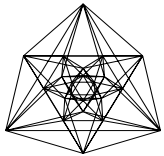


# Parameterizing Meshes with Applications

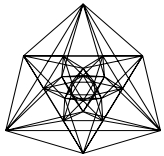
Günther Greiner  
in collaboration with  
Kai Hormann



# Overview

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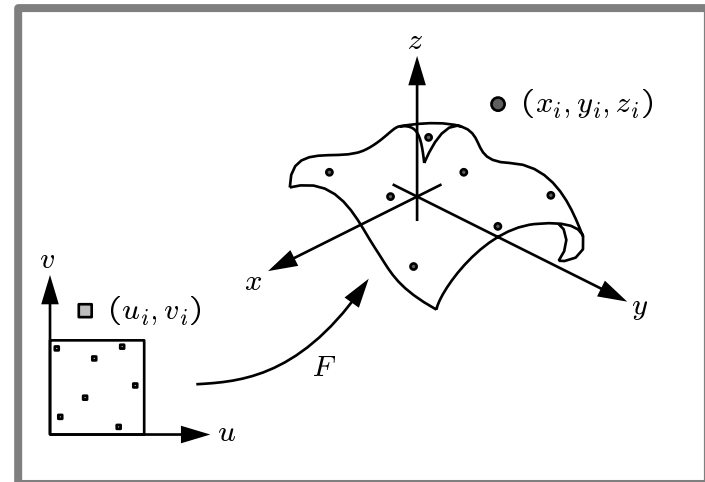
- **Motivation**
- **Parameterizing meshes**
  - projection methods
  - linear energy methods
  - nonlinear methods
- **Applications**
  - texture mapping
  - quadrilateral remeshes and surface fitting
  - regular remeshes and hierarchies
- **Summary**



## Motivation (1)

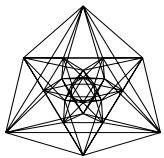
- Analysis on surfaces is usually performed in Euclidean plane, using appropriate (local) coordinates.

⇒ one has to assign to every surface point a parameter value in the plane

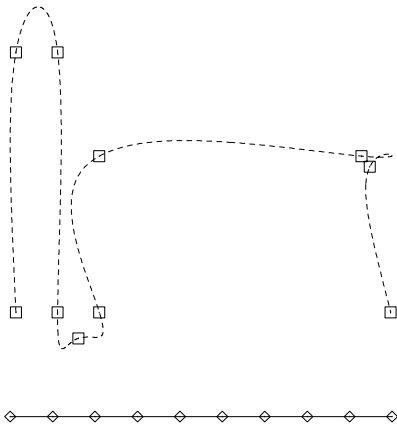


- The result of the analysis often depends on the choice of the parameterization.

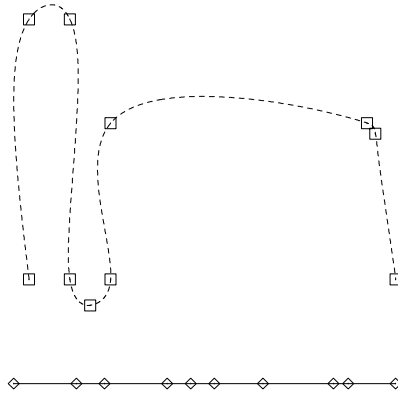
- To obtain **good** results use **good** parameterizations !



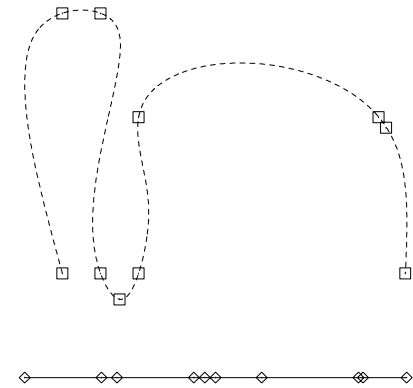
# Example : B-Spline interpolation



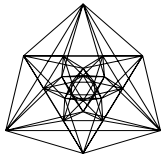
uniform



chord length



centripetal



## Motivation (2)

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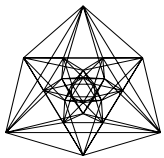
**Q:** What is a **good** parameterization ?

**A:** One that preserve all the (basic) geometry  
length, angles, area, ...

⇒ **isometric parameterization**

**but** : possible only for developable surfaces  
e.g., there will always be distortion !

Try to keep the distortion as small as possible  
(change of length, area, angles,... )



## Motivation (3)

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- Applications

Many operations, manipulations on/with surfaces require a parameterization as a preliminary step.

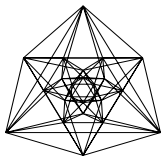
e.g.:

texture mapping

surface fitting

hierarchical representations

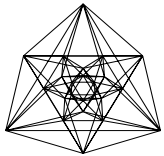
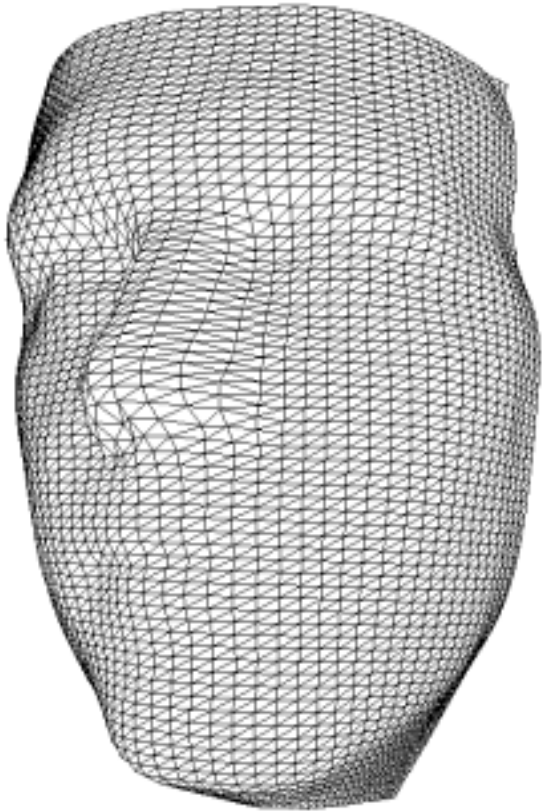
mesh conversion



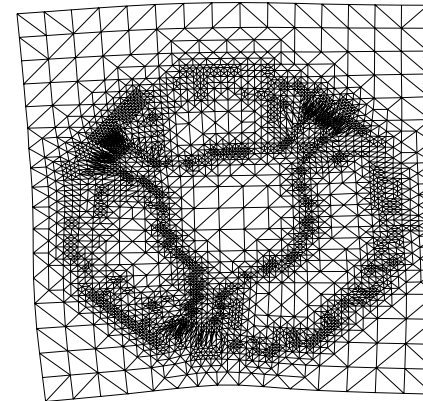
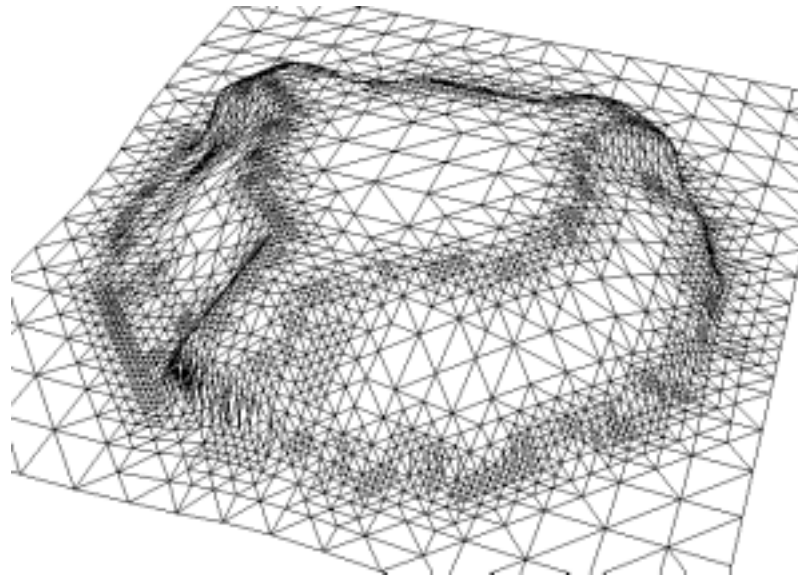
# Motivation: Applications of parameterizations

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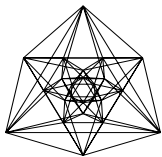
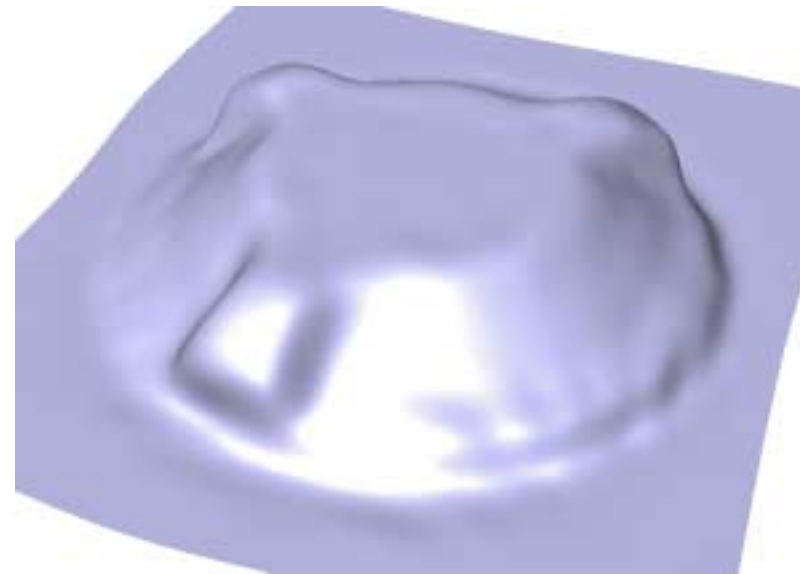
Texture mapping



