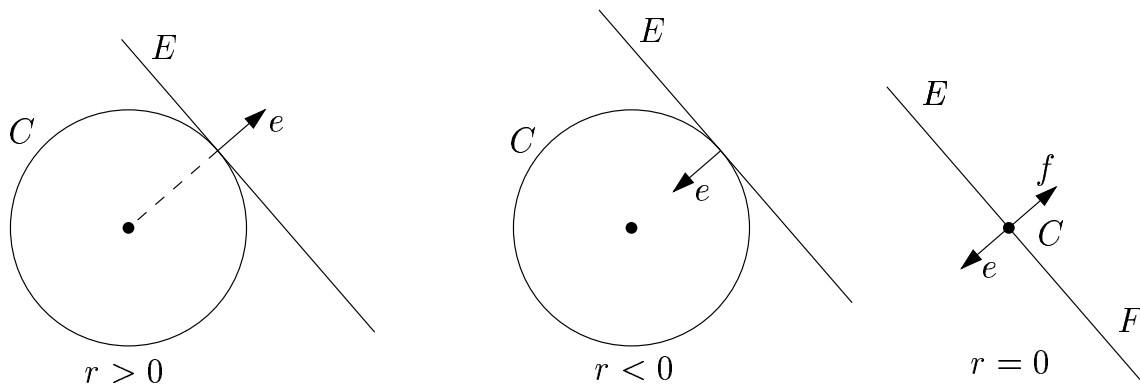


PLANAR EUCLIDEAN LAGUERRE GEOMETRY

A circle C is given by its equation $(\mathbf{x} - \mathbf{m})^2 - r^2 = 0$ in Euclidean plane. Here, \mathbf{m} and r denote its center and radius. A circle C can be oriented by associating a sign to r , in the way that $r > 0$ represents circles with normals pointing outwards, and negative r stands for the opposite. We will study the set of or. circles \mathcal{C} including points as degenerated circles with $r = 0$.

An oriented line E is given by its equation $e_0 + e_1x_1 + e_2x_2 = 0$, where $\mathbf{e} = (e_1, e_2)$ is its unit normal vector ($\|\mathbf{e}\|=1$). \mathcal{E} shall denote the set of or. lines.



An oriented line E is said to be in oriented contact to the or. circle C , if

$$e_0 + e_1 m_1 + e_2 m_2 + r = e_0 + \mathbf{e} \cdot \mathbf{m} + r = 0. \quad (1)$$

A Laguerre transformation (L-trafo) consists of two maps,

$$\alpha_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}, \alpha_{\mathcal{E}} : \mathcal{E} \rightarrow \mathcal{E} \quad (2)$$

which are bijective and preserve or. contact of circles and lines. Motions or similarities are 'point preserving' L-trafos. An example for a not point preserving L-trafo is the following. Let λ_d be a mapping, which adds a signed constant d to the radii of the circles, but leaves their centers unchanged,

$$C : (m_1, m_2, r) \mapsto \lambda_d(C) : (m_1, m_2, r + d). \quad (3)$$

OFFSET GENERATION

Let $X : \mathbf{x}(t) = (x_1, x_2)(t)$ be a smooth planar (non oriented) curve. Interpreting its points as zero-radius circles, the offset curve X_d at distance d is the envelope of circles with centers at $\mathbf{x}(t)$ and radii d . Thus, λ_d maps a curve X to its offset X_d at distance d .

Let $X : \mathbf{x}(t)$ be an or. curve. X can also be generated as envelope of its or. tangent lines

$$\mathbf{y}(t) = (y_0, y_1, y_2)(t) = (x_1\dot{x}_2 - x_2\dot{x}_1, -\dot{x}_2, \dot{x}_1).$$

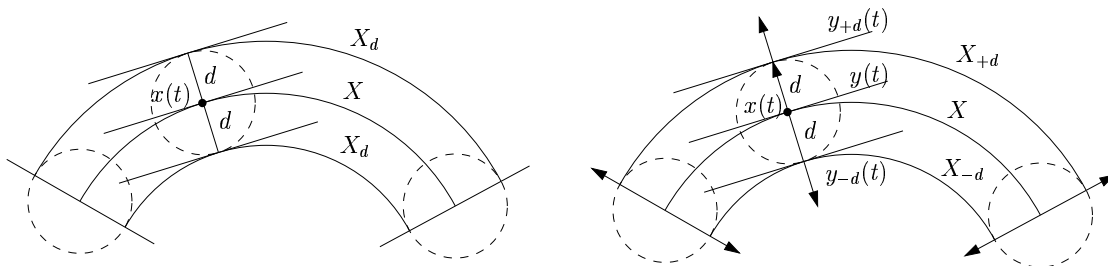
The mapping λ_d operates on or. circles and induces a mapping λ_d^* which acts on \mathcal{E} . A line $G = (g_0, g_1, g_2)$ with unit normal (g_1, g_2) is mapped to

$$\lambda_d^*(G) = (g_0 + d, g_1, g_2).$$

In case of offsets this means that the oriented curve X with tangent lines $\mathbf{y}(t)$ is mapped to the or. onesided offset X_{+d} at signed distance d , whose tangent lines are

$$\mathbf{y}_d(t) = (y_0 + d\sqrt{\dot{x}_1^2 + \dot{x}_2^2}, y_1, y_2).$$

Exchanging $d \mapsto -d$ generates the or. offset at distance $-d$.



