

2. Applications of Duality

Stefan Leopoldseder *

IMA Tutorial on Geometric Design:
Geometries for CAGD
Minneapolis, April 19, 2001

* Institut für Geometrie, TU Wien, Austria.

2. Applications of Duality

2.1. Fundamentals

Scene: Real projective plane P^2

Point X: $x\mathbb{R} = (x_0, x_1, x_2)\mathbb{R} = (x_0 : x_1 : x_2)$
... homogeneous coordinates

$\mathbf{x} = (x_1/x_0, x_2/x_0)$
... inhomogeneous coordinates

$x_0 = 0 \Leftrightarrow x\mathbb{R}$ ideal point

Line L: $u_0x_0 + u_1x_1 + u_2x_2 = 0$
... homogeneous linear equation

$\mathbb{R}u = \mathbb{R}(u_0, u_1, u_2) = (u_0 : u_1 : u_2)$
... homogeneous line coordinates

Incidence Relation:

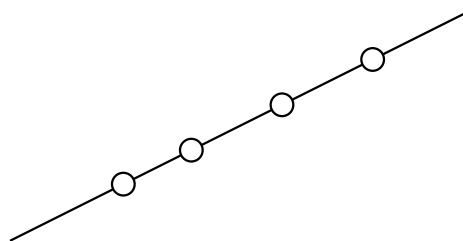
$$u \cdot x = 0 \Leftrightarrow x\mathbb{R} \in \mathbb{R}u$$

This relation is symmetric with respect to point and line coordinates.

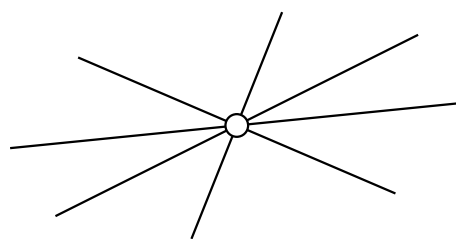
Principle of duality:

If a true statement about geometric objects in the projective plane employs only 'points', 'lines' and 'incidence', one obtains a true statement (*dual statement*) by interchanging the words 'point' and 'line' and leaving 'incidence' unchanged.

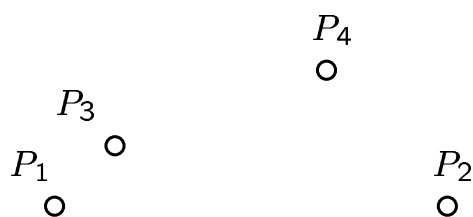
Examples (Dualizing geometric objects):



