# 1. Basics of Projective Geometry

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# 1.1. The real projective plane

Origin of projective geometry:
development of rules of perspective drawing; parallel lines (if not parallel to image plane) are mapped to intersecting lines.

$\longrightarrow$ view parallelity as form of intersection

$\longrightarrow$ *projective extension* of Euclidean space.

First: projective extension of Euclidean plane

Add *ideal point* or *point at infinity* to each line $L$, such that parallel lines share the same ideal point.

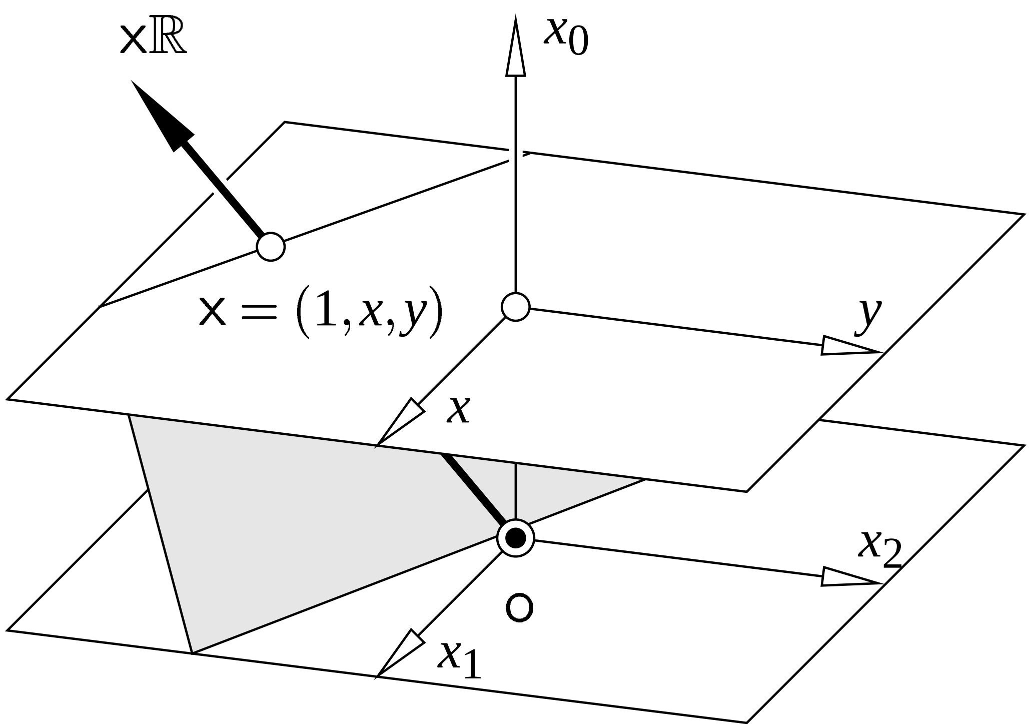
Two lines are parallel $\Longleftrightarrow$ intersect at infinity

*ideal line* $\omega$: consists of all ideal points

# Homogeneous coordinates

Embed Euclidean plane $E^2$ (Cartesian coordinates $x, y$) in $\mathbb{R}^3$ (coordinates $x_0, x_1, x_2$) via

$$\mathbf{x} = (x, y) \mapsto \mathsf{x} = (1, x, y).$$



1-dim. subspace spanned by $\mathsf{x} = (1, x, y)$:

$$\mathsf{x}\mathbb{R} = \{(\lambda, \lambda x, \lambda y), \lambda \in \mathbb{R}\}.$$

Any tripel $(\lambda, \lambda x, \lambda y), \lambda \neq 0$ defines the subspace and the corresponding point in $E^2$: **homogeneous Cartesian coordinates** of point $\mathbf{x} = \mathsf{x}\mathbb{R}$ (write also $(x_0 : x_1 : x_2)$).

Conversion to inhomogeneous Cartesian coordinates:

$$x = \frac{x_1}{x_0}, \; y = \frac{x_2}{x_0}.$$

Ideal point of line $L \subset E^2$ parallel to $(l_1, l_2)$ has homogeneous coordinates

$$(0, l_1, l_2).$$

point $(x_0 : x_1 : x_2)$ is ideal $\Longleftrightarrow$ $x_0 = 0$

One-to-one correspondence between 1-dimensional subspaces of $\mathbb{R}^3$ and points of the extended plane, called *real projective plane* $P^2$.