one-sided and two-sided offset curve

CYCLOGRAPHIC MODEL

Embedding E^2 into affine 3-space A^3 as plane $x_3 = 0$, an or. circle with center \mathbf{m} and radius r is mapped onto the point

$$\zeta(C) = (m_1, m_2, r). \quad (4)$$

The or. lines E shall be mapped to planes

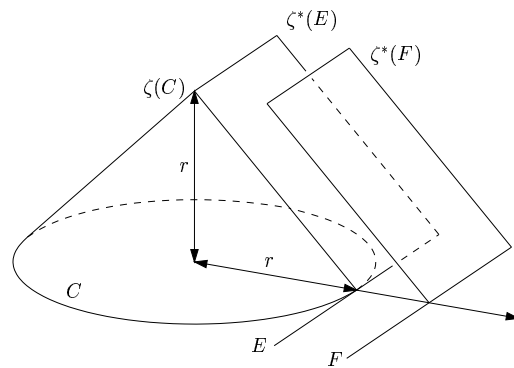
$$\zeta^*(E) : e_0 + e_1x_1 + e_2x_2 + x_3 = 0. \quad (5)$$

Because of $e_1^2 + e_2^2 = 1$, all these image planes $\zeta^*(E)$ form an angle of $\pi/4$ with E^2 .

Parallel or. lines possess parallel image planes in E^3 . All lines E , tangent to a fixed or. circle C are mapped to planes $\zeta^*(E)$ passing through $\zeta(C)$. These planes are tangent to a cone of revolution $\gamma(C)$, with vertex $\zeta(C)$ and axis \parallel to the x_3 -axis.

A plane ε is called γ -plane, if its normal vector satisfies $e_1^2 + e_2^2 - e_3^2 = 0$. The angle formed by E^2 and ε equals $\pi/4$. Similarly, a line G forming an angle of $\pi/4$ with E^2 is called γ -line. Each γ -plane contains a pencil of γ -lines. A cone of revolution, all whose tangent planes

are γ -planes, is called γ -cone. The ζ -images of or. lines E are γ -planes.



CYCLOGRAPHIC MAPPING

The mapping $c = \zeta^{-1}$ is called *cyclographic mapping*. Points in A^3 are mapped to or. circles in E^2 . A real smooth curve $\mathbf{p}(t)$ in A^3 is mapped onto a one parameter family of or. circles, whose envelope $c(\mathbf{p})$ is the image curve of \mathbf{p} . The envelope is not necessarily real, as the following examples show.

Cyclographic Image of a Line

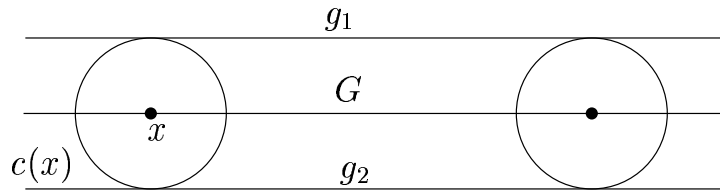
Example 1: The line $G \parallel E^2$ is given by $\mathbf{p}(t) = (t, 0, r)$, with constant r . The circles and its derivatives are

$$C(t) : (x_1 - t)^2 + x_2^2 - r^2 = 0,$$

$$\dot{C}(t) : x_1 - t = 0.$$

The envelope is the pair of lines

$$c(G) = g_1 \cup g_2 : (x_2 - r)(x_2 + r) = 0.$$



Example 2: Assume that G is given by $p(t) = t(0, g_2, g_3)$. The one parameter family of circles and its derivatives are

$$C(t) : x_1^2 + (x_2 - g_2 t)^2 - (g_3 t)^2 = 0,$$

$$\dot{C}(t) : (x_2 - g_2 t)g_2 + g_3^2 t = 0.$$

Eliminating t from above equations gives

$$c(G) : (x_1 \sqrt{d} + x_2 g_3)(x_1 \sqrt{d} - x_2 g_3) = 0,$$

with $d = g_2^2 - g_3^2$.