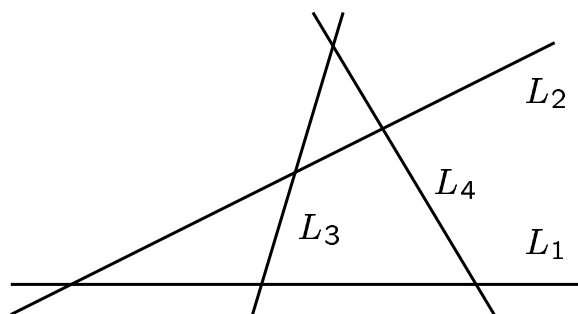
range of points



pencil of lines

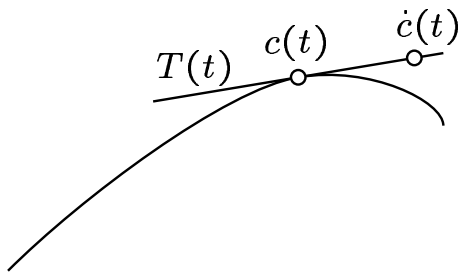


quadrangle

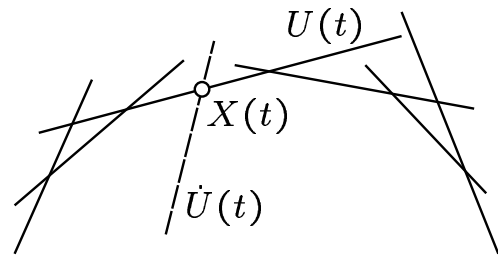


quadrilateral

2.2. Planar curves



planar curve
 $c(t) = c(t)\mathbb{R}$



dual curve
 $U(t) = \mathbb{R}u(t)$

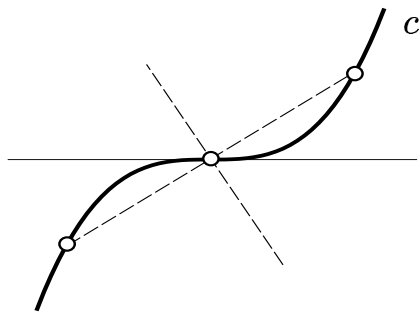
tangent $T(t) = \mathbb{R}t$:
 $T(t) = c(t)\mathbb{R} \vee \dot{c}(t)\mathbb{R}$,
 i.e. $t(t) = c(t) \times \dot{c}(t)$

envelope $X(t) = x\mathbb{R}$:
 $X(t) = \mathbb{R}u(t) \wedge \mathbb{R}\dot{u}(t)$,
 i.e. $x(t) = u(t) \times \dot{u}(t)$

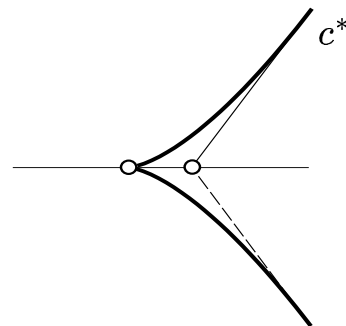
Example: $c(t) = (1, t, t^3)\mathbb{R}$

Let $c^*(t)$ be the family of tangents of $c(t)$
 ($c^*(t) \dots$ dual curve to $c(t)$)

$$c^*(t) = c(t) \times \dot{c}(t) = \mathbb{R}(2t^3, -3t^2, 1)$$



ordinary inflection point



ordinary cusp

2.3. Planar rational Bézier curves

Let $c(t) = c(t)\mathbb{R}$ be a rational Bézier curve with $c(t) = \sum_{i=0}^n B_i^n(t)b_i$. $B_i^n(t)$ denote the Bernstein polynomials.

Bézier points are $B_i = b_i\mathbb{R} = (b_{i0}, b_{i1}, b_{i2})\mathbb{R}$. As the points B_i do not identify b_i uniquely, one can use *weights* of B_i which simply are the first coordinates b_{i0} . As weights are not projectively invariant it is often better to use frame points $F_i = f_i\mathbb{R}$, $f_i = b_i + b_{i+1}$.

The projective control polygon of $c(t)$ is composed of those segments of the projective lines

$B_i \vee B_{i+1}$ in which F_i is lying (see figure). Similar to polynomial Bézier curves we have the

Projective variation diminishing property: A line does not intersect a Bézier curve $c(t)$ more often than its projective control polygon.

2.4. Dual rational Bézier curves

A dual rational Bézier curve $U(t) = \mathbb{R}u(t)$ is given by $u(t) = \sum_{i=0}^n B_i^n(t)b_i$.

Bézier lines $B_i = \mathbb{R}b_i = \mathbb{R}(b_{i0}, b_{i1}, b_{i2})$

Frame lines $F_i = \mathbb{R}f_i$, with $f_i = b_i + b_{i+1}$

Conversion to point representation $c(t)\mathbb{R}$:

The last but one step of the deCasteljau algorithm gives two lines $\mathbb{R}u_l$ and $\mathbb{R}u_r$

$$\begin{aligned}u_l(t) &= \sum_{i=0}^{n-1} B_i^{n-1}(t)b_i \\u_r(t) &= \sum_{j=0}^{n-1} B_j^{n-1}(t)b_{j+1}\end{aligned}$$

which intersect in the enveloping point

$$c(t) = u_l(t) \times u_r(t).$$

In rational Bézier representation we obtain

$$c(t) = \sum_{k=0}^{2n-2} B_k^{2n-2}(t)p_k$$

with

$$p_k = \frac{1}{\binom{2n-2}{k}} \sum_{i+j=k} \binom{n-1}{i} \binom{n-1}{j} b_i \times b_{j+1}.$$

Remark: $c(t) = c(t)\mathbb{R}$ is (in general) of algebraic class n (cf. dual representation) and of algebraic order $2n - 2$ (cf. point representation). Degree reductions are possible.

