6. Level Sets and Applications in Reverse Engineering

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6.1. Introduction

Reverse Engineering: reconstruct a computer representation of an existing object based on measured data.
Data: mostly 3D laser scanner data; point clouds with high density

Create CAD model including the recognition of simple surfaces: planes, cylinders and cones of revolution, spheres, tori, general surfaces of revolution, general cylinders, pipe surfaces (blends), ...

Important for a precise CAD-representation and manufacturing.
Here we do not deal with *free-form surfaces*.

6.2. Plane of regression

Fit plane to data points $\mathbf{p}_i,\ i = 1, \ldots, N$.

Plane equation in Hesse form:

$$\mathbf{u} \cdot \mathbf{x} + u_0 = 0, \quad \|\mathbf{u}\| = 1.$$

Distance (signed) of point $\mathbf{p}$ to plane is $d(\mathbf{p}) = \mathbf{u} \cdot \mathbf{p} + u_0$.

Plane of regression minimizes

$$\sum_{i=1}^{N} d^2(\mathbf{p}_i).$$

Equivalent to minimization of the quadratic form (with $\mathsf{u} := (u_0, \mathbf{u})$)

$$\sum_{i=1}^{N} (\mathbf{u} \cdot \mathbf{p}_i + u_0)^2 =: \mathsf{u}^T \cdot Q \cdot \mathsf{u},$$

under the quadratic constraint

$$\|\mathbf{u}\| = \mathsf{u}^T \cdot \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \cdot \mathsf{u} = 1.$$

leads to an eigenvalue problem

Remark: Minimization of a quadratic form

$$F(\mathsf{x}) = \mathsf{x}^T \cdot Q \cdot \mathsf{x}$$

under a quadratic constraint

$$\mathsf{x}^T \cdot N \cdot \mathsf{x} - 1 = 0. \qquad (N)$$

Lagrange function

$$\mathsf{x}^T \cdot Q \cdot \mathsf{x} - \lambda(\mathsf{x}^T \cdot N \cdot \mathsf{x} - 1).$$

First derivative yields Lagrange equations:

$$\begin{aligned} (Q - \lambda N) \cdot \mathsf{x} &= 0, \qquad (*) \\ \mathsf{x}^T \cdot N \cdot \mathsf{x} - 1 &= 0. \qquad (N) \end{aligned}$$

Linear homogeneous system (*) has solution $\neq 0 \iff$

$$\det(Q - \lambda N) = 0.$$

Zero λ_i of this characteristic equation (gen. eigenvalue) yields solution x_i of (*) (eigenvector). Let x_i be normalized with (N), then

$$F(\mathsf{x}_i) = \mathsf{x}_i^T \cdot Q \cdot \mathsf{x}_i = \lambda_i \mathsf{x}_i^T \cdot N \cdot \mathsf{x}_i = \lambda_i.$$

Hence: solution s is eigenvector to smallest eigenvalue λ_s.

Application of plane fitting

1. *Normal estimation at data points*.
In first segmentation step eliminate points in highly curved regions (no well fitting plane) [Varady, Benkö, Kos]. These are likely to belong to edges or blending surfaces.

2. Gaussian image of planar face is one point. Errors in data and estimated normals: recognize planar faces by small point cluster on Gaussian sphere.

Dimensionality filtering [Varady, Benkö, Kos]

Take concentric balls (radii r_1, r_2) around the point to be classified and count the number of points (n_1, n_2) in the balls. Dimensionality D can be estimated as

$$D = \frac{\log \frac{n_2}{n_1}}{\log \frac{r_2}{r_1}}.$$

For $r_1 : r_2 = 1 : 2$

point cluster: $n_2 \sim n_1 \implies D \sim 0$.
curve-like: $n_2 \sim 2n_1 \implies D \sim 1$.
area-like: n_i proportional to areas of balls $\implies$ $D \sim 2$.

Point clusters appear as Gaussian images of planar faces; curve-like regions characterize developable surfaces, in particular cylinders and cones.

6.3. Cylinder surfaces

Cylinder surface: formed by one-parameter family of parallel lines (generators, rulings). Gaussian image is (part of) a great circle.