$d > 0$: $c(G) =$ pair of (real) lines

$d < 0$: $c(G) =$ conj. complex pair of lines

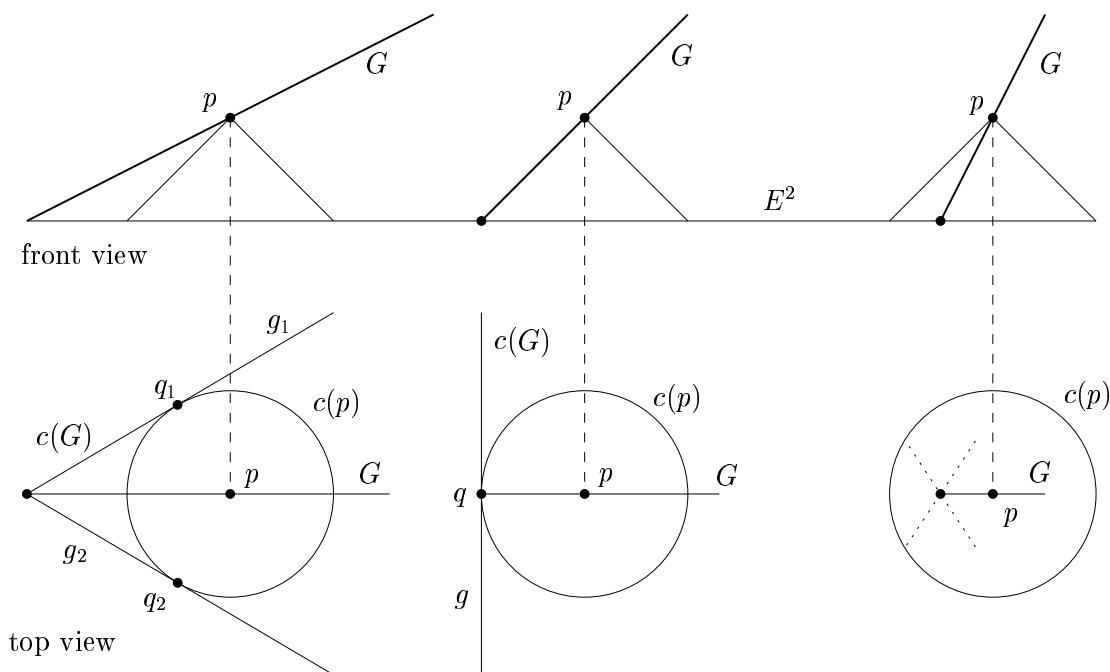
$d = 0$: $c(G) =$ double counted line

Geometrically formulated this says that $c(G)$ is a real pair of lines, a double counted line or a pair of conj. complex lines depending on

whether the inclination angle of G and E^2 is \leq or $> \pi/4$.

Cyclographic Image of a Line Element

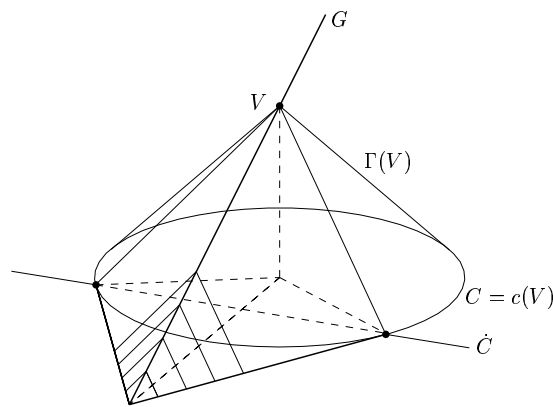
A line G with incident point p is called *line element*. Applying the cyclographic mapping to a line element produces a pair of line elements g_1, q_1, g_2, q_2 , one line element g, q or a single point together with an incident conj. complex pair of lines (represented by dotted lines in the figure) depending on the inclination angle of G against E^2 .



Cyclographic image of a line element

Cyclographic Image of Curves

Let $\mathbf{p}(t)$ be a smooth curve in A^3 . Its cyclographic image curve $c(\mathbf{p})$ can be generated by applying the cyclographic mapping to all lines elements of $\mathbf{p}(t)$, i.e. incident curve points and their tangent lines. If the angle ϕ between the tangent $\mathbf{p} + s\dot{\mathbf{p}}$ and the x_3 -axis is larger than $\pi/4$, $c(\mathbf{p})$ consists locally of two envelopes. If $\phi = \pi/4$ it consists locally of just one branch and otherwise there is no real envelope.



γ -planes through a line, pencils of γ -lines

Using the notion of γ -planes one can formulate the computation of the cyclographic image as follows. Let $\tau(t)$ be the family of all γ -planes passing through tangent lines of $\mathbf{p}(t)$. The envelope $T(\mathbf{p})$ of the family $\tau(t)$ is a developable surface, called γ -developable through $\mathbf{p}(t)$.

The cyclographic image of a curve \mathbf{p} is the intersection curve of the γ -developable $T(\mathbf{p})$ with E^2 .

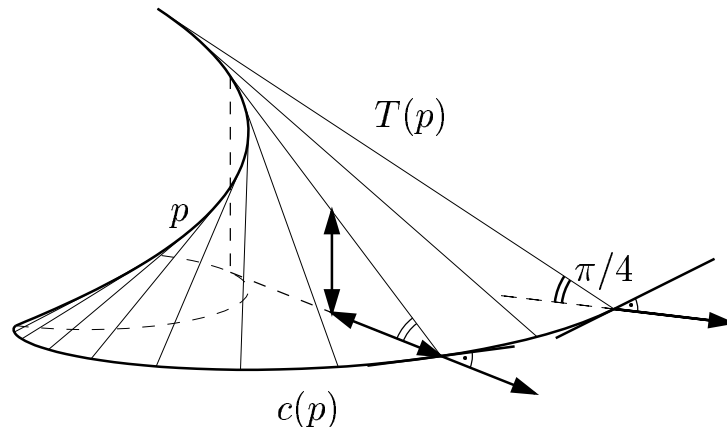
The γ -developable consists of two components for curves, whose tangent lines form angles $< \pi/4$ with E^2 . It is empty for curves, whose tangent lines form angles $> \pi/4$ with E^2 . It consists of one component for curves, whose tangent lines are γ -lines. Later shall be called γ -curves. If a γ -developable is different from a plane or cone, its curve of regression is a γ -curve.

