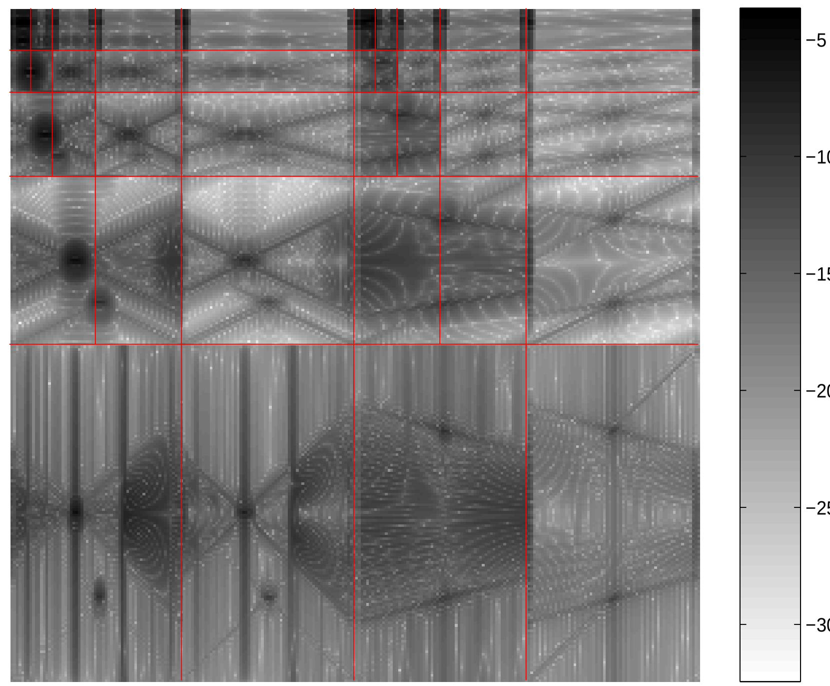
$$(UI)_{j,k;i,l} = \{RI_{j,k;s,\cdot}, W_{i,l}(\cdot)\}.$$