2.5. Rational curves with rational offsets (PH curves)

To a rational curve c with inhomogeneous representation $\mathbf{c}(t)$ the offset at distance d is

$$\mathbf{c}_d(t) = \mathbf{c}(t) + d\mathbf{n}(t)$$

with $\mathbf{n}(t)$ as unit normal vector of $c(t)$.

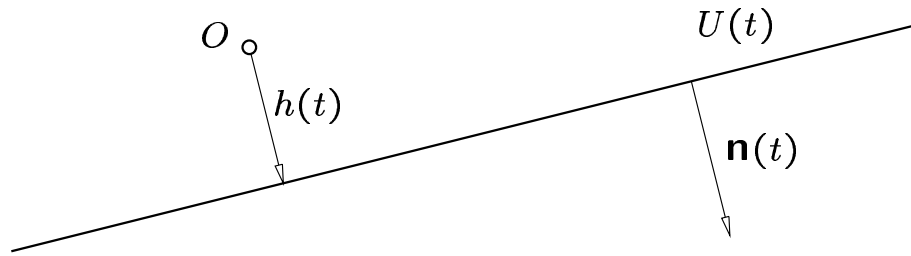
In general $\mathbf{n}(t)$ is not rational. For rational offsets $\mathbf{n}(t) = (n_1(t), n_2(t))$ must be a rational parameterization of the unit circle \bar{c} , thus

$$n_1(t) = \frac{2a(t)b(t)}{a^2(t) + b^2(t)}, n_2(t) = \frac{a^2(t) - b^2(t)}{a^2(t) + b^2(t)}.$$

The tangents at $c(t)$ have line coordinates

$$U(t) = \mathbb{R}(-h(t), n_1(t), n_2(t))$$

with rational $h(t)$. Because of $n_1^2 + n_2^2 = 1$, $h(t)$ is the signed distance of $U(t)$ from the origin.



Replacing $h(t)$ by $h(t) + d$ gives the dual representation of the offset $\mathbf{c}_d(t)$

$$U(t) = \mathbb{R}(-h(t) - d, n_1(t), n_2(t)).$$

Note that the dual representations of c , c_d and of the unit circle \bar{c}

$$U(t) = \mathbb{R}(-1, n_1(t), n_2(t))$$

differ only in the first coordinate. Thus, corresponding tangents (tangents to same parameter t) of c , c_d and \bar{c} are parallel.

After conversion to Bernstein form of same degree n we can state that corresponding Bézier and Farin lines of c , c_d and \bar{c} are parallel.

Example: Rational PH curve c with offset c_d

$c \dots$ algebraic class 3, algebraic order 4

Applications:

G^1 Hermite interpolation with rational PH curves of class 3 (see [13]).

G^2 Hermite interpolation with rational PH curves of class 4 (see [13]).

2.6. Fundamentals of n-dimensional real projective space P^n

Point X: $x\mathbb{R} = (x_0, \dots, x_n)\mathbb{R} = (x_0 : \dots : x_n)$
... homogeneous coordinates

$\mathbf{x} = (x_1/x_0, \dots, x_n/x_0)$
... inhomogeneous coordinates

$x_0 = 0 \Leftrightarrow x\mathbb{R}$ ideal point

Hyperplane U: $u_0x_0 + \dots + u_nx_n = 0$
... homogeneous linear equation

$\mathbb{R}u = \mathbb{R}(u_0, \dots, u_n) = (u_0 : \dots : u_n)$
... homogeneous line coordinates

Incidence Relation:

$$u \cdot x = 0 \Leftrightarrow x\mathbb{R} \in \mathbb{R}u$$

Principle of duality:

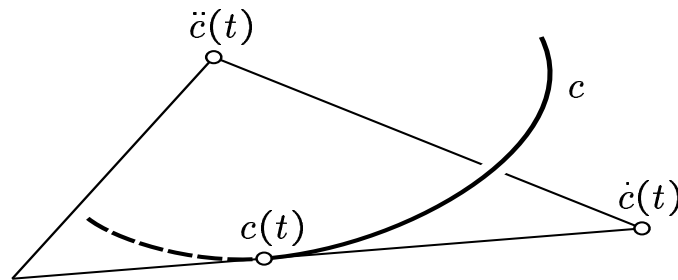
If a statement involving k -dimensional projective subspaces, projective span, intersection, and inclusion of subspaces in P^n is modified by replacing these items by $(n - k - 1)$ -dimensional subspaces, intersection, projective span, and reverse inclusion of subspaces, then this new statement is called *dual* to the original one.