Example 3: Let \mathbf{p} be the helix given by

$$\mathbf{p}(t) = (\cos(t), \sin(t), t).$$

Its derivative vectors $\dot{\mathbf{p}}$ possess constant slope. The γ -developable can be written as $T(\mathbf{p}) : \mathbf{p}(t) + u\dot{\mathbf{p}}(t)$. By setting $u = -t$ we obtain the image curve $c(\mathbf{p}) = T(\mathbf{p}) \cap E^2$,

$$c(\mathbf{p}) : (\cos t + t \sin t, \sin t - t \cos t).$$

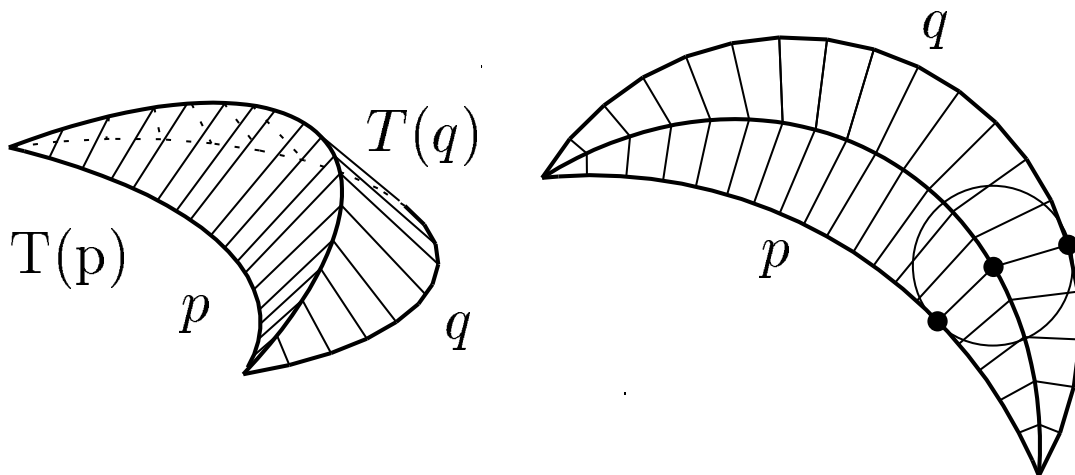


γ -developable as graph of the signed distance function

The γ developable $T(\mathbf{p})$ is the cyclographic preimage of all or. circles being tangent to the planar curve $c(\mathbf{p})$. Thus, it is the graph of the signed distance function to $c(\mathbf{p})$.

BISECTOR and MEDIAL AXIS

The construction of bisectors and medial axes is closely related to Euclidean Laguerre geometry and cyclographic mapping. Usually, the (untrimmed) bisector $B(\mathbf{p}, \mathbf{q})$ of two planar curves \mathbf{p}, \mathbf{q} is defined as locus of points, having same orthogonal distance to both curves.



axonometric view and top view of the bisector construction via
intersection of γ -developables

$T(\mathbf{p})$ as well as $T(\mathbf{q})$ are graphs of the signed distance function to the planar curves \mathbf{p} and \mathbf{q} , respectively. Thus, the untrimmed bisector $B(\mathbf{p}, \mathbf{q})$ is the projection of the intersection curve $T(\mathbf{p}) \cap T(\mathbf{q})$ into E^2 .

LAGUERRE GEOMETRY in the CYCLOGRAPHIC MODEL

Introducing a scalar product \langle, \rangle in affine space A^3 leads to a Euclidean space E^3 . The cyclographic mapping does not introduce a Euclidean geometry in A^3 , but a pseudo-Euclidean (pe). This is motivated by the fact that we have to distinguish between lines, whose angle against E^2 is smaller, equal or larger than $\pi/4$.

The natural, predefined bilinear form in A^3 is

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3, \quad (6)$$

and defines orthogonality and a distance function via $\|\mathbf{p} - \mathbf{q}\| = (\langle \mathbf{p} - \mathbf{q}, \mathbf{p} - \mathbf{q} \rangle)^{1/2}$ on lines with slope $< \pi/4$.