# Lines

Lines of $P^2$: represented by 2-dim. subspaces of $\mathbb{R}^3$.

Parametric representation of line $L$ spanned by points $\mathsf{a}\mathbb{R}, \mathsf{b}\mathbb{R}$:

$$\mathsf{x}\mathbb{R} = (\lambda_0 \mathsf{a} + \lambda_1 \mathsf{b})\mathbb{R}, \ \ \lambda_0, \lambda_1 \in \mathbb{R}.$$

This 2-dim. subspace may also be written as

$$u_0 x_0 + u_1 x_1 + u_2 x_2 = 0. \qquad (*)$$

$(u_0, u_1, u_2) = \mathsf{u}$: *homogeneous line coordinates* (write $L = \mathbb{R}\mathsf{u}$)
(*) with canonical scalar product:

$$\mathsf{u} \cdot \mathsf{x} = 0.$$

*Incidence relation* of line $\mathbb{R}\mathsf{u}$ and point $\mathsf{x}\mathbb{R}$.

$\Longrightarrow$ Connecting line $L = \mathbb{R}\mathsf{u}$ of points $\mathsf{a}\mathbb{R}$ and $\mathsf{b}\mathbb{R}$:

$$\mathsf{u} = \mathsf{a} \times \mathsf{b}.$$

# Duality

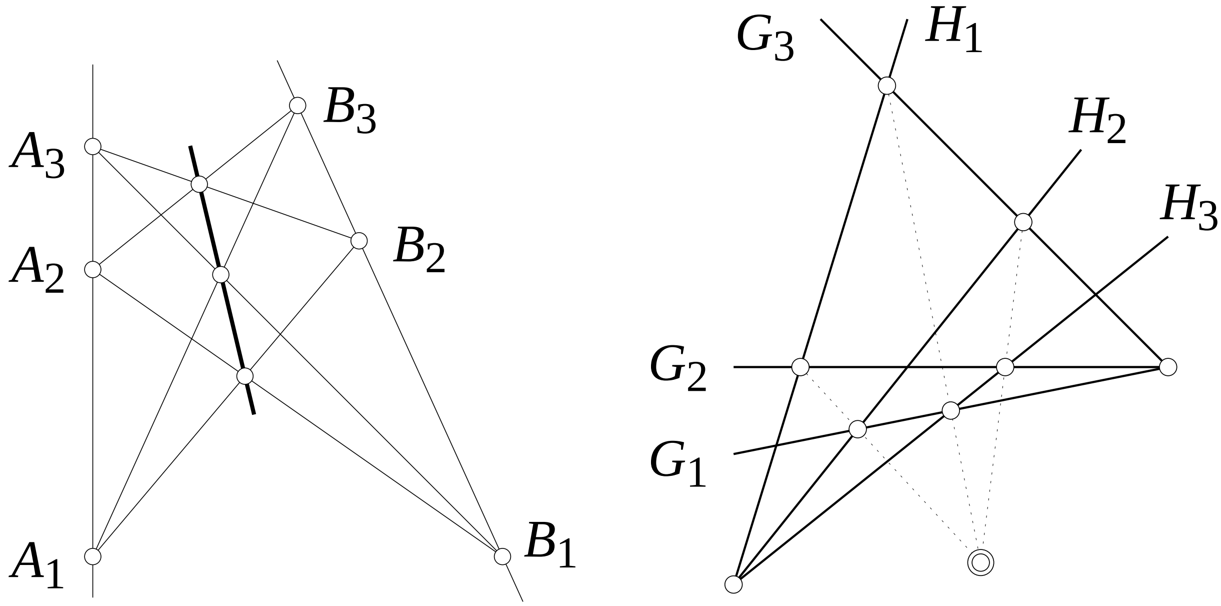
Incidence condition is symmetric: remains unchanged if point and line coordinates are interchanged. $\Longrightarrow$

Def.: 'Line' and 'point' are dual to each other in $P^2$. If in a statement about points and lines and their incidences every occurrence is replaced by its dual, this new statement is called the dual of the original one.

**Principle of Duality:** *If a true statement about geometric objects in $P^2$ employs only 'points', 'lines' and 'incidence', its dual is also true, and vice versa.*

Examples:

- range of points dual to pencil of lines
- connecting line dual to point of intersection
- Point and incident line (line element): self-dual figure.