Recognition:
A. Estimate normals $\mathbf{n}_i$ at data points $\mathbf{d}_i$.
B. Fit great circle c to Gaussian image by fitting plane through 0 to its points $\mathbf{n}_i$ (eigenvalue problem). Cylinder surface if Gaussian image close to great circle c.

Reconstruction:
Rulings of the surface are parallel to the normal $\mathbf{n}$ of the plane of c. Orthogonal projection parallel to $\mathbf{n}$ yields planar point cloud which is approximated by a curve $\mathbf{p}(u)$ [Lee], [Randrup]. Approximating surface is $\mathbf{x}(u,v) = \mathbf{p}(u) + v\mathbf{n}$. (needs trimming!)

Right circular cylinder: fit circle $\mathbf{p}$ to projected data.

Alternatively, there exists a nonlinear least squares approach in [Lukacs, Marshall, Martin]. This method is also appropriate for *circular cone* and *torus* reconstruction.

We can recognize a right circular cone by its Gaussian image (small circle; does characterize a developable surface of constant slope, but most likely belongs to a cone of revolution).

Both a cone of revolution and a torus may be fitted using the kinematic method for approximation with *surfaces of revolution* [Pottmann, Randrup].

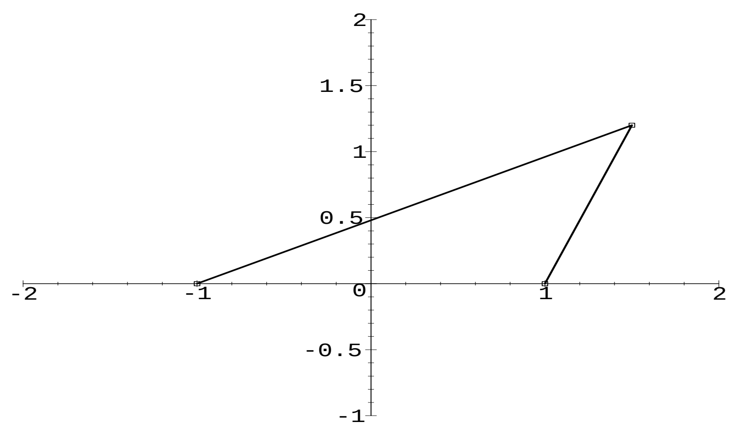
6.4. Level Sets – Fundamentals

Given: function $f(x_1, \dots, x_n)$

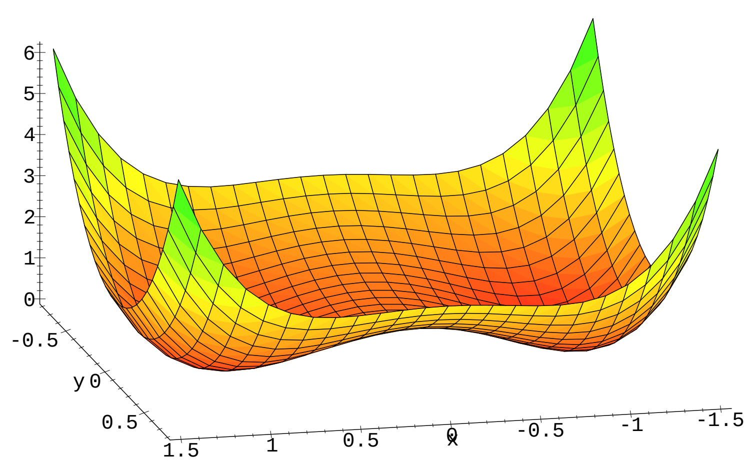
Level Sets:

$$f(x_1, \dots, x_n) = c, \qquad c = \text{const}$$

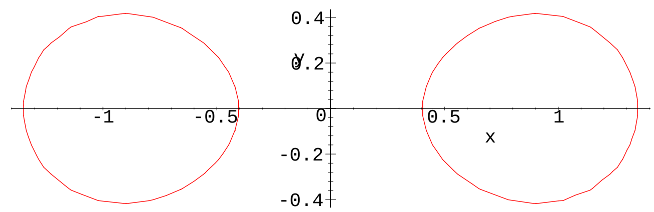
Example 1: $\quad f(x,y) = \left((x-1)^2 + y^2\right)\left((x+1)^2 + y^2\right)$