The main theorem of Laguerre geometry says that planar L-trafos are induced by linear mappings in A^3 which leave pe-distances invariant. Analytically, any L-trafo possesses a representation in A^3 of the form

$$\mathbf{x}' = \mathbf{c} + A \cdot \mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^3,$$

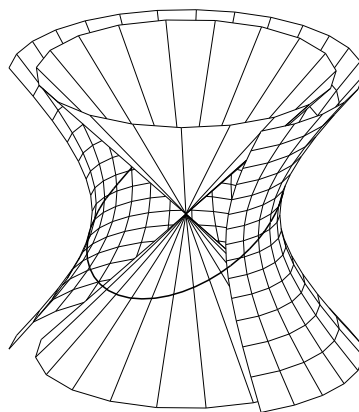
with a 3×3 matrix A , which satisfies

$$A^T \cdot I_{pe} \cdot A = I_{pe}, \text{ with } I_{pe} = \text{diag}(1, 1, -1).$$

A pe -sphere is a surface whose points possess constant pe -distance from a given center \mathbf{m} . Analytically this reads as

$$\langle \mathbf{x} - \mathbf{m}, \mathbf{x} - \mathbf{m} \rangle - r^2 = 0. \quad (7)$$

With $\mathbf{m} = (0, 0, 0)$ we get $x_1^2 + x_2^2 - x_3^2 - r^2 = 0$, which describes one-sheet hyperboloids. Their axis is x_3 and the angle formed by the asymptotic cone and E^2 equals $\pi/4$. (If the square of the constant distance r^2 is negative, we obtain also two-sheet hyperboloids.) Vanishing distance $r = 0$ characterizes γ -cones.

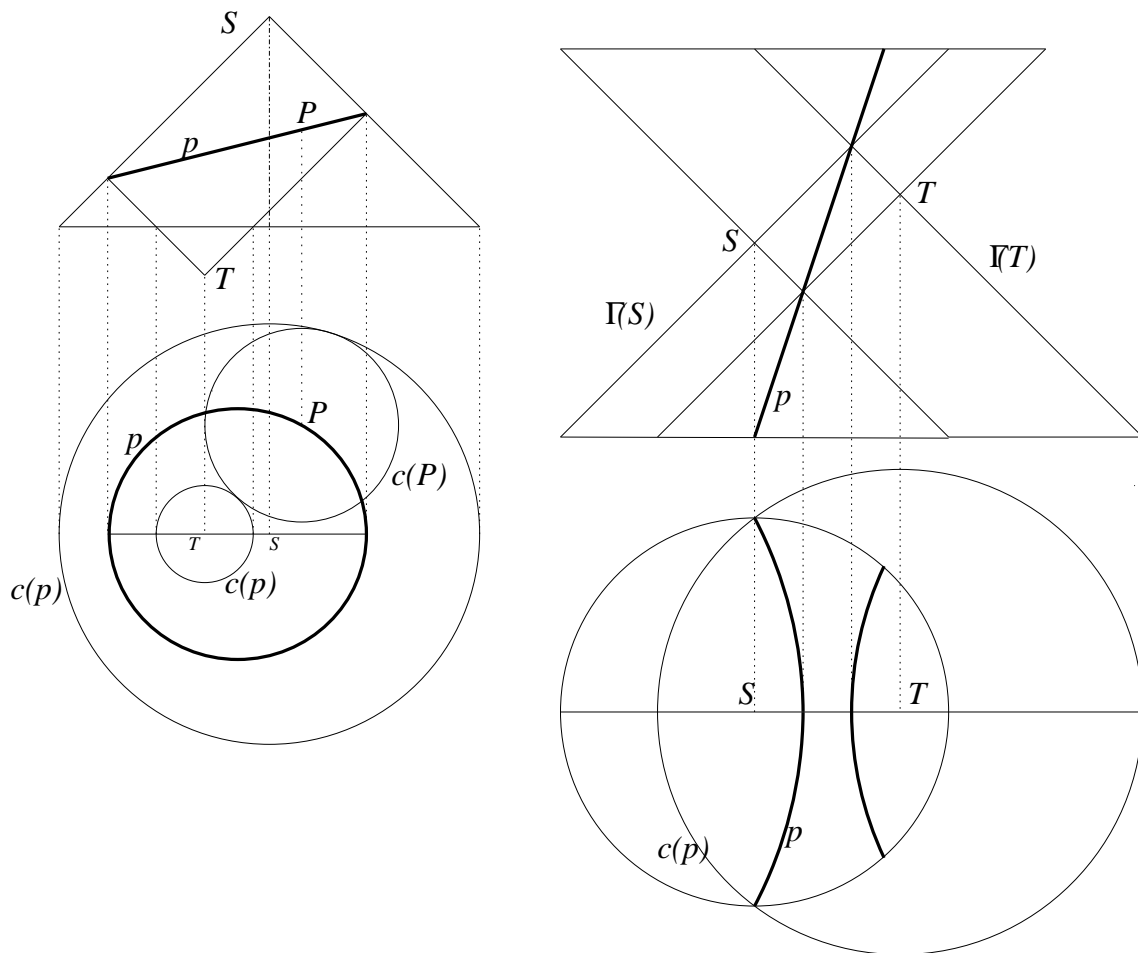


pe -sphere with asymptotic cone and planar section

A planar intersection of a pe -sphere is a pe -circle. We are only interested in those pe -circles, which are either ellipses or hyperbolas,

from the Euclidean viewpoint and do not pay attention to intersections of pe-spheres with tangent planes.