Left: Pappos' theorem. Right: its dual

# 1.2. Projective $n$-space

Extend Euclidean $n$-space $E^n$ analogously to extension of plane by adding ideal points, one for each class of parallel lines.

Homogeneous coordinates $(x_0 : x_1 : \ldots : x_n)$.

For proper points, inhomogeneous coordinates are recovered as

$$\left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \ldots, \frac{x_n}{x_0}\right).$$

Ideal point of line $L \subset E^n$, parallel to $(l_1, \ldots, l_n)$ has homogeneous coordinates

$$(0 : l_1 : \ldots : l_n).$$

Ideal points lie in *ideal hyperplane* $\omega : x_0 = 0$.

One-to-one correspondence between **1-dimensional subspaces** of $\mathbb{R}^{n+1}$ and **points** of the extended space $P^n$, called *n-dimensional real projective space*.

# Projective subspaces

$(k+1)$-dim. linear subspaces of $\mathbb{R}^{n+1}$ define $k$-dim. projective subspaces of $P^n$:

$$\begin{array}{rl} k = \quad -1: & \text{empty subspace} \\ k = \quad 0: & \text{point} \\ k = \quad 1: & \text{line} \\ k = \quad 2: & \text{plane} \\ k = n-1: & \text{hyperplane.} \end{array}$$

Def.: points $P_0 = \mathsf{p}_0\mathbb{R}, \ldots, P_k = \mathsf{p}_k\mathbb{R}$ are *projectively independent* $\Longleftrightarrow$ vectors $\mathsf{p}_0, \ldots, \mathsf{p}_k$ are linearly independent.

Indep. points $P_0 = \mathsf{p}_0\mathbb{R}, \ldots, P_k = \mathsf{p}_k\mathbb{R}$ span proj. $k$-space $P_0 \vee \ldots \vee P_k$; parameterization with $k+1$ homogeneous parameters $\lambda_0, \ldots, \lambda_k \in \mathbb{R}$:

$$\mathsf{x}\mathbb{R} = (\lambda_0 \mathsf{p}_0 + \ldots + \lambda_k \mathsf{p}_k)\mathbb{R}.$$

Represent hyperplane by one linear equation,

$$u_0 x_0 + \ldots + u_n x_n = 0 \quad \text{or} \quad \mathsf{u} \cdot \mathsf{x} = 0. \quad (*)$$

$(u_0 : \ldots : u_n)$ are *homogeneous hyperplane coordinates*. (*) is incidence relation of hyperplane $\mathbb{R}\mathsf{u}$ and point $\mathsf{x}\mathbb{R}$.

# Duality

Projective span $U \vee V$ (intersection $U \cap V$) of projective subspaces $U, V$ is determined by linear hull (intersection, resp.) of corresponding linear subspaces of $\mathbb{R}^{n+1}$.

Def.: If a statement involving $k$-dim. subspaces, proj. span, intersection and inclusion of subspaces in $P^n$ is modified by replacing these items by $n-k-1$-dim. subspaces, intersection, proj. span, and reverse inclusion of subspaces, this new statement is called dual to the original one.

**Principle of Duality:** A statement which fulfills the criteria of the above definition is true if and only if its dual is.

(Proof by orthogonality relation (*))

Duality in $P^3$: Points are dual to planes, lines are dual to lines, a range of points is dual to a pencil of planes,. . .

# 1.3. Projective maps

Projective geometry of $P^n$ studies properties invariant under *projective maps*. These are induced by linear maps $f$ in $\mathbb{R}^{n+1}$,

$$f : \mathrm{x} \mapsto \mathrm{x}' = A \cdot \mathrm{x},$$

with a regular $(n+1) \times (n+1)$ matrix $A$. $f$ determines a projective map $\phi : P^n \to P^n$ by

$$\phi : \mathrm{x}\mathbb{R} \mapsto (A \cdot \mathrm{x})\mathbb{R}.$$

$\phi$ is one-to-one on the set of $k$-dim. subspaces of $P^n$, for all $k = 0, \ldots, n-1$.

Def.: A set of $n+2$ points of $P^n$ is called a *fundamental set* if every subset of $n+1$ points is proj. independent.

Theorem: If $P_0, \ldots, P_{n+1}$ and $P'_0, \ldots, P'_{n+1}$ are fundamental sets in $P^n$, there exists a unique projective map $P^n \to P^n$ which maps $P_i$ to $P'_i$ for $i = 0, \ldots, n+1$.