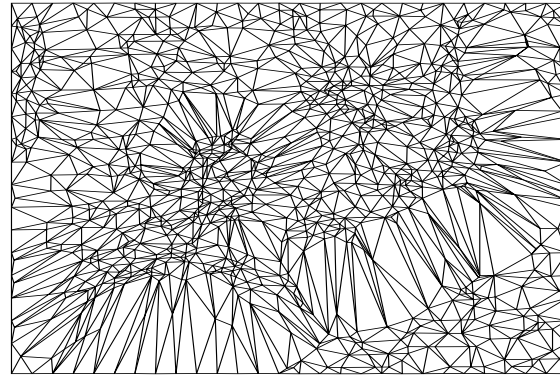
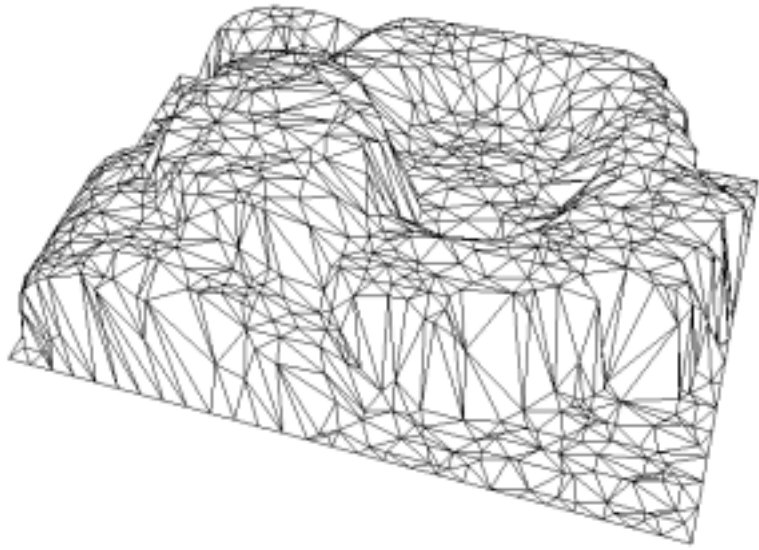
# Motivation: Applications of parameterizations



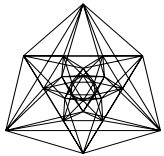
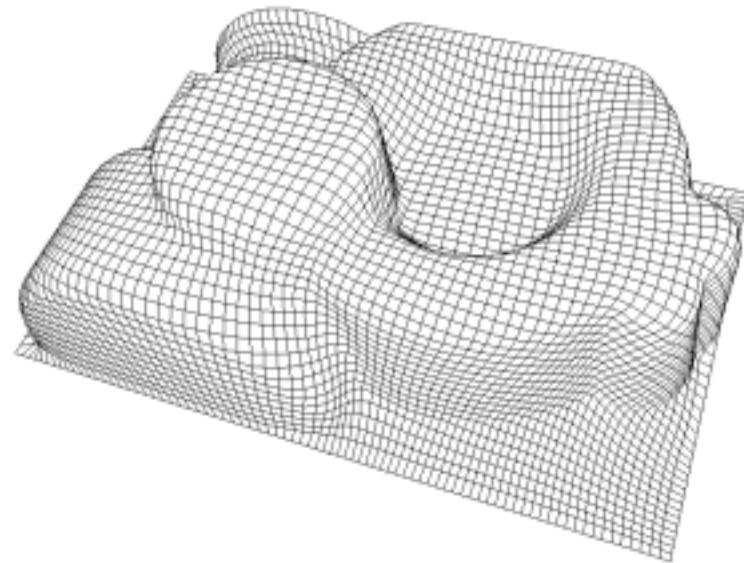
Surface fitting



# Motivation: Applications of parameterizations



mesh conversion



# Parameterization of 3D data points

## PROBLEM:

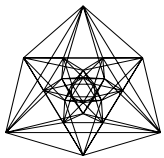
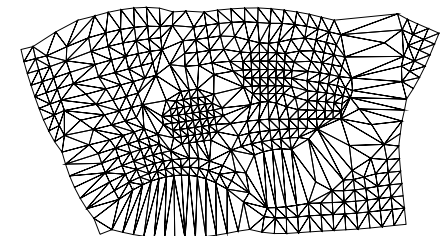
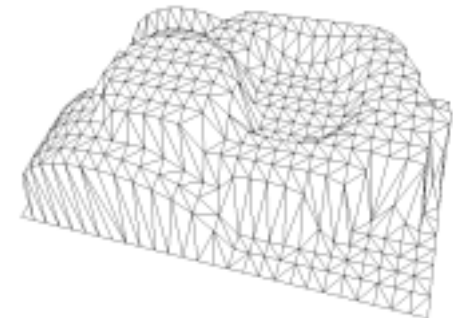
For a triangulated set of data points

$$P_i \in \mathbb{R}^3, T_j = \Delta(P_{j_0}, P_{j_1}, P_{j_2})$$

find a planar parametrization

$$p_i \in \mathbb{R}^2, t_j = \Delta(p_{j_0}, p_{j_1}, p_{j_2})$$

with minimal distortion.

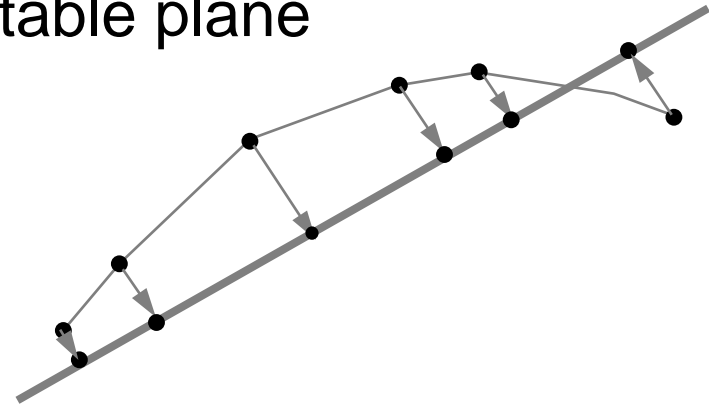


## Parameterizations: Projection methods

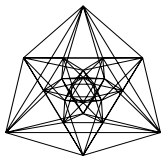
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- Project data points onto a suitable plane

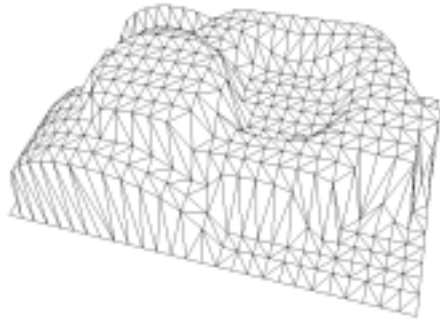
e.g. least square fitting  
plane of the data points



- very simple, works well only for planar geometries
- Generalization (Ma, Kruth: CAD 1995)  
Project on other simple surfaces,  
e.g. Coons patches

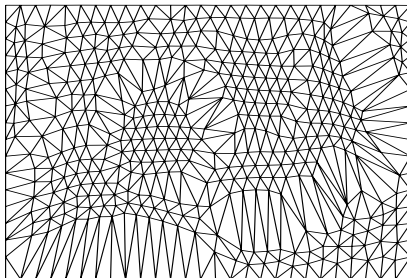


# Parameterizations: Linear energy methods

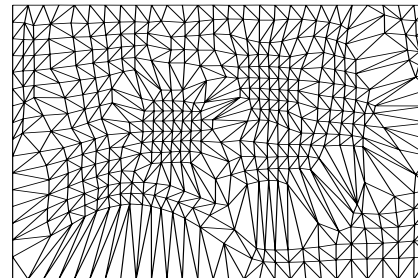


*linear spring-energy:*

$$E = \frac{1}{2} \sum_{\{i,j\} \in \text{Edges}} c_{ij} \|p_i - p_j\|^2$$

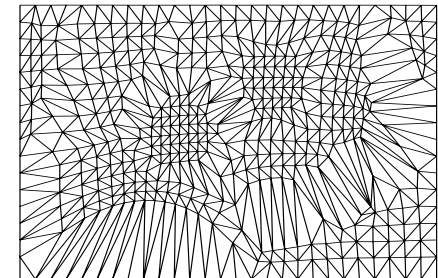


Hooke's law



shape-preserving

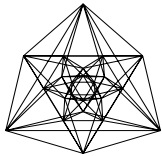
*Floater '96*



harmonic energy

*Pinkall/Polthier '93  
Eck et.al. SIGGRAPH 95*

Note: boundary points have to be fixed!



## Parameterization: Linear methods

$$E = \frac{1}{2} \sum_{\{i,j\} \in \text{Edges}} c_{ij} \|p_i - p_j\|^2$$

- Spring constants for **Hooke**:

chord length:  $c_{ij} = \|P_i - P_j\|^{-1}$

centripetal:  $c_{ij} = \|P_i - P_j\|^{-1/2}$

uniform:  $c_{ij} = 1$

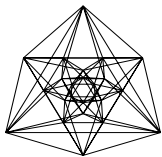
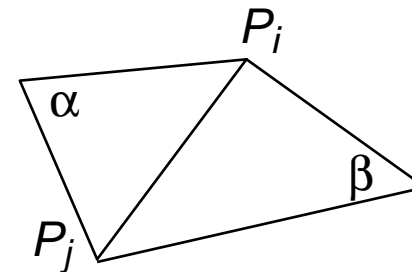
- **harmonic energy**:

minimize *Dirichlet energy*

$$\Rightarrow c_{ij} = \cot(\alpha) + \cot(\beta)$$

– negative weights ? –

$$E_D(f) := \frac{1}{2} \int_M \|\text{grad} f\|^2 dA$$



## Parameterization: Linear methods

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- **shape-preserving**

solve the linear system

with positive weights

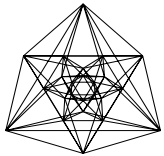
$\lambda_{ij}$  that sum to 1.

$$p_i = \sum_{\{j:[i,j] \in \text{edges}\}} \lambda_{ij} p_j$$

- **modifications:** more realistic energy functional

e.g. 
$$E = \sum_{[i,j] \in \text{edges}} c_{ij} (\|p_i - p_j\| - \ell_{ij})^2$$

⇒ better results, but no longer linear !



# Parameterizations: Systematic approach

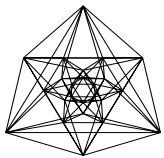
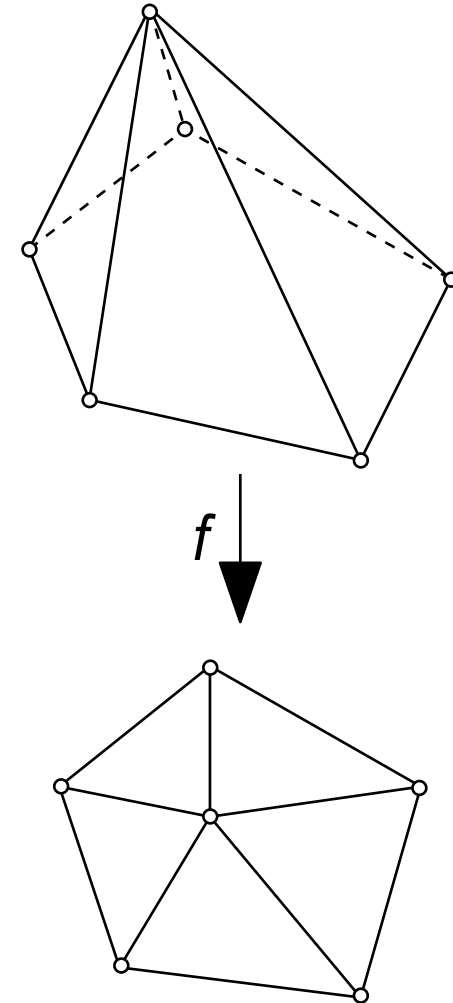
## FACT:

Any parametrization deforms the shape of the triangles.

except for developable surfaces  
(e.g., planes, cylinders, conics)

## QUESTION:

How to measure this deformation?