A statement which fulfills above criteria is true if and only if its dual is.

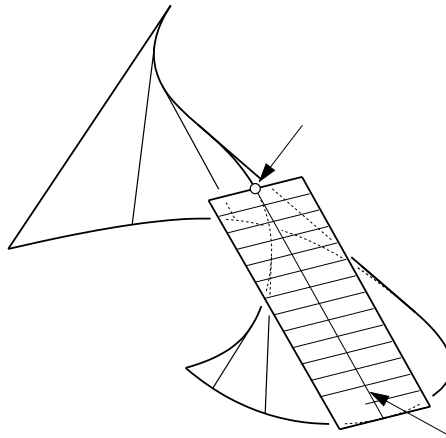
Examples in real projective 3-space P^3 :

- o) Points are dual to planes.
- o) Lines are dual to lines (self-dual).
- o) 'Intersecting lines' and 'skew lines' are self-dual terms.
- o) The points of a line are dual to the planes which contain a line.

2.7. Dualizing spatial curves $c(t)$:



spatial curve $c(t) = c(t)\mathbb{R}$
 curve tangent $c(t) \vee \dot{c}(t)$
 osculating plane $c(t) \vee \dot{c}(t) \vee \ddot{c}(t)$



developable surface (as envelope of
 1-parameter family of planes $U(t) = \mathbb{R}u(t)$)
 generator line $U(t) \wedge \dot{U}(t)$
 singular point $U(t) \wedge \dot{U}(t) \wedge \ddot{U}(t)$

We define the *dual curve* c^* to c by

$$c^*(t) = c(t) \vee \dot{c}(t) \vee \ddot{c}(t).$$

The envelope of these osculating planes of c is the tangent surface of c . $c^*(t)$ is the tangent plane in all points of the tangent at $c(t)$. Thus we have $c^{**} = c$, if c is sufficiently smooth.

Example (Twisted cubic $c(t) = c(t)\mathbb{R}$):

$$\begin{aligned}c(t) &= (1, t, t^2, t^3) \\ \dot{c}(t) &= (0, 1, 2t, 3t^2) \\ \ddot{c}(t) &= (0, 0, 2, 6t)\end{aligned}$$

The tangent surface of the twisted cubic $c(t)$ has the dual representation $c^*(t) = \mathbb{R}c^*(t)$ with $c^*(t) = c(t) \times \dot{c}(t) \times \ddot{c}(t) = (2t^3, -6t^2, 6t, -2)$

This surface is of algebraic class 3.

We have

$$\begin{aligned}c^*(t) &= (2t^3, -6t^2, 6t, -2) \\ \dot{c}^*(t) &= (6t^2, -12t, 6, 0) \\ \ddot{c}^*(t) &= (12t, -12, 0, 0) \\ c^{**}(t) &= c^*(t) \times \dot{c}^*(t) \times \ddot{c}^*(t) = (1, t, t^2, t^3) = c(t)\end{aligned}$$

2.8. Developable rational Bézier surfaces

The dual representation of a developable rational Bézier surface $U(t) = \mathbb{R}u(t)$ is

$$u(t) = \sum_{i=0}^n B_i^n(t) b_i$$

with Bézier planes $B_i = \mathbb{R}b_i$ and frame planes $F_i = \mathbb{R}(b_i + b_{i+1})$ (cf. section 2.4.).

Developable surface of class 3 (left figure) and its Bézier planes (right figure). The frame planes have been chosen as planes of symmetry of consecutive Bézier planes.