Sketch of proof: Represent $P_i$ as $\mathsf{b}_i\mathbb{R}$ with $\mathsf{b}_{n+1} = \sum_{i=0}^{n} \mathsf{b}_i$. Same for $P_i'$ and define $f$ by $\mathsf{b}_i \mapsto \mathsf{b}_i'$, $i = 0, \dots, n$.

A fundamental set defines a *projective coordinate system* as follows: Fundamental set $(P_0, \dots, P_n, E)$ can be written as

$$(\mathsf{b}_0\mathbb{R}, \dots, \mathsf{b}_n\mathbb{R}, \mathsf{e}\mathbb{R} = (\mathsf{b}_0 + \dots + \mathsf{b}_n)\mathbb{R}).$$

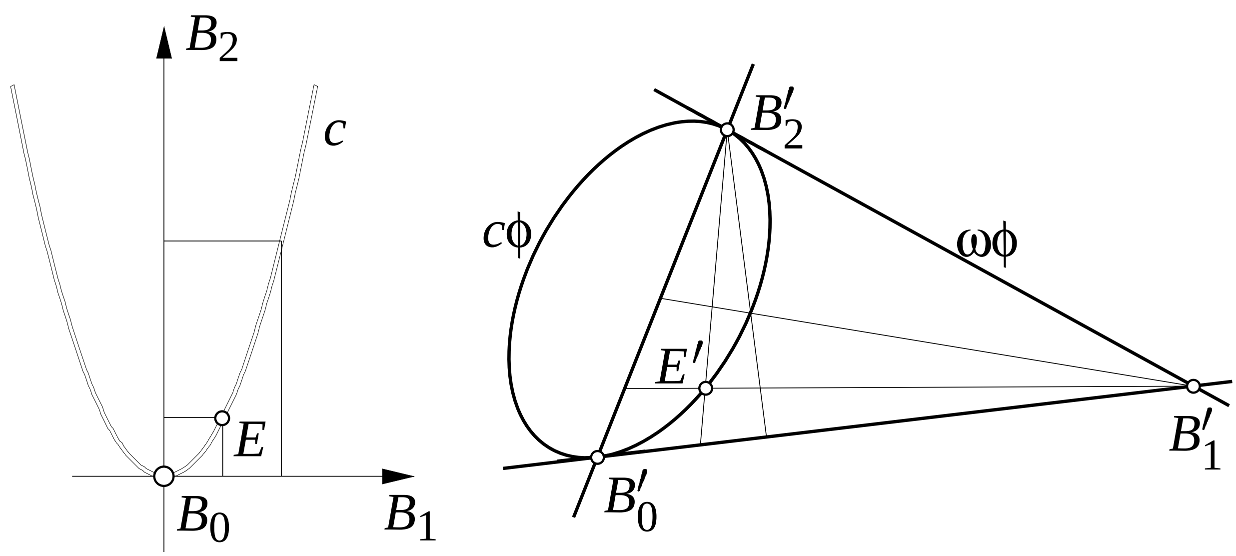
Points $\mathsf{b}_i\mathbb{R}$ are called *fundamental points*, $\mathsf{e}\mathbb{R}$ is called *unit point*. All vectors $\mathsf{p} \in \mathbb{R}^{n+1}$ can be expressed as $\mathsf{p} = x_0\mathsf{b}_0 + \dots + x_n\mathsf{b}_n$.
$(x_0, \dots, x_n)$ are called *projective coordinates* of point $\mathsf{p}\mathbb{R}$ with respect to the projective frame $(P_0, \dots, P_n; E)$. To indicate homogeneity, we write also $(x_0 : \dots : x_n)$.

Fundamental points and unit point have coordinates

$$(1:0:\cdots:0), \dots, (0:\cdots:0:1), \ (1:1:\cdots:1).$$

Cartesian coordinate system $\Sigma$ defines proj. coord. system: $P_0$ is origin of $\Sigma$, $E$ is unit point of $\Sigma$, $P_1, \dots, P_n$ are ideal points of coordinate axes.