It is not very difficult to check that the cyclographic image $c(\mathbf{p})$ of a pe-circle consists of two circles C_1, C_2 . These are the intersection curves of the two possible γ -cones, passing through \mathbf{p} .



Cyclographic image of a pe-circle (ellipse and hyperbola)

SPATIAL EUCLIDEAN LAGUERRE GEOMETRY

In analogy to the previous sections, the geometric objects under consideration are oriented spheres C and oriented planes E ,

$$C : (\mathbf{x} - \mathbf{m})^2 - r^2 = 0,$$

$$E : e_0 + \mathbf{e} \cdot \mathbf{x} = 0.$$

The orientation is defined by orienting the unit normal of sphere or plane. The set of or. spheres contains the point set (radius=0). Oriented contact of C and E is analogously defined by

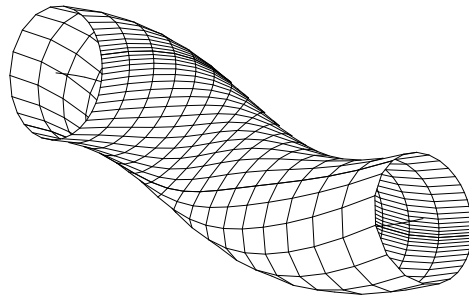
$$e_0 + \mathbf{e} \cdot \mathbf{m} + r = 0. \quad (8)$$

Via mappings ζ, ζ^* , or. spheres are represented by points $\zeta(C) = (\mathbf{m}, r)$ in 4-space A^4 , and or. planes by those 3-spaces which are of the form

$$\zeta^*(E) : e_0 + e_1x_1 + e_2x_2 + e_3x_3 + x_4 = 0,$$

where we assumed $\mathbf{e} = (e_1, e_2, e_3)$ to be normalized. In the planar case the cyclographic mapping of curves $\mathbf{p} \subset A^3$ generated envelopes of one parameter families of circles.

Now, $c : A^4 \rightarrow E^3$ maps curves $\mathbf{p}(t)$ to envelopes of one parameter families of spheres, so called *canal surfaces*. If the fourth coordinate (\mapsto radius) is constant, these surfaces are called *pipe* or *tubular surfaces*. Latter are envelopes of a moving sphere.



Further, c maps surfaces $\mathbf{p}(u, v)$ in A^4 to envelopes $c(\mathbf{p})$ of a two parameter family of spheres. But we will concentrate on the first case and give some examples.

Cyclographic Image of a Line

We have to distinguish between two cases where line G intersects Euclidean 3-space $E^3 : x_4 = 0$, or G is parallel to E^3 .

Example 4: Assume that line G is parametrized by $\mathbf{p}(t) = t(0, 0, g, r)$. The family of spheres C and its derivatives are

$$C(t) : x_1^2 + x_2^2 + (x_3 - gt)^2 - (rt)^2 = 0,$$

$$\dot{C}(t) : (x_3 - gt)g + r^2t = 0.$$

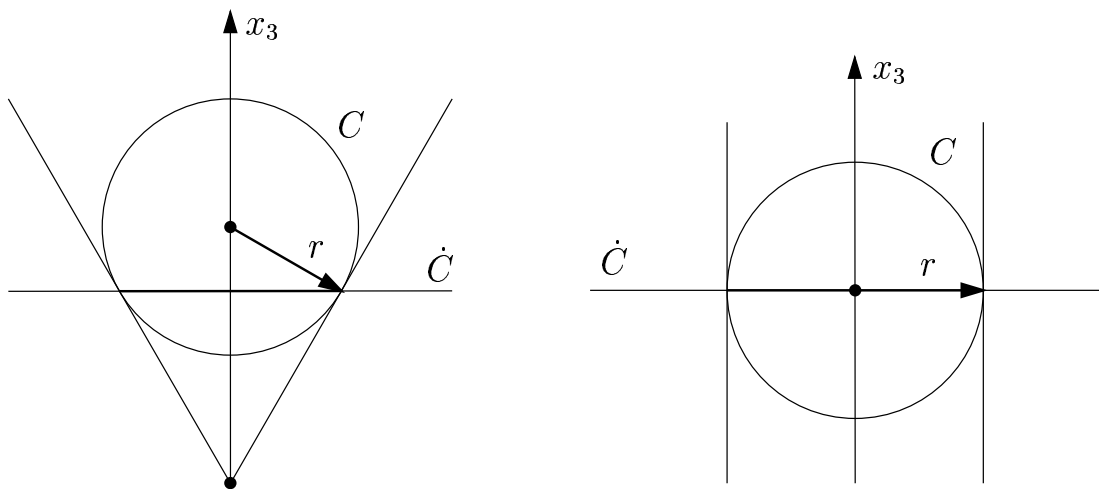
By eliminating t we obtain the equation of the envelope

$$c(\mathbf{p}) : x_1^2 + x_2^2 - x_3^2 \frac{r^2}{g^2 - r^2} = 0,$$

which is a real cone of revolution with axis x_3 , if $g^2 > r^2$. Its vertex is the origin and the angle between x_3 and G satisfies

$$\tan(\angle) = \frac{r}{\sqrt{g^2 - r^2}}.$$

For $g^2 = r^2$ we obtain the double counted plane $x_3 = 0$ and for $g^2 < r^2$ there is no real envelope.