# Systematic approach: How to measure the distortion?

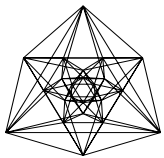
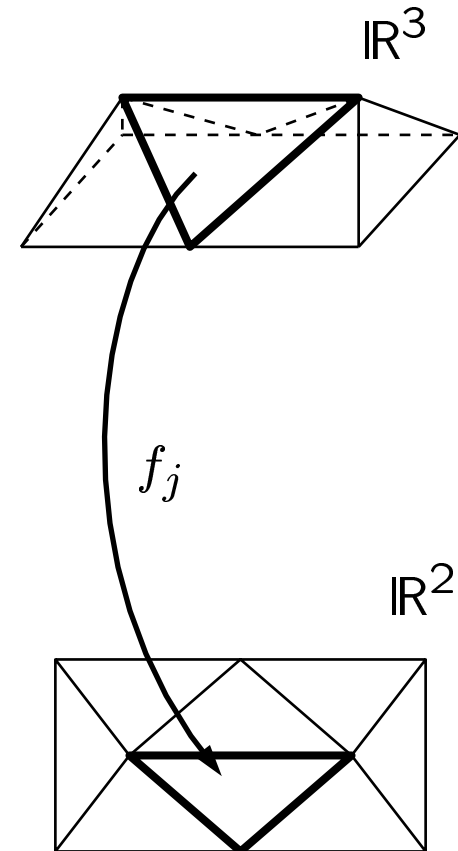
## IDEA:

Find a functional that measures the distortion of the *atomic linear maps*

$$f_j : T_j \rightarrow t_j$$

and minimize

$$\sum_j E(f_j)$$

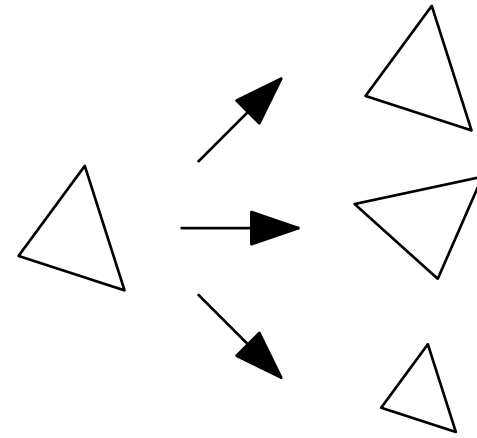


## **Systematic approach:** properties of functional

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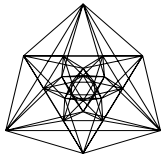
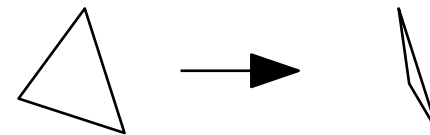
The deformation functional should be invariant to

1. translations,
2. orthogonal transformations,
3. scalings,



and should avoid degeneracies by

4. punishing collapsing triangles very badly.



## Systematic approach: derivation of functional

Consider the atomic linear map

$$g : \mathbb{R}^2 \rightarrow \mathbb{R}^2, x \mapsto Ax + b$$

Singular value decomposition

$$U^t A V = \Sigma = \begin{pmatrix} \sigma_1 & \\ & \sigma_2 \end{pmatrix}$$

2-norm condition

$$\kappa_2(A) = \|A\|_2 \|A^{-1}\|_2 = \frac{\sigma_1}{\sigma_2}$$

Properties:

1. translations

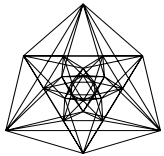
→ ignore  $b$

2. orthogonal transf.

→ ignore  $U$  and  $V$

3. scalings

4. collapsing triangles



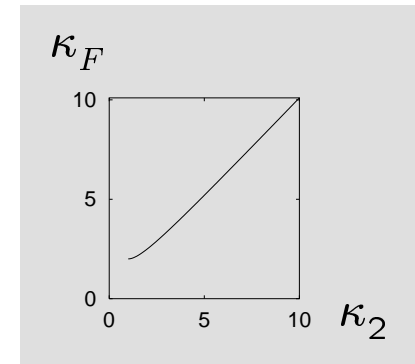
## Drawback of the 2–norm condition

### PROBLEM:

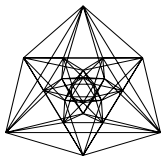
2–norm condition not easy to compute

### SOLUTION:

Use *Frobenius Norm* condition instead



$$\begin{aligned}\kappa_F(A) &= \|A\|_F \|A^{-1}\|_F = \sqrt{\sigma_1^2 + \sigma_2^2} \sqrt{1/\sigma_1^2 + 1/\sigma_2^2} \\ &= \frac{\sigma_1^2 + \sigma_2^2}{\sigma_1 \sigma_2} = \frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1} = \kappa_2(A) + \frac{1}{\kappa_2(A)} \\ &= \frac{\text{trace}(A^t A)}{\det A}\end{aligned}$$



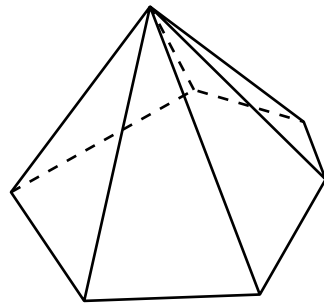
# Minimization of the deformation functional

Use a "Gauss–Seidel–like" method:

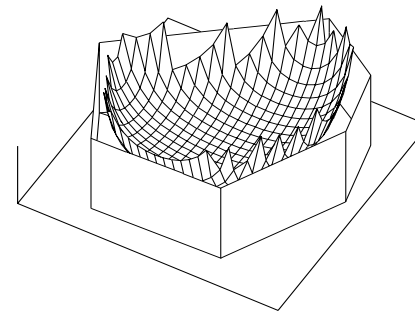
pick successively one vertex and position it optimal.

Local functional is convex!

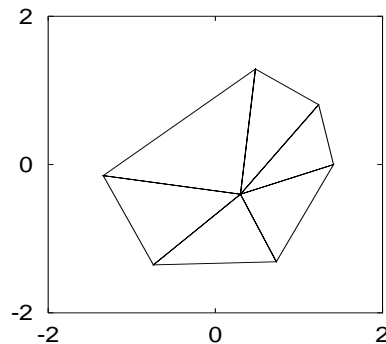
Local configuration



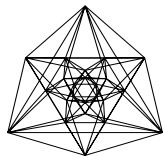
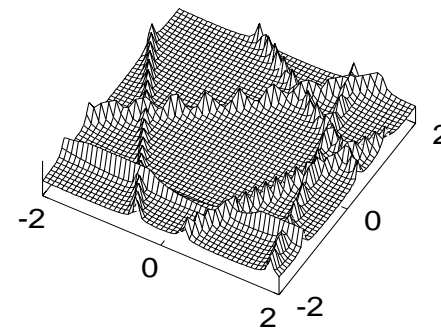
Graph of local functional



Local parametrization



Singularities along boundary edges



# Parameterization: References & further methods

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Ma, Lin: Optimal texture mapping, in: *Proceedings Eurographics '88*, 421–428

Bennis, Vézien, Iglésias: Piecewise surface flattening for non-distorted texture mapping, in: *Proceedings Siggraph '91*, 237–246

Pinkall, Polthier: Computing discrete minimal surfaces and their conjugates, *Experimental Mathematics*, vol. 2, 1993, 15–36

Maillot, Yahia, Verroust: Interactive texture mapping, in: *Proceedings Siggraph '93*, 27–34

Ma, Kruth: Parameterization of randomly measured points for least squares fitting of B-spline curves and surfaces, *CAD*, vol. 27, 1995, 663–675

Eck, DeRose, Duchamp, Hoppe, Lounsbery, Stuetzle: Multiresolution analysis of arbitrary meshes, in: *Proceedings Siggraph '95*, 173–182

Greiner, Hormann: Interpolating and approximating scattered 3D data with hierarchical tensor product B-splines, in: *Surface Fitting and Multiresolution Methods*, 1997, 163–172

Floater: Parameterization and smooth approximation of surface triangulations, *CAGD*, vol. 14, 1997, 231–250

Duchamp, Certain, DeRose and Stuetzle: Hierarchical computation of PL harmonic embeddings, preprint, 1997

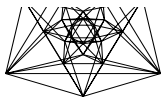
Lee, Sweldens, Schröder, Coswar, Dobkin: MAPS: Multiresolution adaptive parameterization of surfaces, in: *Proceedings Siggraph '98*, 95–104

Lévy, Mallet: Non-distorted texture mapping for sheared triangulated meshes, in: *Proceedings Siggraph '98*, 343–352

Hormann, Greiner: MIPS: An efficient global parametrization method, in: *Curve and Surface Design: Saint-Malo 1999*, 2000, 153–162

Sheffer, de Sturler: Surface parameterization for meshing by triangulation flattening, in: *Proceedings 9th International Meshing Roundtable*, 2000, 161–172

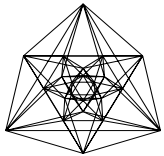
Zigelman, Kimmel, Kiryati: Texture mapping using surface flattening via multi-dimensional scaling, preprint, 2000