Digital OrthoRidgelet Transform of Half Dome

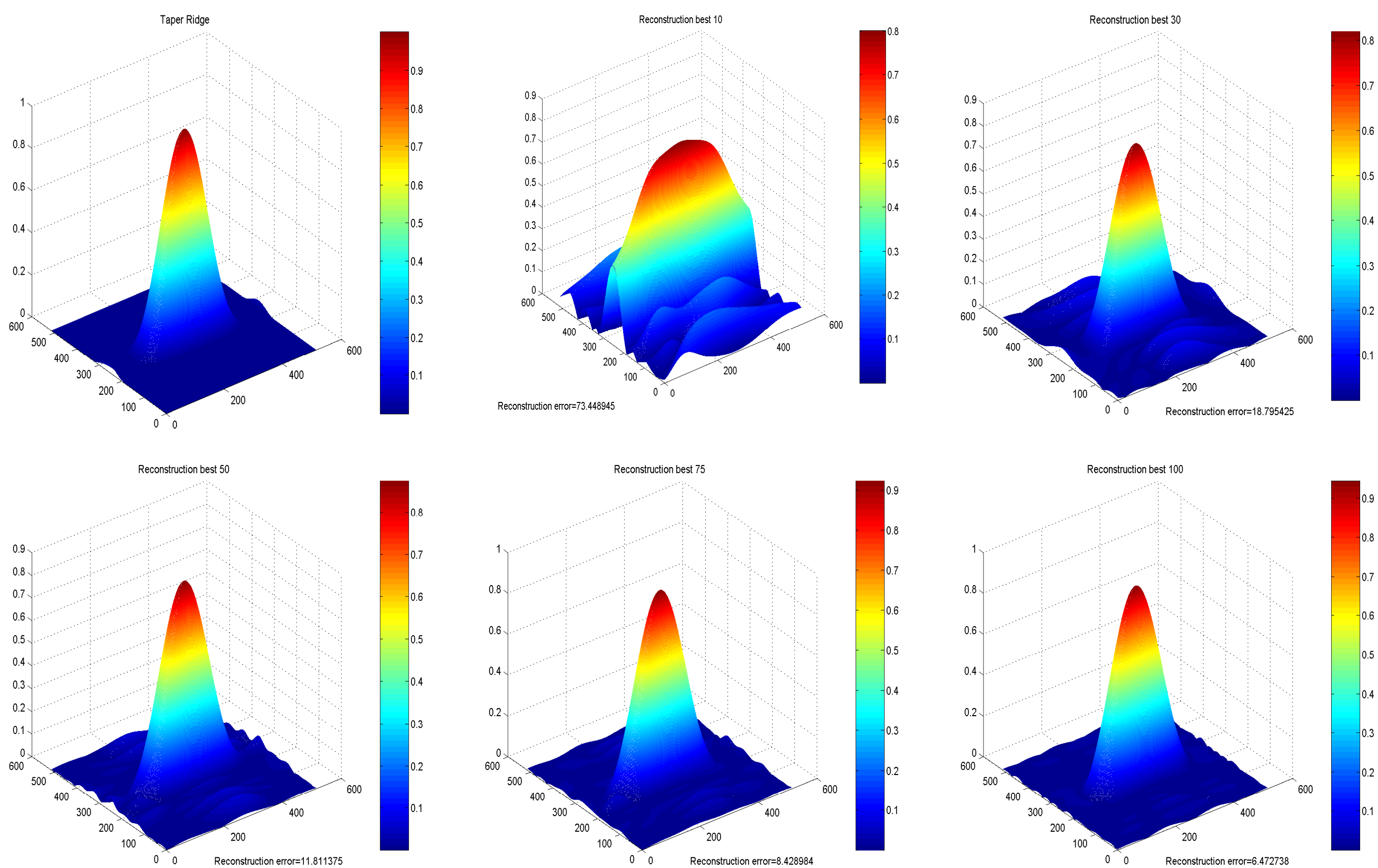
HalfDome



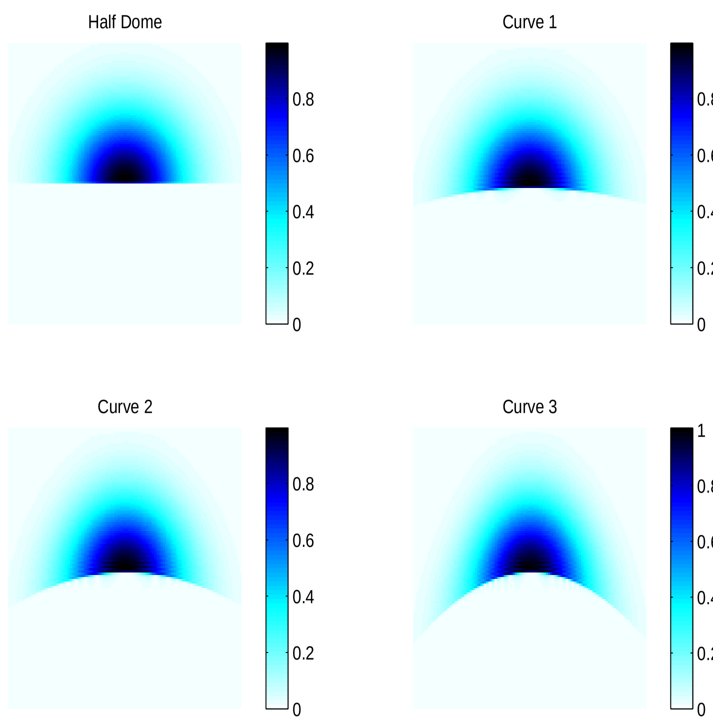
log(Ortho Ridgelet Transform) of HalfDome

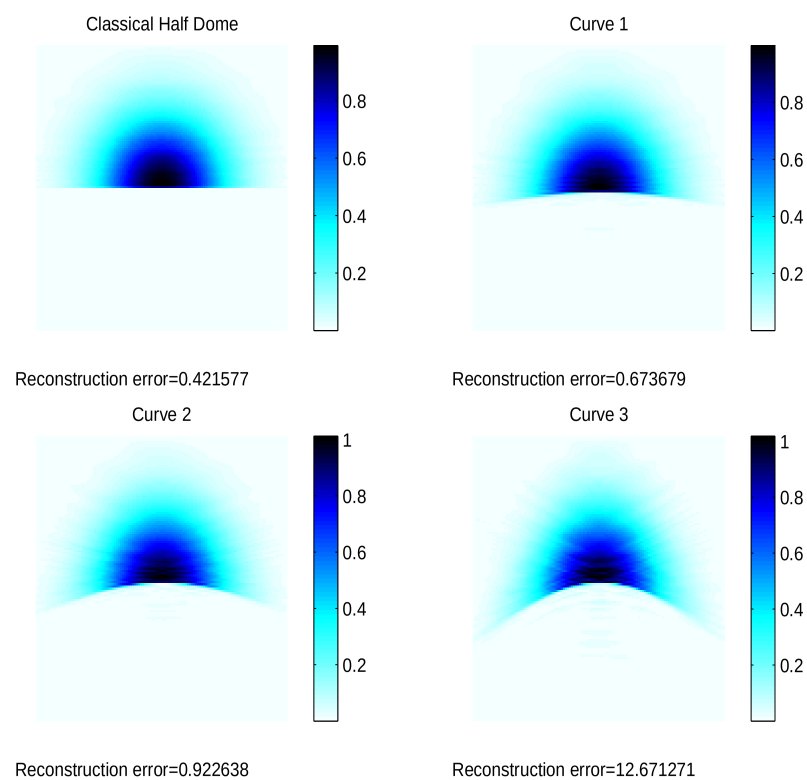


Study of OrthoRidgelet Transform of a Tapered Ridge Function



Digital Ridgelet Transform of Objects with curved singularities





Digital OrthoNormal Ridgelets vs Z_p^2 Ridgelets