Example 5: Let G be parallel to $x_4 = 0$ and parametrized by $\mathbf{p}(t) = (0, 0, 0, r) + t(0, 0, g, 0)$. The family of spheres and its derivatives are

$$C(t) : x_1^2 + x_2^2 + (x_3 - gt)^2 - r^2 = 0,$$

$$\dot{C}(t) : (x_3 - gt) = 0,$$

The envelope is a cylinder of revolution with axis x_3 and radius r ,

$$c(\mathbf{p}) : x_1^2 + x_2^2 - r^2 = 0.$$

DUPIN CYCLIDES

Using Laguerre geometry, Dupin cyclides can be generated quite simple. For that we need the notion of 'pe-circles' in A^4 . Introducing the bilinear form (see also (6))

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_2 + x_2y_2 + x_3y_3 - x_4y_4, \quad (9)$$

we define the pe-distance of points \mathbf{p} and \mathbf{q} by

$$\|\mathbf{p} - \mathbf{q}\| = (\langle \mathbf{p} - \mathbf{q}, \mathbf{p} - \mathbf{q} \rangle)^{1/2}.$$

A pe-sphere is the set of points possessing same pe-distance to a center \mathbf{m} . Its equation is

$$\langle \mathbf{x} - \mathbf{m}, \mathbf{x} - \mathbf{m} \rangle - r^2 = 0.$$

Pe-circles are defined as planar section curves of pe-spheres or, equivalently, as planar curves possessing equal pe-distance to a center.

Example 6: Let a pe-circle be given by the parametrization

$$\mathbf{p}(t) = (R \cos t, R \sin t, 0, r).$$

The corresponding family of spheres and its derivatives are

$$C(t) : (x_1 - R \cos t)^2 + (x_2 - R \sin t)^2 + x_3^2 = r^2,$$
$$\dot{C}(t) : x_1 \sin t - x_2 \cos t = 0.$$

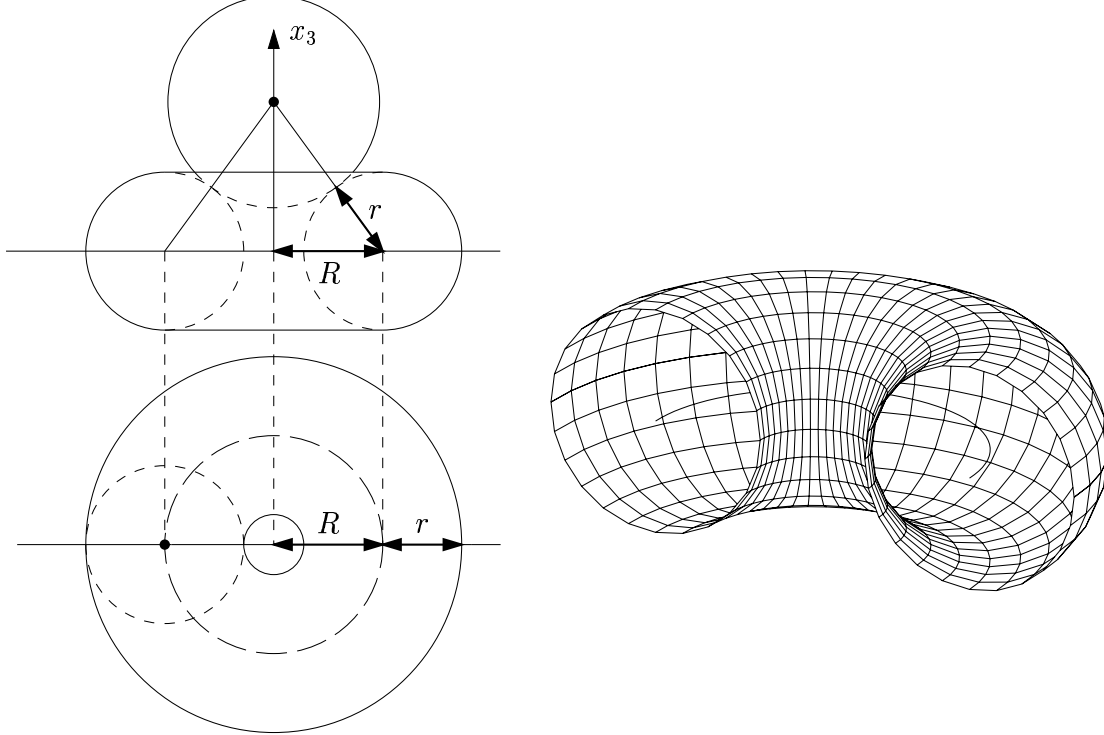
The second equation leads to

$$\cos t = \frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \sin t = \frac{x_2}{\sqrt{x_1^2 + x_2^2}}.$$

Inserting this into $C(t)$ gives

$$c(\mathbf{p}) : (x_1^2 + x_2^2 + x_3^2 + R^2 - r^2)^2 = 4R^2(x_1^2 + x_2^2),$$

which is the equation of a torus with rotational axis x_3 , radii R, r .