## Comparison

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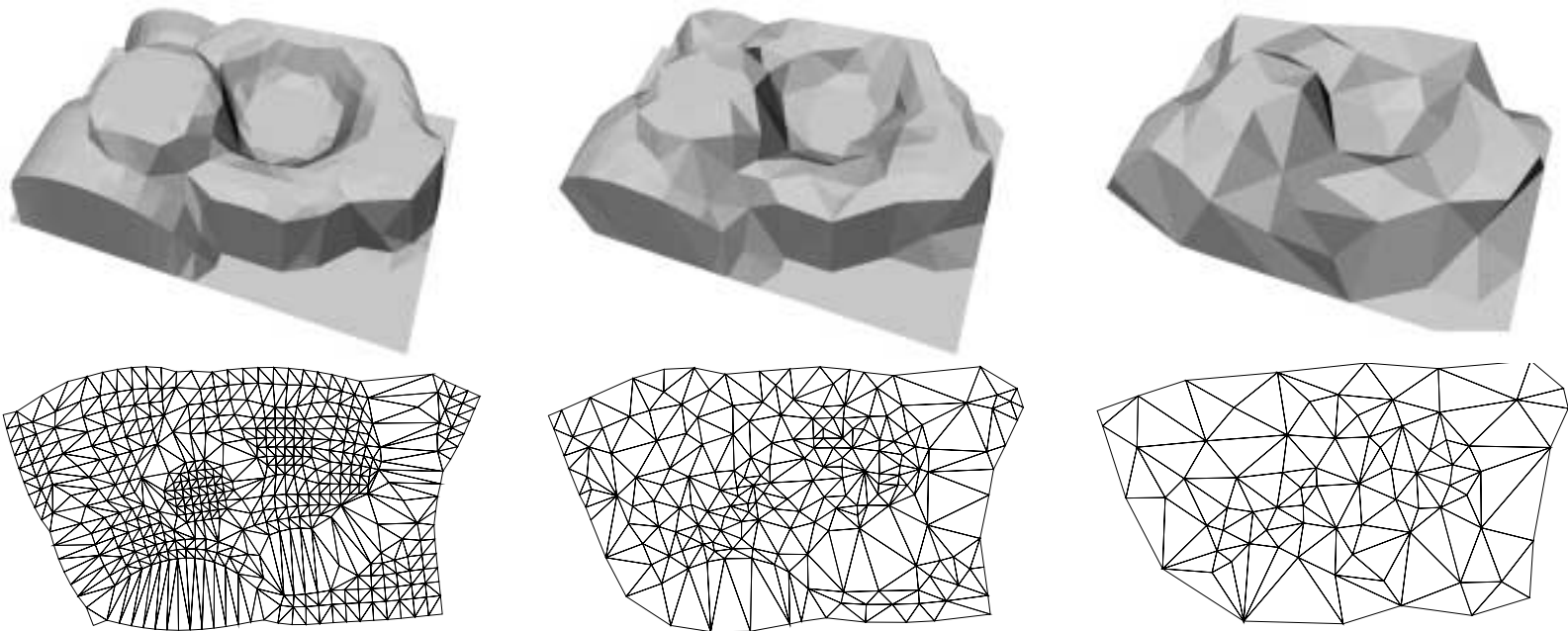
- linear methods can be realized fast and easy
- harmonic energy can produce foldovers (neg. weights)
- nonlinear methods (realistic springs, MIPS) often (not always) produce better results
- most methods require to specify a parameterization of the boundary, except MIPS
- **The optimal parameterization method has not been found yet, or it does not exist !**



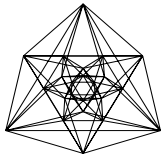
## **Parametrizations: Hierarchical methods**

For large data sets (> 20k points): hierarchical approach

**build mesh hierarchy**



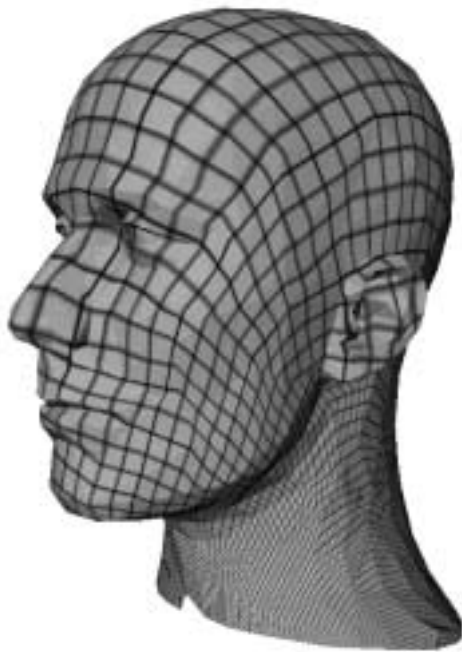
**optimize each level**



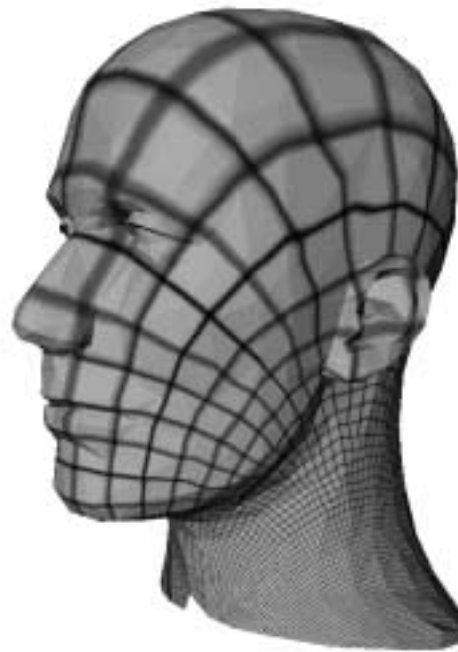
# Applications: texture mapping

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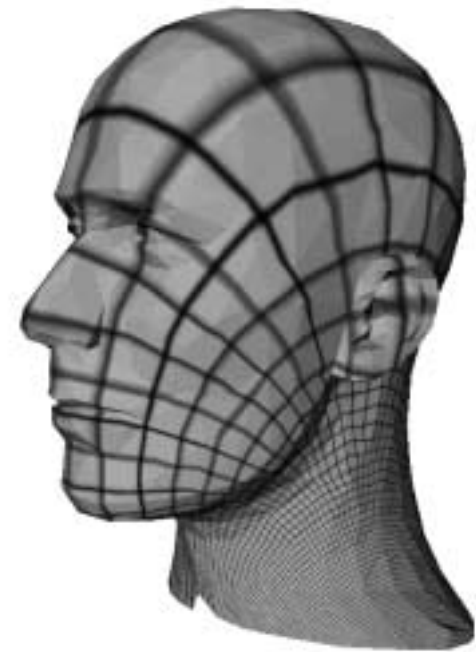
texture mapping of uniform rectangular grid



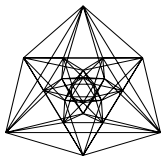
real spring



MIPS



harmonic

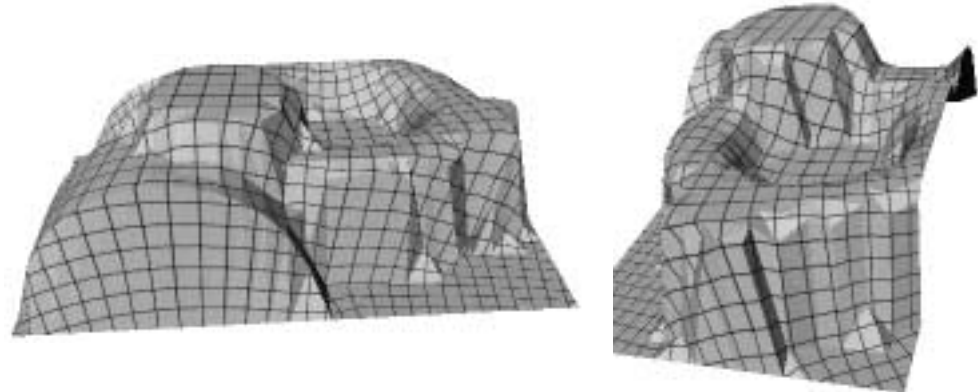
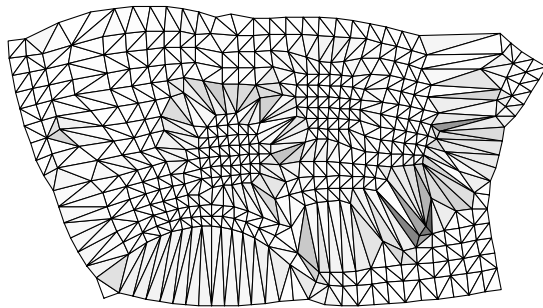


# Applications: texture mapping

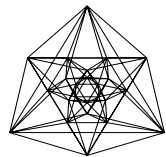
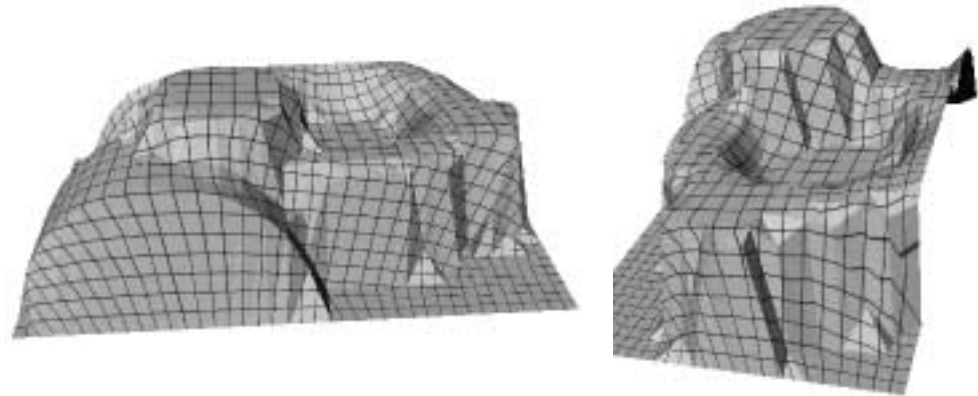
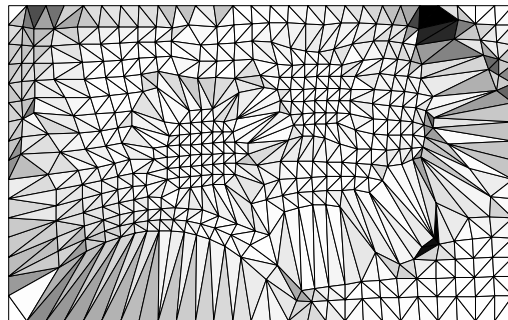
gray-coded deformation  
energy per triangle