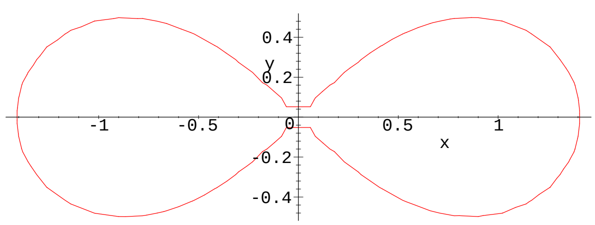
Graph of function f:

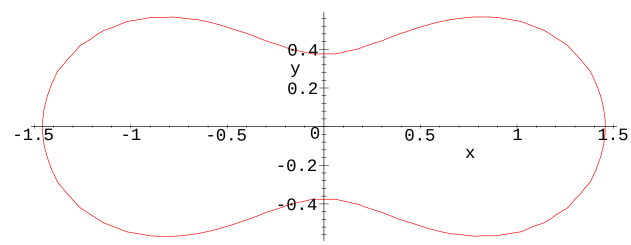


Level sets $f(x,y) = c$ are special algebraic curves (Cassinian curves) which are real only for $c \geq 0$.



$c = 0$: 2 points $\qquad$ $0 < c < 1$: 2 convex curves





$c = 1$: Lemniscate of B. $\qquad$ $c > 1$: 1 convex curve

Thus: level sets are an easy tool to describe families of curves with different *topological structures*.

Tangent in a point of $f(x,y) - c = 0$ possses the normal vector $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$.

Example 2:

$$f(x,y,z) = \left((x-1)^2 + y^2 + z^2\right)\left((x+1)^2 + y^2 + z^2\right).$$

Here the level sets are surfaces of revolution. They are just the curves from example 1 rotated about the x-axis.

Again: *change in the topology of the surfaces!*

Tangent plane in a point of $f(x,y,z) - c = 0$ has normal vector

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right).$$

Algebraic Curves and Surfaces

If f is a *polynomial function* its level sets are called algebraic curves or surfaces, resp.

Without loss of generality: $c = 0$

Planar algebraic curves:

$$f(x,y) = \sum_{i+j \le n} a_{ij} x^i y^j = 0$$

Geometric meaning of *degree* n: maximal number of intersection points with a test line.

Examples:

degree 1: straight line,
degree 2: conic (in general),
degree 4: e.g.: curves of example 1.

Algebraic surfaces:

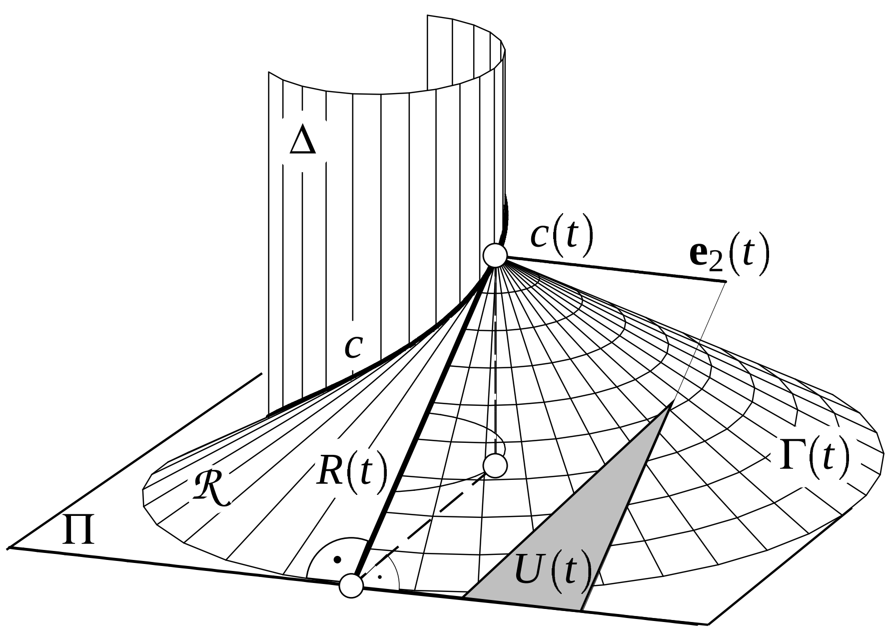
$$f(x,y,z) = \sum_{i+j+k \le n} a_{ijk} x^i y^j z^k = 0$$

Examples:

degree 1: plane,
degree 2: quadrics, quadratic cones (in general).