There is also a second pe-circle

$$\mathbf{q} = (0, 0, u, \sqrt{R^2 - u^2} - r),$$

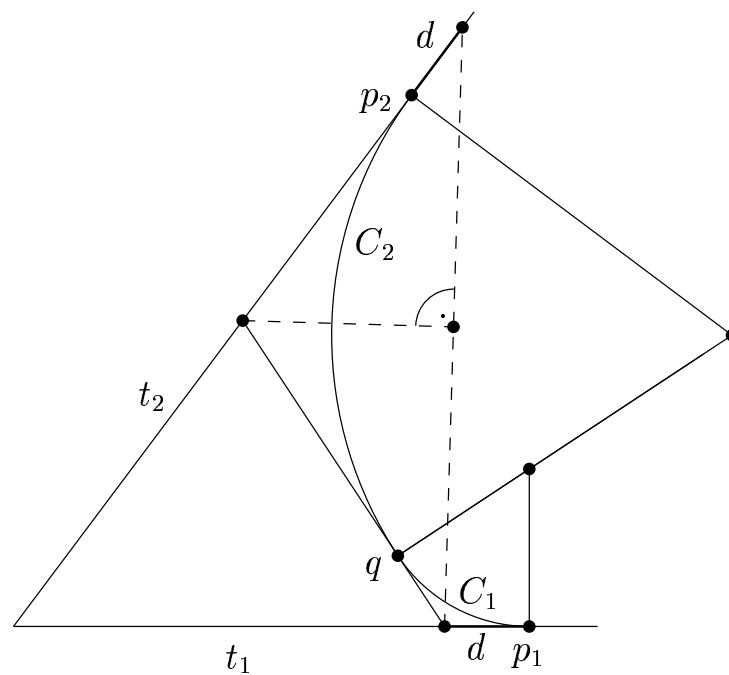
whose cyclographic image is $c(\mathbf{p})$. It represents a second family of spheres

$C_2(u) : x^2 + y^2 - (z - u)^2 - (\sqrt{R^2 - u^2} - r)^2 = 0,$
with centers at x_3 -axis and tangent to $c(\mathbf{p})$.

Approximating Canal Surfaces by Dupin Cyclide Pairs

Since a real canal surface F is envelope of a one parameter family of spheres, it can be represented in A^4 by a curve \mathbf{p} , whose derivative vectors satisfy $\langle \dot{\mathbf{p}}, \dot{\mathbf{p}} \rangle > 0$.

There are curve approximation techniques in E^3 using biarcs (pairs of circles). More precisely, given two points p_1, p_2 and tangent vectors t_1, t_2 , it is possible to find a pair of circles C_1, C_2 in E^3 , which interpolate the given data and possess common tangent line at the junction point q . Moreover, there is a one parameter family of such biarcs.

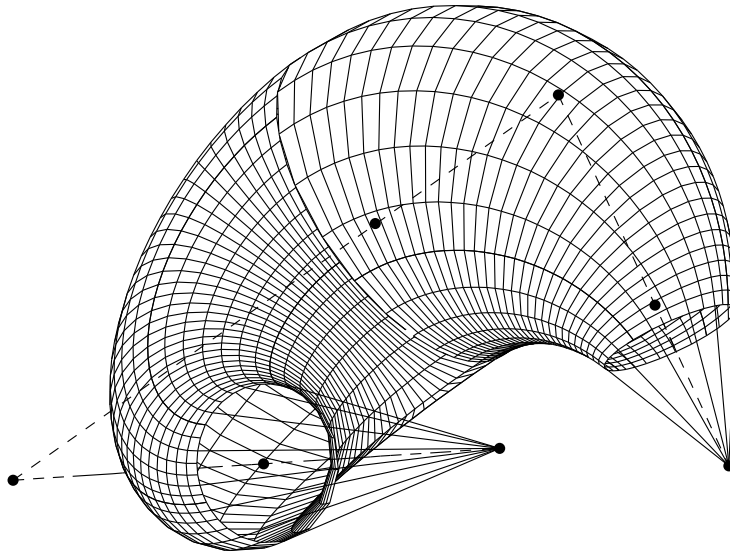


pair of circles interpolating two line elements

An analogous construction can be applied in pseudoeuclidean A^4 , if both tangent vectors satisfy

$$\langle t_i, t_i \rangle > 0.$$

Applying this procedure, the given curve \mathbf{p} will be approximated by an p -arc spline, a sequence of biarcs, in the p -sense. The cyclographic mapping approximates the given canal surface F by a sequence of pairs of Dupin cyclides, which join smoothly. A more detailed description is given in [21].



Touching pair of Dupin cyclides, tangent to two cones of revolution along prescribed circles

CONCLUSION

We discussed some tools of Laguerre geometry to get a better insight to several of CAGD. Of course, there are more applications, for instance the representation of curves and surfaces with rational offsets or error propagation in geometric constructions.

There are also other sphere geometries, due to Möbius and Lie, which will apply to several questions in surface modeling and geometric computing.

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