texture mapping of uniform rectangular grid

MIPS



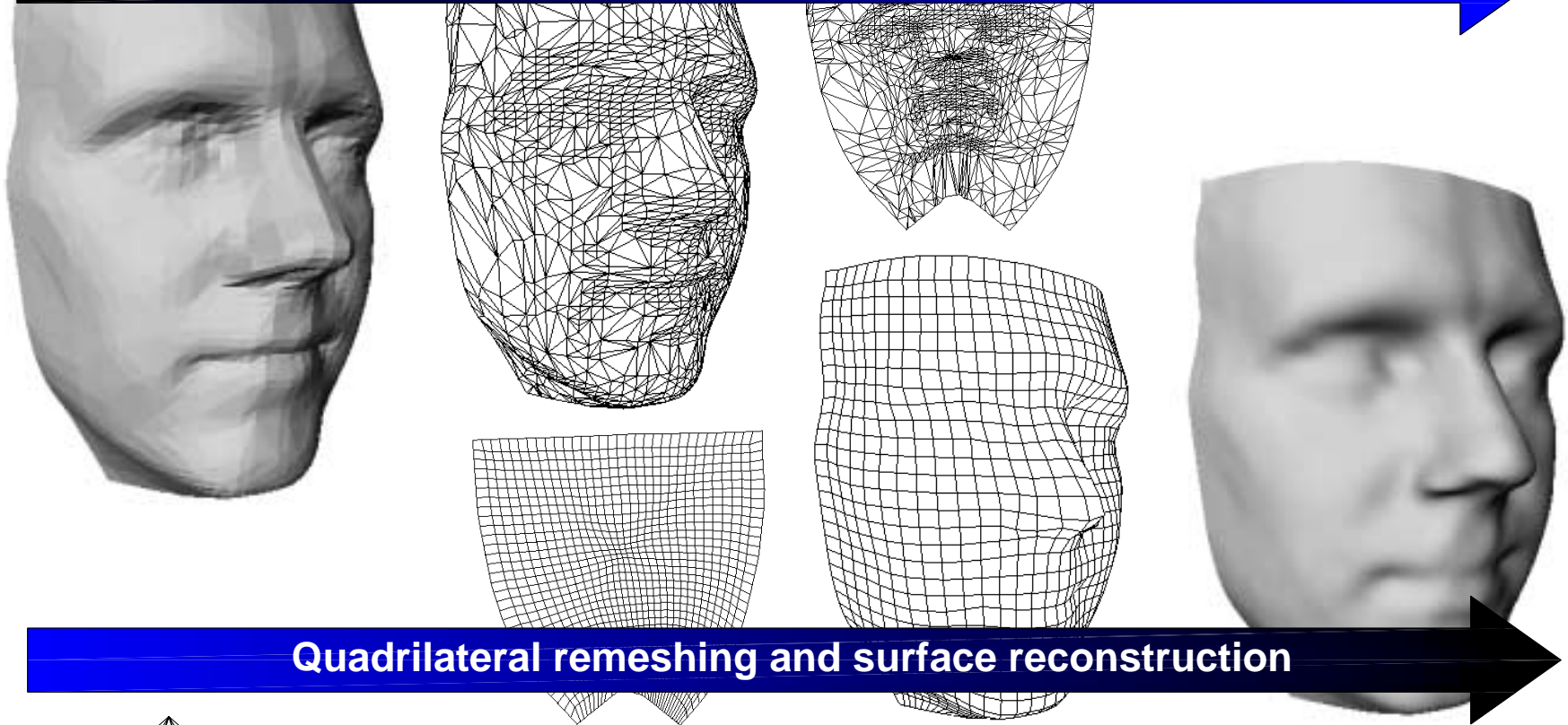
harmonic



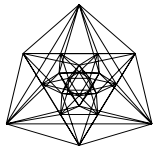
# Applications: Quadrilateral Remeshing

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Parameterizing triangle meshes



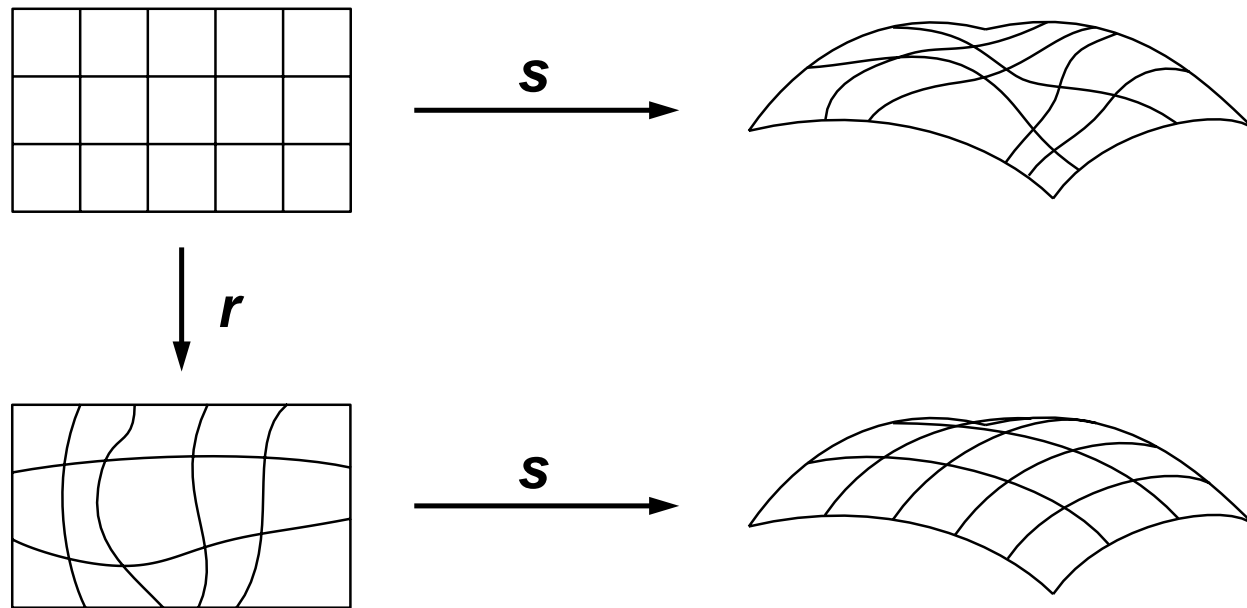
Quadrilateral remeshing and surface reconstruction



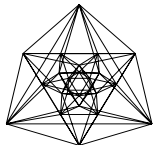
# The Problem of Badly Shaped Iso-Curves

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**Problem:** smooth surface  $s$ , but badly shaped iso-curves



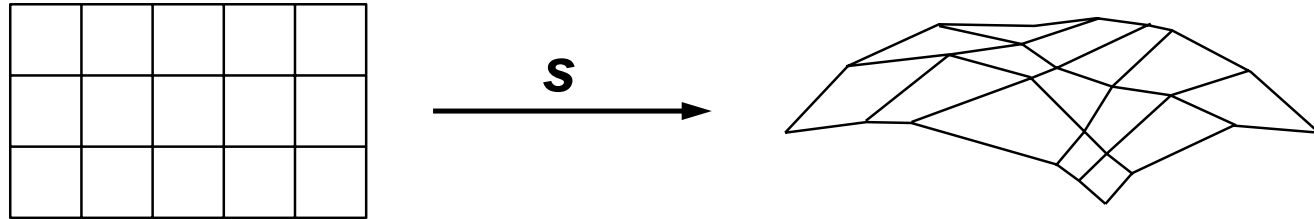
**Task:** find  $r$ , such that the iso-curves of  $s \circ r$  are smooth



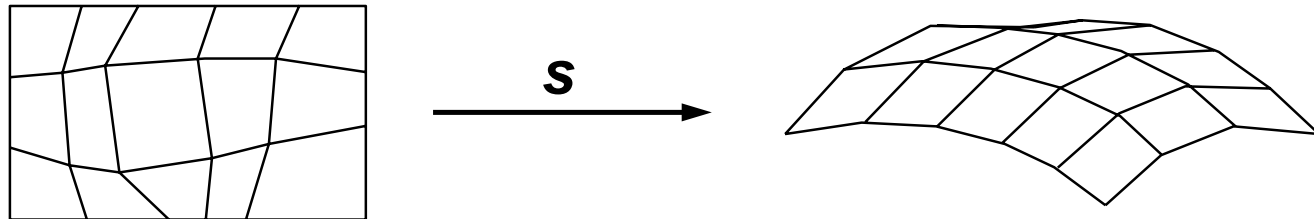
# Quadrilateral Remeshing of Smooth Surfaces

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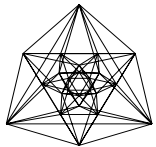
1) evaluate  $s$  at the knots of a uniform 2D grid



2) use shape information of 3D grid to move knots of the 2D grid



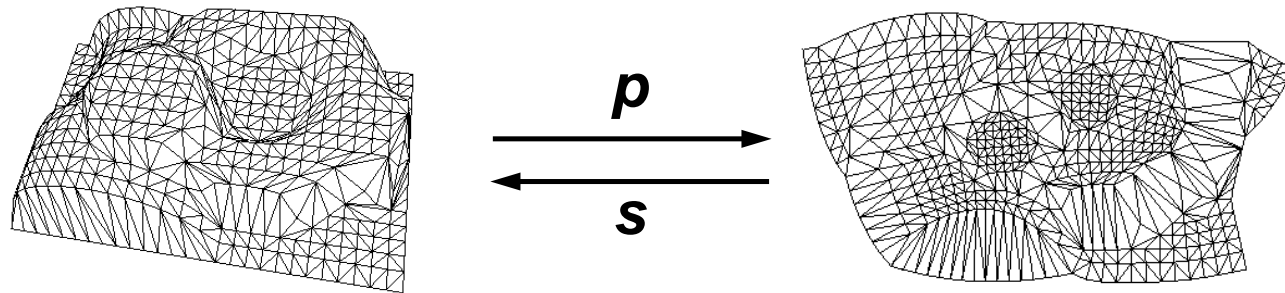
3) let  $r$  be the piecewise bilinear mapping between the 2D grids



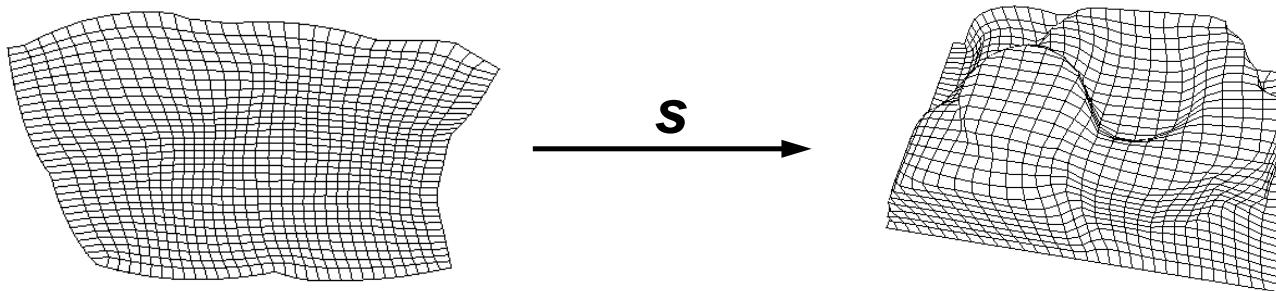
# Applying these Methods to Triangle Meshes

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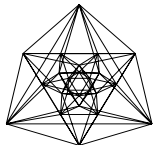
1) parameterize the triangle mesh