For applications the **distance function to a curve/surface** and its level sets are of interest.

The graph of the oriented distance function to a curve l carries a family of rulings. These stem from the curve normals.

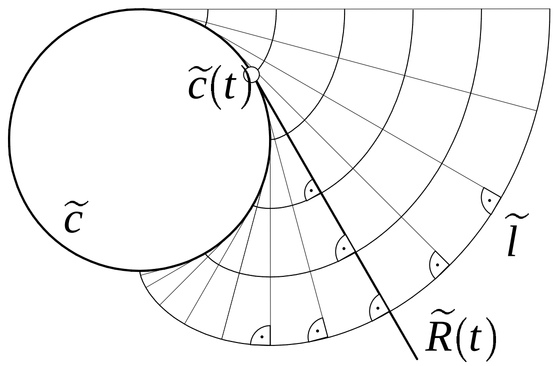


The rulings of this graph surface have an inclination of 1. They are the tangents of a twisted space curve (of constant slope).

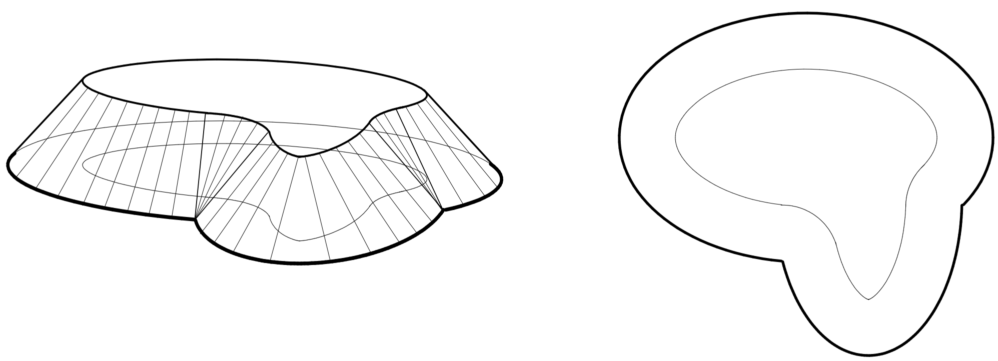
Thus:

$$\|\nabla f\| = 1$$

Top projection of the above figure:

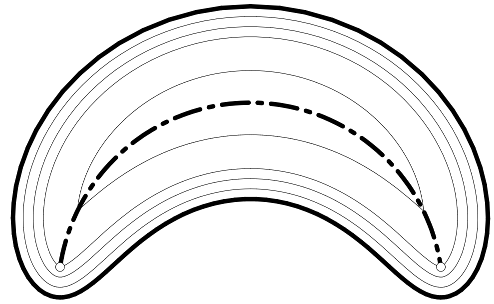


Level sets of the graph surface are *offsets* of the original curve.

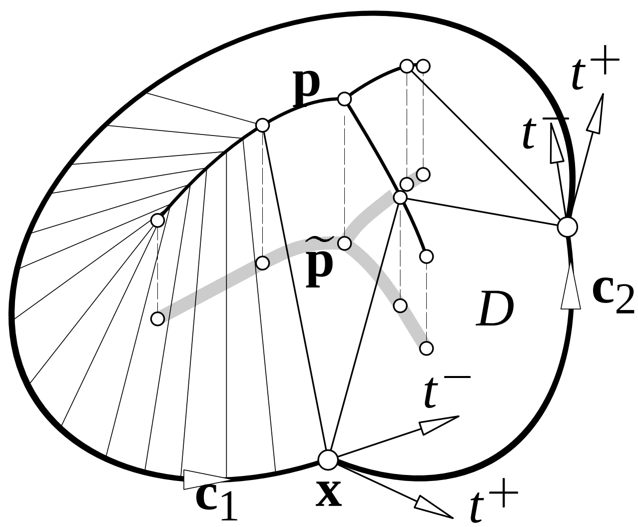


The *medial axis* of a planar region is defined as the loci of centers of maximum circles. The medial axis is related to the self-intersections of the graph surface.