# Example in $P^2$




Homogeneous Cartesian System $(B_0, B_1, B_2; E)$ is mapped under projective map $\phi$ onto proj. system $(B'_0, B'_1, B'_2; E')$. Image of ideal line $\omega = B_1 B_2$ is $B'_1 B'_2$. Image of parabola $c: \ y = x^2$, whose homog. equation is

$$x_1^2 - x_0 x_2 = 0,$$

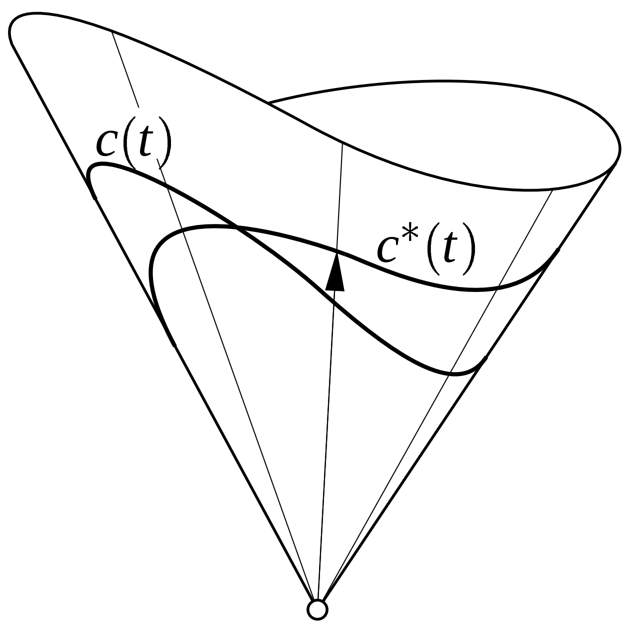
is a conic with the same equation in the image frame $(B'_0, \ldots, E')$.

# 1.4. Curves

Curve in $P^n$ given by parameterization in homogeneous coordinates:

$$\mathsf{c}(t) = (c_0(t), c_1(t), \ldots, c_n(t)) \neq (0, \ldots, 0) \text{ for all } t \in I.$$

Multiplication of $\mathsf{c}(t)$ with scalar function $\rho(t)$ ('renormalization') yields parameterization $\mathsf{c}^\star(t) = \rho(t)\mathsf{c}(t)$ of the same curve: view curve in $P^n$ as cone in $\mathbb{R}^{n+1}$.



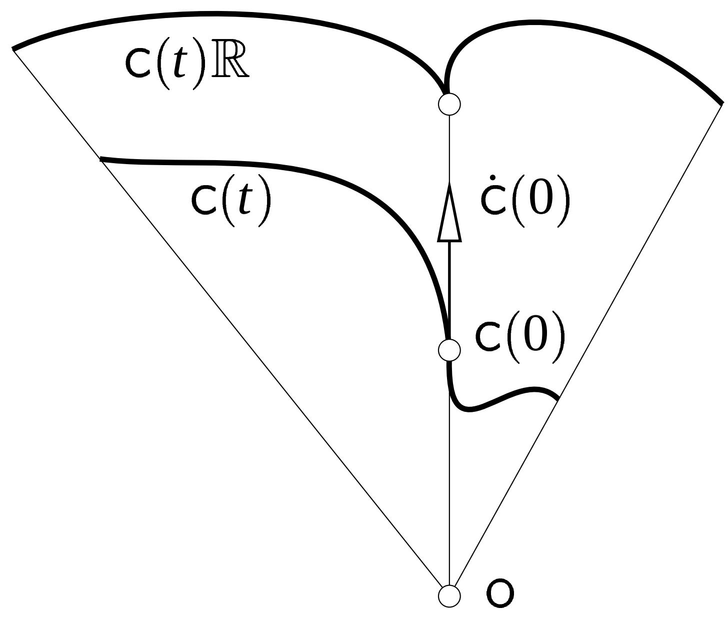

Advantage of homog. coordinates in CAGD: rational curve has polynomial homog. representation.

Application of renormalization: degree elevation of a rational curve $\mathsf{c}(t)$, obtained by multiplication with linear function $at+b \Longrightarrow$ degree elevation for rational curves not unique.

## Curve tangents and singularities

First derivative $\dot{\mathsf{c}}(t)$ (if $\neq 0$) determines a POINT $c^1(t) = \dot{\mathsf{c}}(t)\mathbb{R}$.

Curve is regular at point $c(t)$ if $c^1(t) \neq c(t)$. Then the curve tangent there is the line $c(t) \vee c^1(t)$ (view tangent plane of cone in $\mathbb{R}^{n+1}$). At a singular point, $c(t) = c^1(t)$.