2) quadrilateral remeshing of the triangle mesh



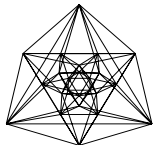
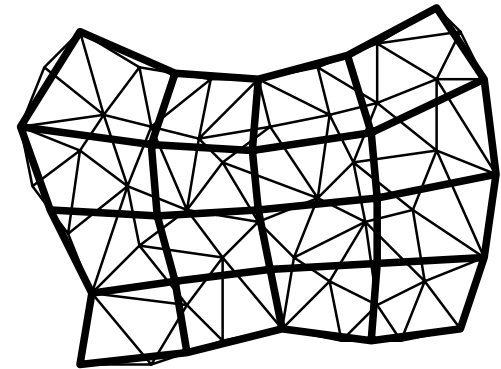
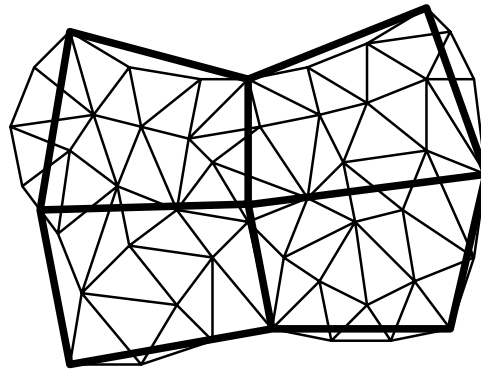
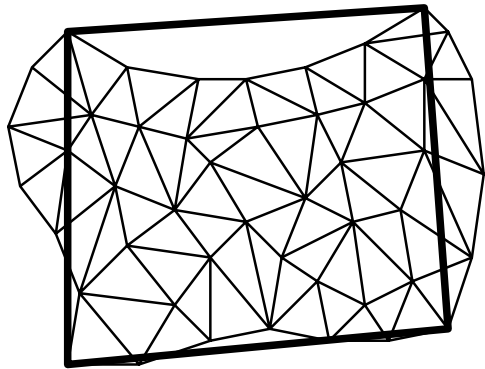
3) interpolate the remesh with a TP-B-Spline



# Quadrilateral Remeshing of Triangle Meshes (1)

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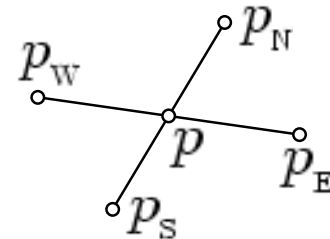
- 1) find an initial quadrilateral
- 2) iteratively split each quadrilateral into four
- 3) use weighted barycentres for new vertex positions



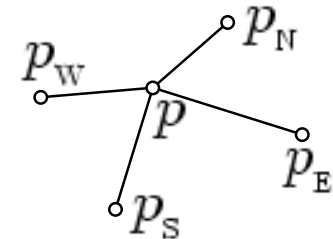
# Quadrilateral Remeshing of Triangle Meshes (2)

Weighted barycentres:

$$p = \frac{w_N p_N + w_E p_E + w_S p_S + w_W p_W}{w_N + w_E + w_S + w_W}$$



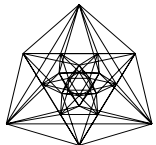
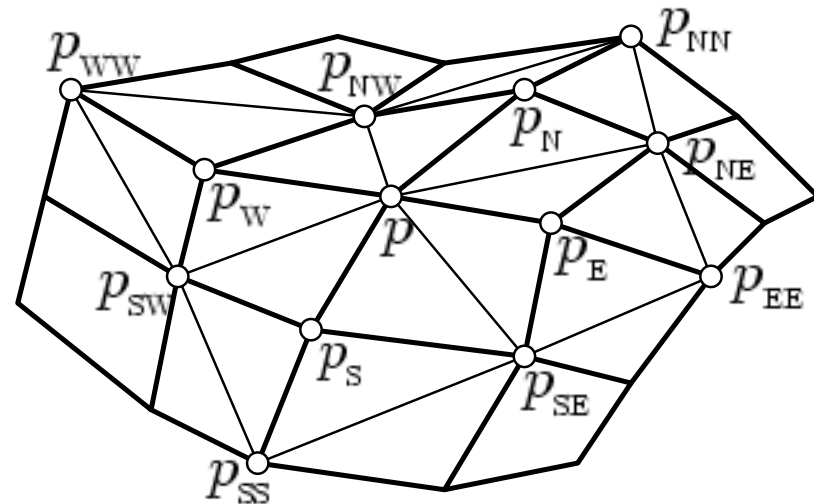
*uniform barycentre*



*weighted barycentre*

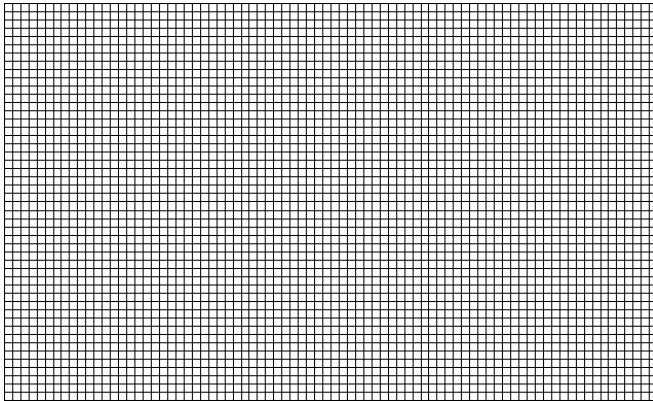
How to compute the weights?

$$w_E = \text{area } \triangle(p_E, p_{NE}, p_{EE}) + \text{area } \triangle(p_E, p_{EE}, p_{SE}) + \text{area } \triangle(p_E, p_{SE}, p) + \text{area } \triangle(p_E, p, p_{NE})$$

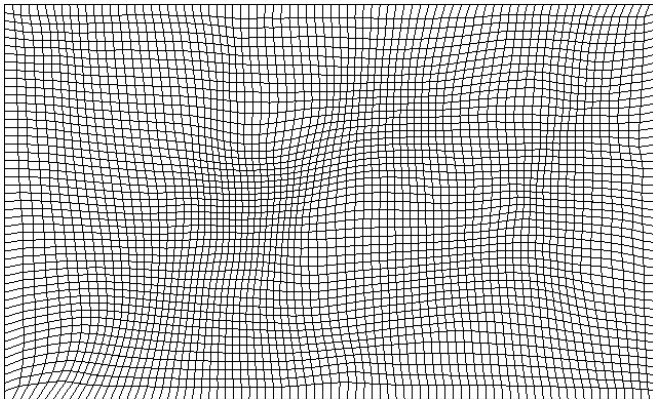
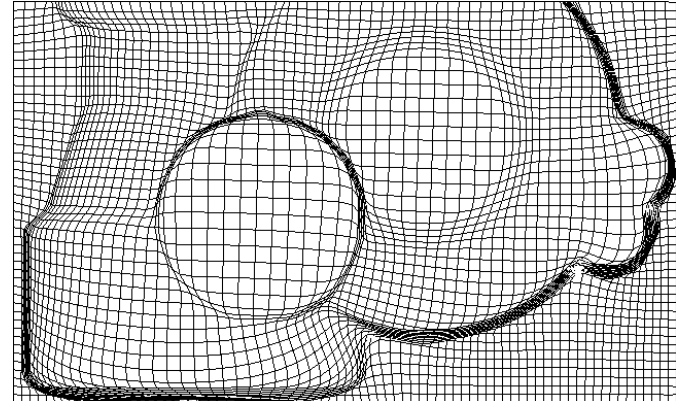


# Quadrilateral Remeshing of Triangle Meshes (3)

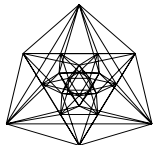
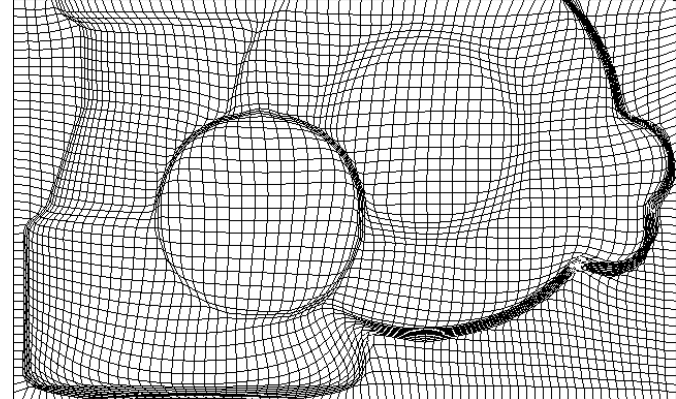
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using  
uniform  
barycentres



using  
weighted  
barycentres



# Interpolating the Quadrilateral Remesh

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Computing the interpolating TP-B-Spline  $t$

Vertices of the quadrilateral remesh:  $P_{00}, \dots, P_{mn}$

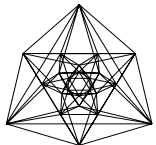
Uniform knot vectors:  $\mathbf{u} = \{0, 0, 0, 0, 1, 2, \dots, m-1, m, m, m, m\}$