Example:



Example:



Problem: The distance function of an algebraic curve is *not* polynomial (except the curve is a straight line). The *algebraic distance* of a point $P(x_P, y_P)$ to the curve $f(x,y) = 0$ is defined by $f(x_P, y_P)$. This algebraic distance is different from the Euclidean one.

The above considerations for distance functions of *curves* can be extended to *surfaces* straightforwardly.

6.5. Fitting Curves and Surfaces with Level Sets

Given: points $P_i = (x_i, y_i), \qquad i = 1, \ldots, N$

Goal: find a curve $f(x,y) = 0$ from a chosen family of curves that approximates $\{P_i\}$ best.

$$\sum d_i^2 \longrightarrow \min$$

Example: Best-fitting circles and spheres

A *circle* in the plane is given by

$$f(x,y) := a(x^2+y^2)+bx+cy+d=0.$$

The graph surface of the 'algebraic distance function' f is a paraboloid of revolution,

$$z(x,y) = a(x^2+y^2)+bx+cy+d.$$

For points (x,y) close to the circle, the algebraic distance is close to the geometric distance if the paraboloid intersects the plane $z=0$ at an angle of $\pi/4$. [Pratt]

$\iff \|\nabla f\| = 1$ at $f=0$.

$$
\begin{aligned}
(\nabla f)^2 &= (2ax+b)^2+(2ay+c)^2 \\
&= 4a[a(x^2+y^2)+bx+cy]+b^2+c^2.
\end{aligned}
$$

Along the curve $f=0$, we have $a(x^2+y^2)+bx+cy=-d$ and thus $(\nabla f)^2=1$ is equivalent to

$$
b^2+c^2-4ad=1. \quad (N)
$$

Circle fitting to points $P_i=(x_i,y_i)$ amounts to minimization of

$$
\sum_{i=1}^{N} f^2(x_i,y_i)=\mathsf{x}^T\cdot Q\cdot\mathsf{x}, \text{ with } \mathsf{x}=(a,b,c,d),
$$

under the constraint (N) (eigenvalue problem).

Example: Sphere fitting

A *sphere* is written as

$$a(x^2+y^2+z^2)+bx+cy+dz+e=0.$$

Here, the normalization reads

$$b^2+c^2+d^2-4ae=1.$$

More generally: Method of Taubin

$$\sum_i \frac{f^2(\mathsf{x}_i)}{\|\nabla f(\mathsf{x}_i)\|^2} \Longrightarrow \min.$$

Find solution e.g. by weighted iteration.

Further examples:

Approximation with

∗) cylinders of revolution,
∗) cones of revolution,
∗) tori

have been studied by Varady et al.

Level Set Method of Osher and Sethian

Active contour $\Gamma(t)$ (curve, surface) as a level set of a time-dependent function

$$f(\mathbf{x},t) = 0.$$

Differentiation of $f(\mathbf{x}(t),t) = 0$ leads to

$$f_t + f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt} = 0.$$

$$\Longleftrightarrow f_t + \nabla f \cdot \mathbf{v} = 0, \qquad \mathbf{v} := \dot{\mathbf{x}}(t).$$

With $v_n := \mathbf{v} \cdot \mathbf{n} = \mathbf{v} \cdot \frac{\nabla f}{\|\nabla f\|}$ we obtain

Transport Equation:

$$f_t + v_n \|\nabla f\| = 0.$$

Hamilton-Jacobi equation (1. order PDE).

Numerical solution with *finite differences*

Example 1:

Scene $\mathbb{R}^2$; $K \ldots$ curvature of $f = 0$.

$$K = \nabla \cdot \frac{\nabla f}{\|\nabla f\|} = \mathrm{div}\left(\frac{\nabla f}{\|\nabla f\|}\right).$$

Transport equation $f_t + F(K)\|\nabla f\| = 0 \quad (*)$

Note: for the solution of $(*)$ the function $F(K)$ must be extended to a neighborhood of the moving contour.

$F(K) = 1 \ldots$ family of offsets

$F(K) = -K\ldots$ "curvature flow":

shrinks a simply closed curve to a "circular" point (i.e. a circle when performing an appropriate simultaneous scaling) [Grayson, 1987].

Scene $\mathbb{R}^3$; $F(K) = -K$, where K is the mean curvature:

shrinks a convex surface to a sphere (with simultaneous scaling).

Active Contours (2D and 3D) in Image Processing