Dual to the fact that the central projection of a Bézier curve is a Bézier curve we get:

The intersection of a developable Bézier surface with a plane α gives a Bézier curve. The intersection of the Bézier and frame planes with α yields the Bézier and frame lines of the boundary curve in α .

Conversion to tensor product form:

The intersection with two planes α, β gives two boundary curves of degree $2n - 2$ whose Bézier representations can be found according to section 2.4. Thus we obtain Bézier points

$B_{i,0} \in \alpha, B_{i,1} \in \beta, i = 0, \dots, 2n - 2$ and frame points $F_{i,0}, F_{i,1}, i = 0, \dots, 2n - 3$.

For any parameter t , we get points in α and β which belong to the same generator. Thus the developable surface patch can be represented as rational $(2n - 2, 1)$ -tensor product surface

$$x(t, u) = \sum_{i=0}^{2n-2} \sum_{j=0}^1 B_i^{2n-2}(t) B_j^1(u) b_{i,j}.$$

Applications:

Connection of two planar curves with a developable surface and its approximation by a *rational* developable surface (see [14]).

References

- [1]. Bodduluri, R.M.C. and Ravani, B., Geometric design and fabrication of developable surfaces, *ASME Adv. Design Autom.* **2** (1992), 243–250.
- [2]. Bodduluri, R.M.C. and Ravani, B., Design of developable surfaces using duality between plane and point geometries, *Computer-Aided Design* **25** (1993), 621–632.
- [3]. Farin, G., Rational curves and surfaces, in: *Mathematical Methods in CAGD*, T. Lyche and L.L. Schumaker, eds., Academic Press, Boston, MA, 1989, 215–238.
- [4]. Farin, G., *Curves and Surfaces for Computer Aided Geometric Design*, Academic Press, Boston, 3rd edn., 1992.
- [5]. Farin, G., *NURBS for Rational Curve and Surface Design*, A K Peters, Wellesley, MA, 1994.
- [6]. Farouki, R.T., Pythagorean-hodograph curves in practical use, in *Geometry Processing for Design and Manufacturing*, R.E. Barnhill, ed., SIAM, Philadelphia, 1992, pp. 3–33.
- [7]. Hoschek, J., Dual Bézier curves and surfaces, in *Surfaces in Computer Aided Geometric Design*, R.E. Barnhill and W. Boehm, eds., North Holland, 1983, pp. 147–156.
- [8]. Hoschek, J. and Pottmann, H., Interpolation and approximation with developable B-spline surfaces, in *Mathematical Methods for Curves and Surfaces*, M. Daehlen, T. Lyche and L.L. Schumaker, eds., Vanderbilt University Press, Nashville 1995, pp. 255–264.
- [9]. Hoschek, J. and Schneider, M., Interpolation and approximation with developable surfaces, in *Curves and Surfaces with Applications in CAGD*, A. Le Méhauté, C. Rabut and L.L.

Schumaker, eds., Vanderbilt University Press, Nashville 1997, pp. 185–202.

- [10]. Pottmann, H., Applications of the dual Bézier representation of rational curves and surfaces, in *Curves and Surfaces in Geometric Design*, P.J. Laurent, A. Le Méhauté and L.L. Schumaker, eds., A K Peters, Wellesley, MA, 1994, 377–384.
- [11]. Pottmann, H., Rational curves and surfaces with rational off-sets, *Computer Aided Geometric Design* **12**, (1995), 175–192.
- [12]. Pottmann, H., Studying NURBS curves and surfaces with classical geometry, in *Mathematical Methods for Curves and Surfaces*, M. Daehlen, T. Lyche and L.L. Schumaker, eds., Vanderbilt University Press, Nashville 1995, pp. 413–438.
- [13]. Pottmann, H., Curve design with rational Pythagorean-hodograph curves, *Advances in Computational Mathematics* **3** (1995), 147–170.
- [14]. Pottmann, H. and Farin, G., Developable rational Bézier and B-spline surfaces, *Computer Aided Geometric Design* **12** (1995), 513–531.
- [15]. Pottmann, H. and Wallner, J., Approximation algorithms for developable surfaces, *Computer Aided Geometric Design* **16** (1999), 539–556.