Regular curve in $\mathbb{R}^{n+1}$ representing a curve with singularity in $P^n$.

# Osculating spaces

For sufficiently differentiable curve $c(t) = \mathsf{c}(t)\mathbb{R}$, the linear subspaces

$$[\mathsf{c}(t)],\ [\mathsf{c}(t), \dot{\mathsf{c}}(t)],\ [\mathsf{c}(t), \dot{\mathsf{c}}(t), \ddot{\mathsf{c}}(t)], \ldots$$

define projective subspaces

$$c(t),\ c(t) \vee c^1(t),\ c(t) \vee c^1(t) \vee c^2(t), \ldots,$$

where

$$c(t) = \mathsf{c}(t)\mathbb{R},\ c^1(t) = \dot{\mathsf{c}}(t)\mathbb{R},\ c^2(t) = \ddot{\mathsf{c}}(t)\mathbb{R}, \ldots.$$

The projective subspace $c(t) \vee \ldots \vee c^k(t)$ is called *osculating (sub)space of order* $k$ (dimension $\leq k$). Sequence of dimensions of osculating spaces characterizes special curve points:

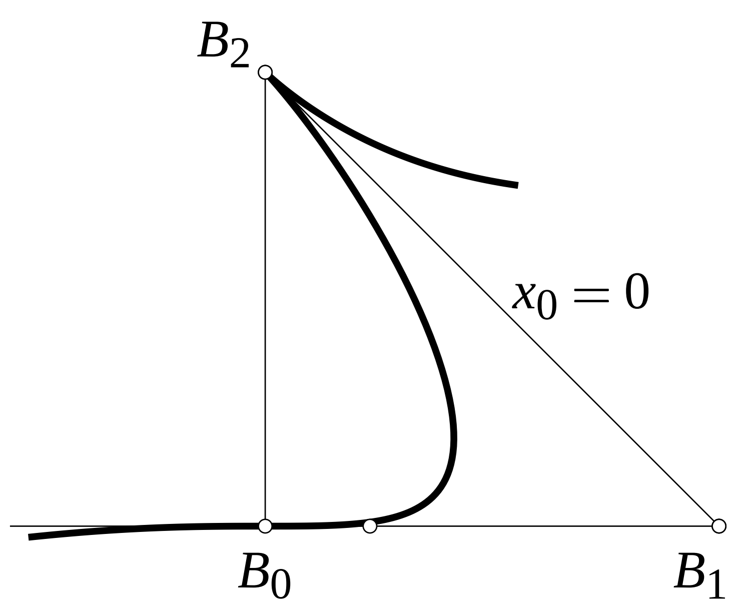
| | |
|---|---|
| $0,1$ | regular point |
| $0,1,\ldots,n$ | point of main type |
| $0,1,1$ | inflection point (ordinary if 0,1,1,2) |
| $0,1,1,1$ | flat point |
| $0,1,2,2$ | point with stationary osculating plane |
| $0,0$ | cusp (ordinary if 0,0,1) |

Example

Semicubic parabola $y^2 = x^3$ has homogeneous parameterization $t \mapsto (1, t^2, t^3)$. ($\Longrightarrow$ ordinary cusp at $t = 0$). Behaviour at infinity? Reparameterization $t = 1/u$ ($u \neq 0$) yields $u \mapsto (1, 1/u^2, 1/u^3)$. Continuous extension by

$$\mathsf{c}(t) = (u^3, u, 1),\ \ u \in \mathbb{R}.$$

Point $c(0)$ is ideal point. It is an inflection point because $c^1(u) = (3u^2, 1, 0)\mathbb{R}$, $c^2(u) = (6u, 0, 0)\mathbb{R}$ and $\dim(c(0) \vee c^1(0)) = \dim(c(0) \vee c^1(0) \vee c^2(0)) = 1$. Figure shows projectively equivalent curve (also projectively equivalent to $y = x^3$).



# Contact order

Def.: Two curves c$(t)\mathbb{R}$, d$(u)\mathbb{R}$ are said to have contact of order $k$ at parameter values $t_0, u_0$ if after a suitable regular parameter transform $u = u(t)$ with $u_0 = u(t_0)$ and re-normalization the derivatives of order $0, 1, \ldots, k$ agree.

The order of contact of two curves is the maximum $k$ such that the two curves have contact of order $k$ there.