$\mathbf{v} = \{0, 0, 0, 0, 1, 2, \dots, n-1, n, n, n, n\}$

Interpolation conditions:  $t(i, j) = P_{ij}$

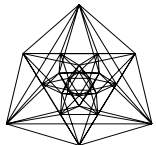
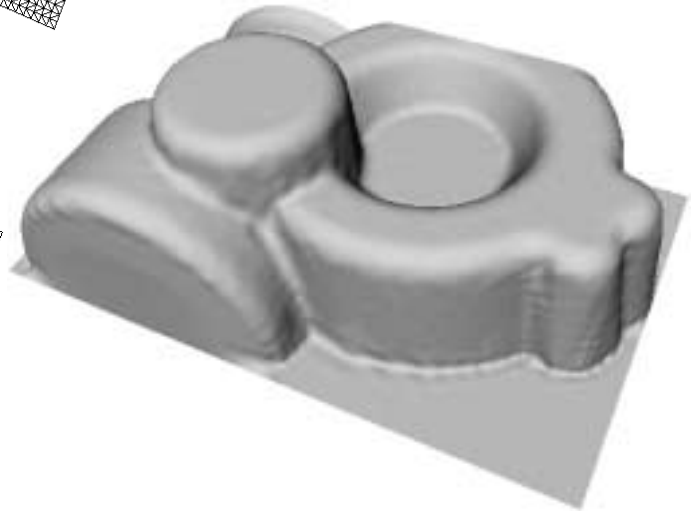
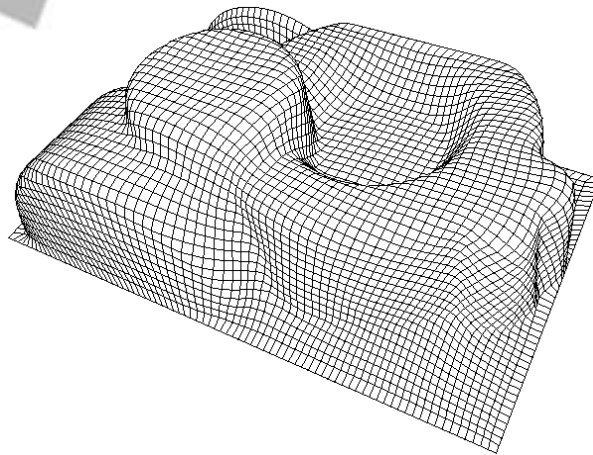
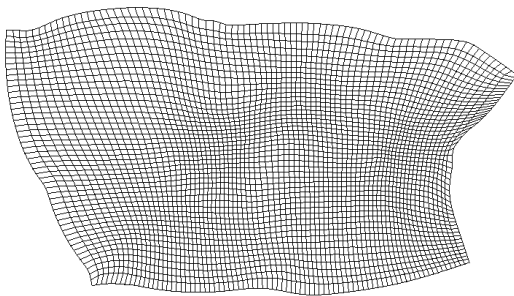
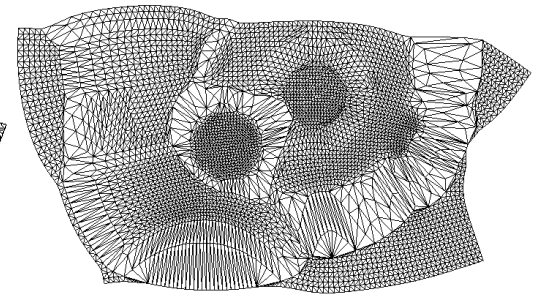
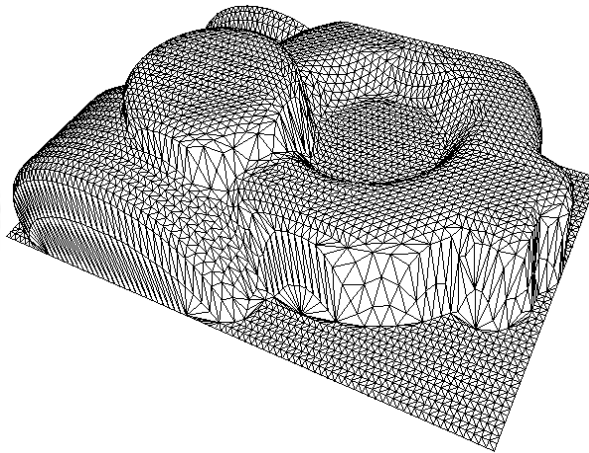
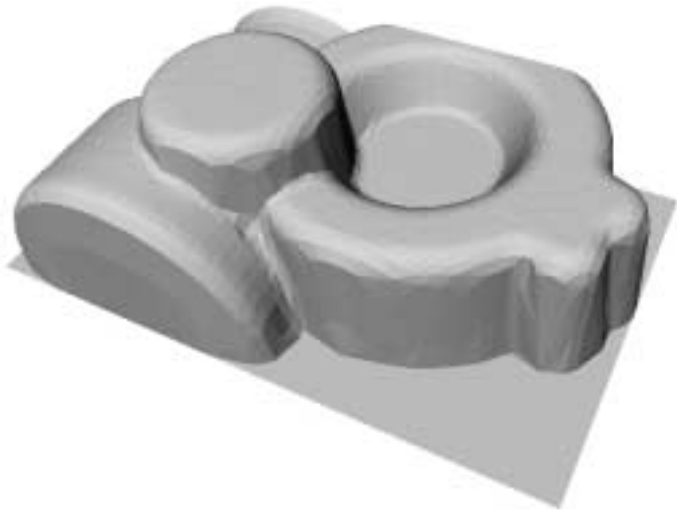
Natural end conditions:  $t'' = 0$  along the boundary

Requires solving  $(m+3) + (n+3)$  linear, tridiagonal systems



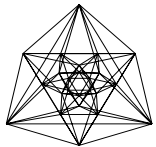
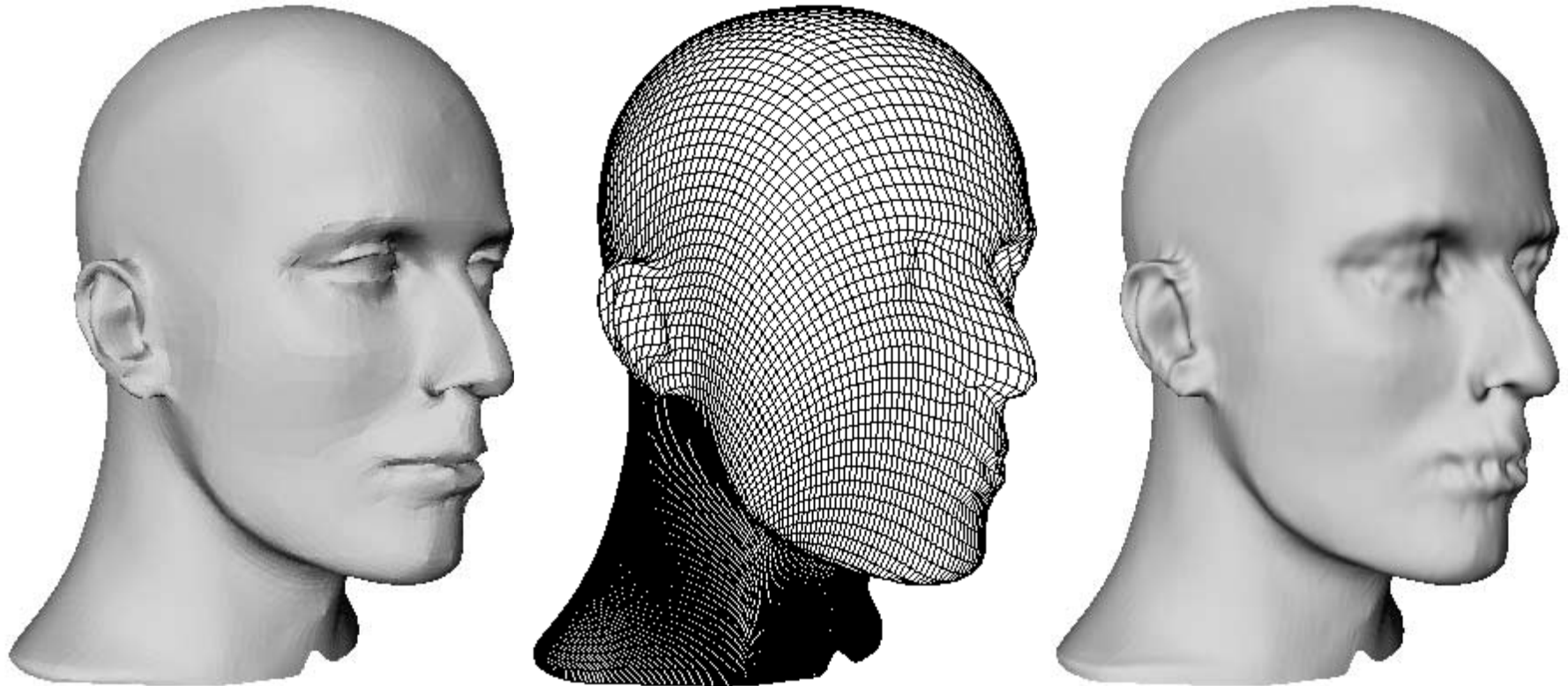
# Examples (1)

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## Examples (2)

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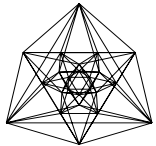
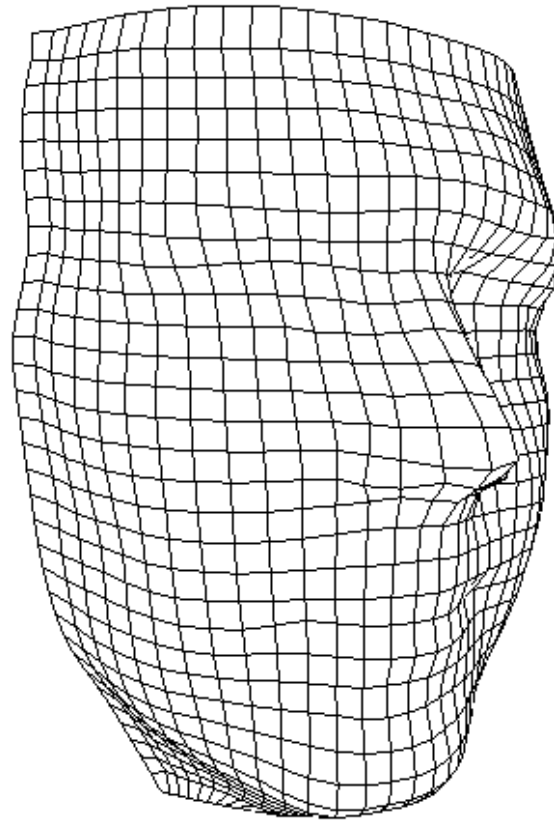
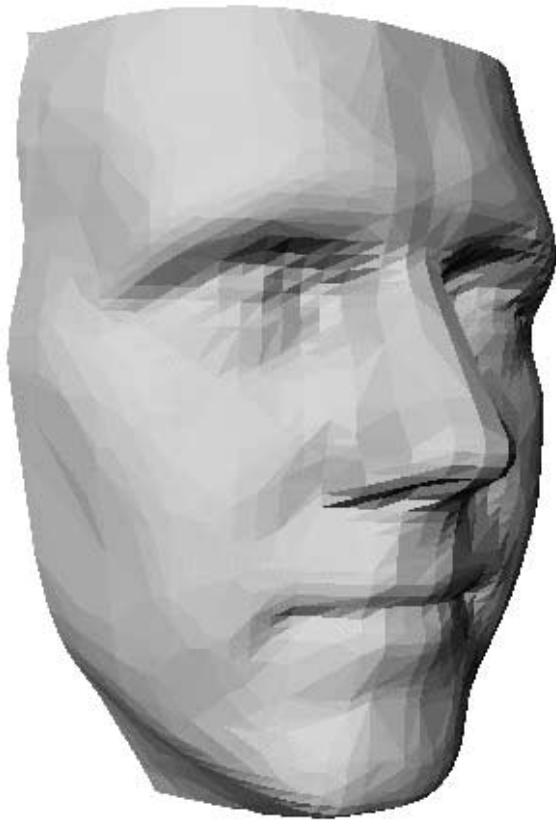


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## Examples (3)