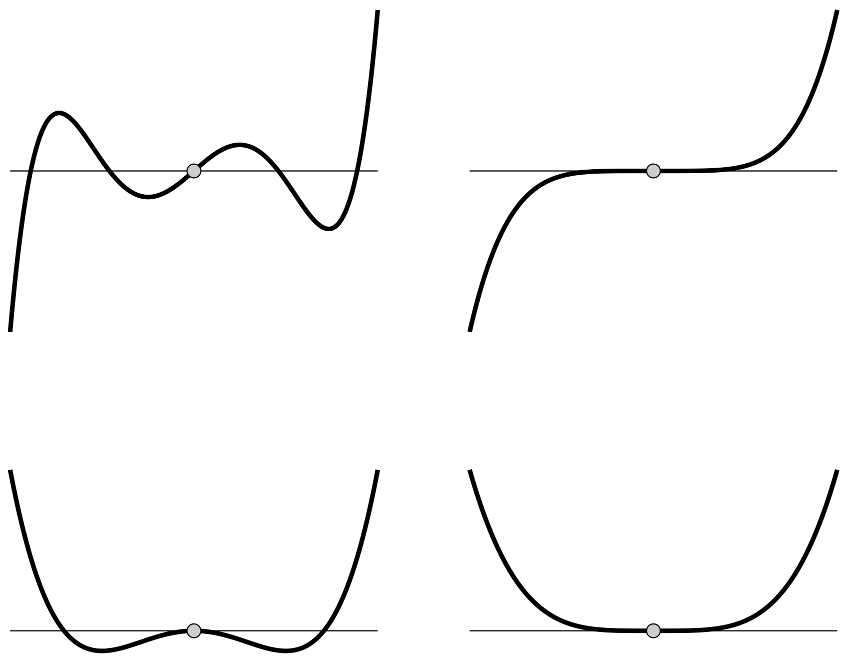
This definition is projectively invariant (even invariant with respect to $C^k$ diffeomorphisms).

Example: Curve (higher parabola)

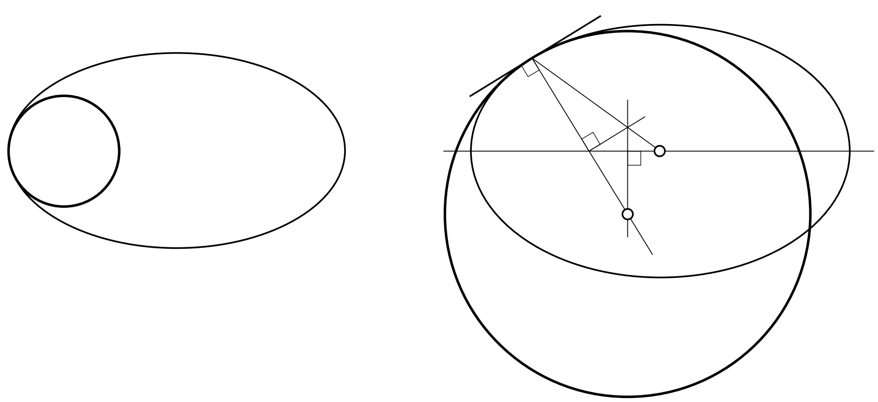
$$y = x^n, \quad (n \geq 3),$$

with inhomogeneous parametrization $(t, t^n)$ has contact of order $n-1$ with the $x$-axis $(t, 0)$ at the origin $(0, 0)$.

Example (contd.) $(0,0)$ is an inflection point according to its definition, but side change of the tangent only for $n$ odd. Visualize contact of order $k$ as limit of $k+1$ intersection points.



Local behavior at higher inflection points



Osculating circle at vertex (left) has contact of order 3 with ellipse and at a general point (right) has contact of order 2.

# Surfaces in $P^3$

Homogeneous parameterization

$(u,v) \mapsto \mathsf{s}(u,v) = (s_0(u,v), \ldots, s_3(u,v)),\ (u,v) \in D \subset \mathbb{R}^2.$

Projective subspace spanned by $s(u,v) = \mathsf{s}(u,v)\mathbb{R}$ and the 'partial derivative points'

$$s_u(u,v) = \frac{\partial}{\partial u}\mathsf{s}(u,v)\mathbb{R},\ s_v(u,v) = \frac{\partial}{\partial v}\mathsf{s}(u,v)\mathbb{R},$$

is called tangent space at $s(u,v)$. If its dimension equals 2 (tangent plane) the surface point is called regular.

Analogously as for curves: contact order between surfaces at a point. Contact order $k$ between curve $c$ and surface $s$: on the surface $s$ there exists a curve $\bar{c}$ which has contact of order $k$ with $c$.

Example: asymptotic tangents (at parabolic or hyperbolic surface points) have contact of order $\geq 2$ with the surface.

# Dual curves in $P^2$

Curve $c(t) = \mathsf{c}(t)\mathbb{R}$ described as family of its tangents (dual curve),

$$c^\star(t) = \mathbb{R}\mathsf{u}(t) = \mathbb{R}(\mathsf{c}(t) \times \dot{\mathsf{c}}(t)).$$

The iterated dual $c^{\star\star}$ is calculated as

$$\begin{aligned} c^{\star\star} &= (\mathsf{c}^\star \times \dot{\mathsf{c}}^\star)\mathbb{R} = (\mathsf{c} \times \dot{\mathsf{c}}) \times (\dot{\mathsf{c}} \times \dot{\mathsf{c}} + \mathsf{c} \times \ddot{\mathsf{c}})\mathbb{R} \\ &= \det(\mathsf{c}, \dot{\mathsf{c}}, \ddot{\mathsf{c}})\mathsf{c}\mathbb{R} = \mathsf{c}\mathbb{R} = c. \end{aligned}$$

The fact that iterated duality gives the original curve again leads to the notion of curve as a self-dual object, which can be described either as family of points or family of lines (i.e., tangents).

Remark: Dual parameterization to a rational parameterization of degree $n$ is rational of degree $\leq 2n-2$ ($\Longrightarrow$ dual Bézier form).

Example

The curve

$$c(t) = \mathsf{c}(t)\mathbb{R} = (1, t^m, t^n)\mathbb{R}, \ \ (m < n)$$

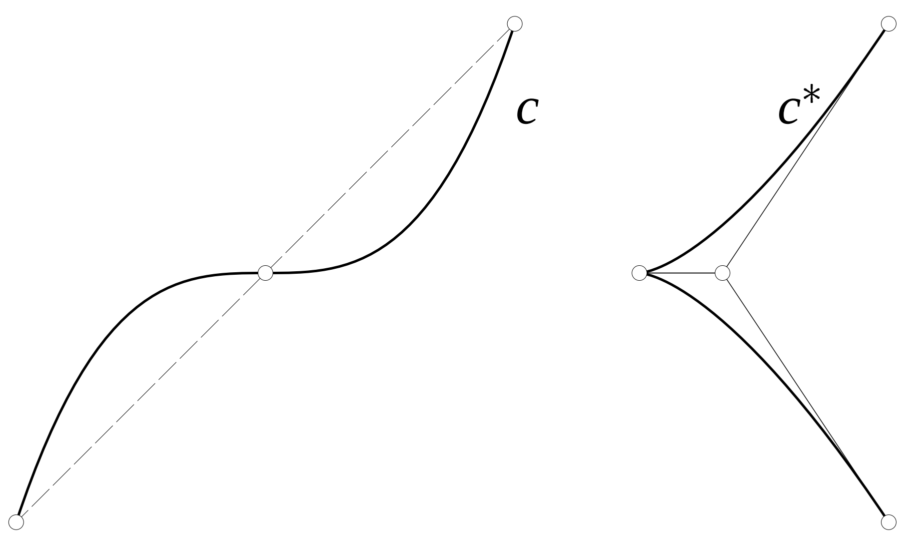
has the line representation

$$c^\star(t) = \mathbb{R}\mathsf{c}(t) \times \dot{\mathsf{c}}(t) = \mathbb{R}((n-m)t^{m+n-1}, -nt^{n-1}, mt^{m-1}).$$

Renormalization yields the equivalent form

$$c^\star(t) = \mathbb{R}((n-m)t^n, -nt^{n-m}, m).$$

$(m,n) = (1,3)$ gives a curve with an inflection at $t = 0$. Dual representation is $(2t^3, -3t^2, 1)$; it has a cusp at $t = 0$ (if we view it as point set; see Figure). ($\Longrightarrow$ algorithms of J. Hoschek for detection of inflection points at hand of cusps of their duals).




Inflection and cusp are dual to each other. Left: a line intersects in three points. Right: A point is incident with three tangents.

# References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