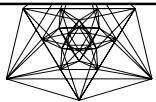
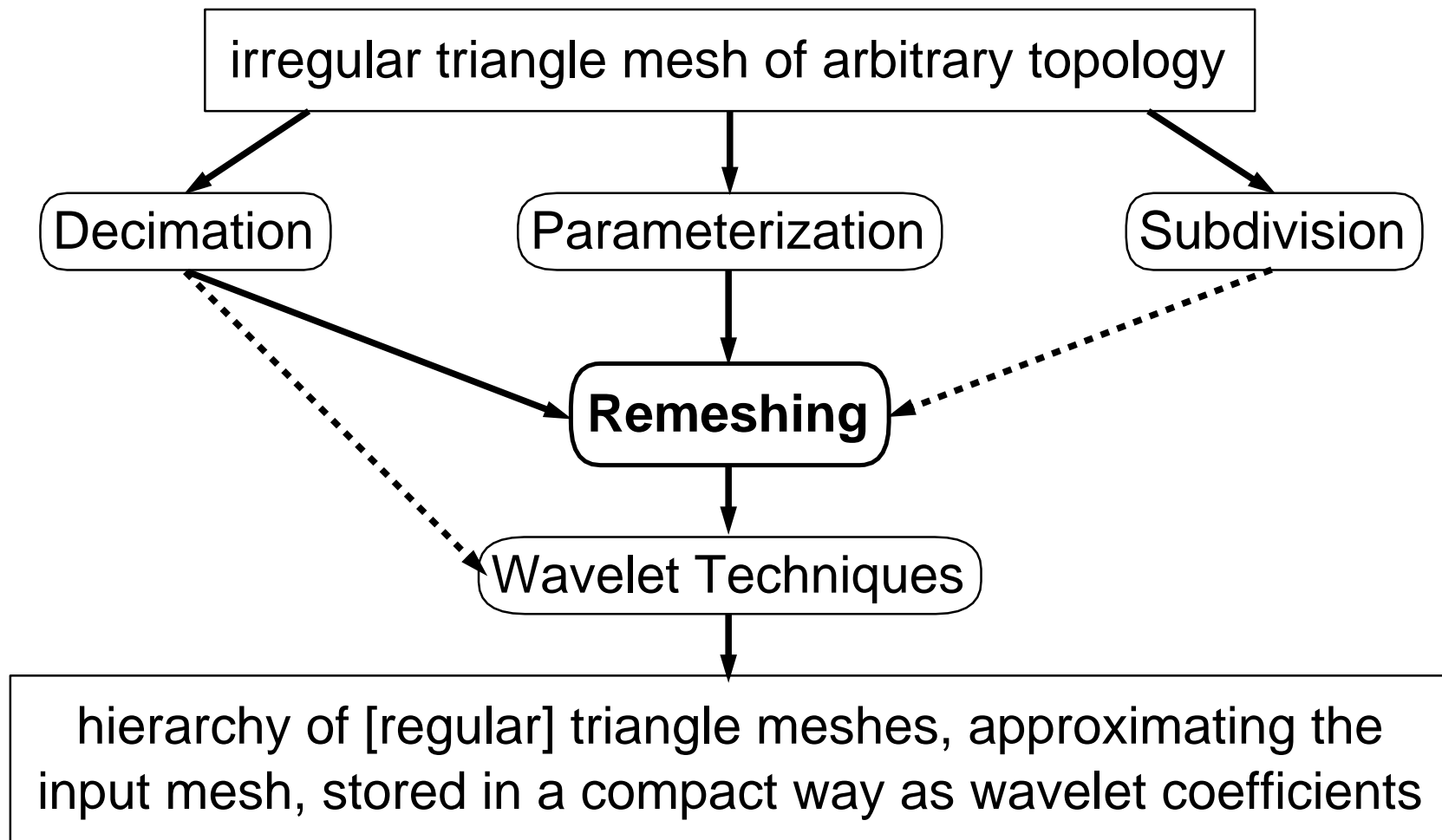
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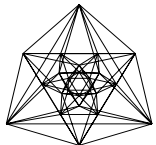
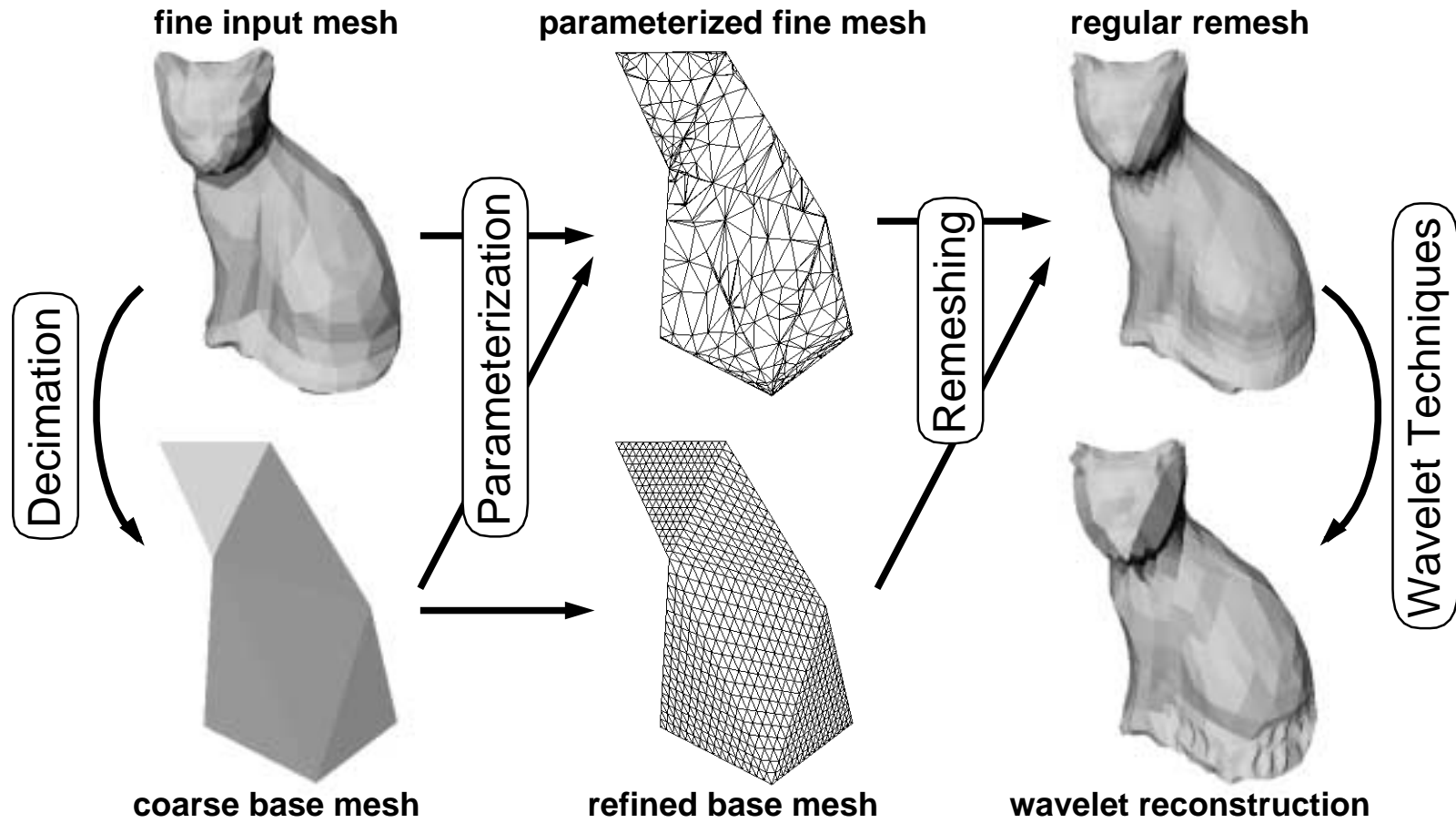
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# Application: regular remeshes and hierarchies



# Application: regular remeshes and hierarchies



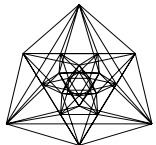
## Application: regular remeshes and hierarchies

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- we know very well how to parameterize disk-like objects
- but how to parameterize objects with arbitrary topology?

dissect the object into disk-like patches  
and parameterize each patch separately

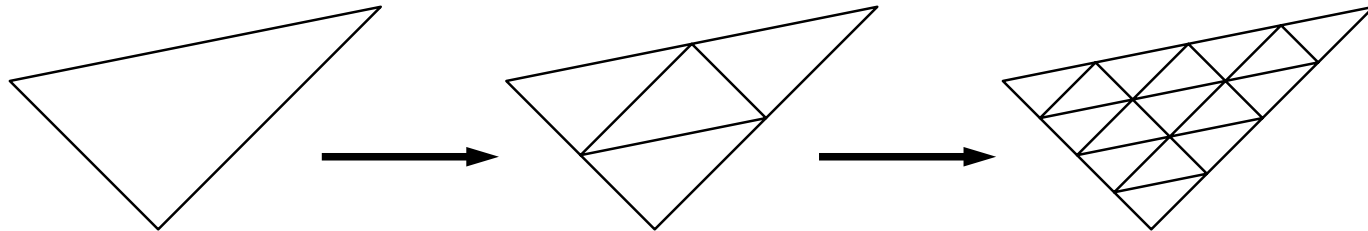
- for each edge in the coarse mesh, find a corresponding path in the fine mesh
- for each triangle in the coarse mesh, parameterize the corresponding region of the fine mesh



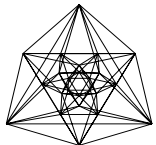
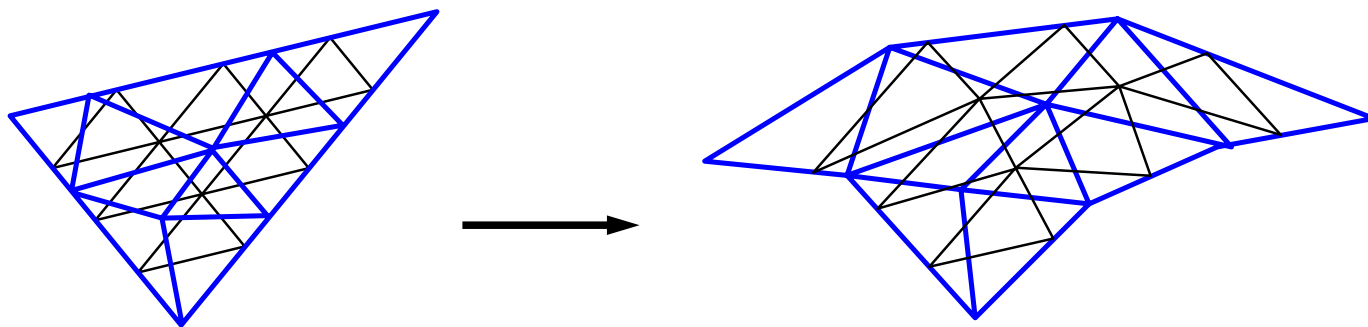
## Application: regular remeshes and hierarchies

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- refine the base mesh by successive 1-to-4-splits



- use the parameterization to lift the refined base mesh onto the surface of the input mesh



# Regular remeshes and hierarchies: Example

