Decomposition of optimal investments.

- $$\pi_s^* = \pi_s^m + H_s$$
  $$\pi_s^m \downarrow \text{ myopic}, \quad H_s \downarrow \text{ "hedge"}$$

- Supporting claim

$$\Lambda(Y) = -\int_t^T \frac{1}{2} \frac{\lambda^2(Y_s, s)}{\gamma}\, ds$$

$$\downarrow$$

$$h_t(\Lambda) = \mathcal{E}_{\tilde{Q}}\left(-\int_t^T \frac{1}{2} \frac{\lambda^2(Y_s, s)}{\gamma}\, ds \right)$$

$$= -\mathcal{E}_{\hat{Q}}\left(\int_t^T \frac{1}{2} \frac{\lambda^2(Y_s, s)}{\gamma}\, ds \right)$$

- Variational

$$h_t + \frac{1}{2} a^2(y,t) h_{yy} + \left(b(y,t) - \rho \lambda(y,t) a(y,t) \right) h_y$$
$$+ \frac{1}{2} \gamma (1-\rho^2) a^2(y,t) h_y^2 = \frac{1}{2} \frac{\lambda^2(y,t)}{\gamma}$$

Payoff decomposition

$$\Lambda(Y_s\,;\, t \le s \le T) = \int_t^T -\frac{1}{2}\,\frac{\lambda^2(Y_s,s)}{\gamma}\,ds$$

$$= h(Y_t,t) + \int_t^T H_s\,\frac{dS_s}{S_s}$$

$$-\int_t^T \frac{1}{2}\gamma(1-\rho^2)\,a^2 h_y^2(Y_s,s)\,ds + \int_t^T \sqrt{1-\rho^2}\,a\,h_y(Y_s,s)\,dW_s^{1,1}$$

Optimal investment

$$\pi_s^* = \pi_s^m + H_s = \frac{\mu(Y_s,s)}{\gamma\sigma^2(Y_s,s)} + \rho\,\frac{a(Y_s,s)}{\sigma(Y_s,s)}\,h_y(Y_s,s)$$

$$h_y(y,t) = \frac{\partial}{\partial y}\,\mathcal{E}_{\tilde{Q}}\left(-\int_t^T \frac{1}{2}\,\frac{\lambda^2(Y_s,s)}{\gamma}\,ds \,\Big/\, Y_t = y\right)$$

Two interesting observations for "hedges" under indifference

$X_s^{0,*}$, $X_s^{\Lambda,*}$ optimal wealth without/with claim

- Residual wealth : $L_s = X_s^{0,*} - X_s^{\Lambda,*}$

  $L_t = h(Y_t, t) \qquad t \le s \le T$

- Residual risk : $R_s = L_s - h(Y_s, s)$

  $R_t = 0 \qquad t \le s \le T$

$$\downarrow$$

- $dL_s = \rho \dfrac{a(Y_s,s)}{\sigma(Y_s,s)} h_y(Y_s,s)\left(\mu(Y_s,s)ds + \sigma(Y_s,s)dW_s^1\right)$

- $E_{\mathbb{P}}\left(-e^{-\gamma R_s}\right) = 0$

Sensitivity analysis of the excess risky demand

$$H_s = H(Y_s, s) = \rho \frac{a(Y_s, s)}{\sigma(Y_s, s)} h_y(Y_s, s)$$

$$\begin{cases} h_t + \frac{1}{2} a^2(y,t) h_{yy} + \left(b(y,t) - \rho \lambda(y,t) a(y,t)\right) h_y \\ \quad + \frac{1}{2} \gamma (1-\rho^2) a^2(y,t) h_y^2 = \frac{1}{2} \frac{\lambda^2(y,t)}{\gamma} \\ h(y,T) = 0 \end{cases}$$

- Reduction if Y-coefficients autonomous.

$$dY_s = k(Y_s) ds + dW_s$$

$$h_t + \frac{1}{2} h_{yy} + \left(k(y) - \rho \lambda(y)\right) h_y + \frac{1}{2} \gamma (1-\rho^2) h_y^2 =$$

$$= \frac{1}{2} \frac{\lambda^2(y,t)}{\gamma}$$

- Term $g = h_y$ solves a Burger's eqn.

$$\pi_s^* = \frac{\mu(Y_s,s)}{\gamma\sigma^2(Y_s,s)} + \rho \frac{1}{\sigma(Y_s,s)} h_y(Y_s,s)$$

$$H(y,t) = \rho \frac{1}{\sigma(Y_s,s)} h_y(y,t)$$

i) Sign of H w.r.t. monotonicity of λ

$\rho > 0$ $\quad \lambda \uparrow (\downarrow)$ $\quad H < 0$

$\rho < 0$ $\quad \lambda \downarrow (\uparrow)$ $\quad H > 0$

ii) Behavior of H w.r.t. trading horizon

$\rho > 0$ $\quad \lambda \uparrow (\downarrow)$ $\quad H \downarrow (\uparrow)$ w.r.t. $(T-t)$

$\rho < 0$ $\quad \lambda \downarrow (\uparrow)$ $\quad H \uparrow (\downarrow)$ w.r.t. $(T-t)$

extension of results of Campbell-Viceira, Wachter.

iii) Monotonicity of H w.r.t y

$\rho > 0 \qquad (\lambda^2)_{yy} > 0 \qquad H \downarrow$ w.r.t y

$\rho < 0 \qquad (\lambda^2)_{yy} > 0 \qquad H \uparrow$ w.r.t. y

iv) Bounds on investment rules

$$H(y,t) = \frac{\rho}{1-\rho^2} \frac{1}{\sigma(y)} \frac{\hat{u}_y(y,t)}{\hat{u}(y,t)}$$

$$\hat{u}_t + \frac{1}{2}\hat{u}_{yy} + (b(y) - \rho\lambda(y))\hat{u}_y = \frac{1}{2}(1-\rho^2)\lambda^2(y)\hat{u}$$

$$H(y,t) = \frac{\rho}{1-\rho^2} E_{\overline{Q}}\left(-\int_t^T e^{\overline{C}(Y_s)}(1-\rho^2)(\lambda\lambda_y)(Y_s)\left(\frac{\hat{u}(Y_s,s)}{\hat{u}(y,t)}\right)ds \Big/ Y_t = y\right)$$

$\overline{Q}$, $\overline{C}(\cdot)$ known

Equivalent classes under incompleteness

(I) Stochastic factor – time additive utility

$$dS_s = \mu(Y_s,s)S_s\,ds + \sigma(Y_s,s)\,S_s\,dW_s^1$$

$$U(X_T) = -e^{-\gamma X_T}$$

$$V(x,y,t) = \sup E\left[U(X_T)/X_t=x, Y_t=y\right] = -e^{-\gamma x}\,\hat{u}(y,t)^{1/1-\rho^2}$$

(II) Constant Sharpe ratio – recursive utility

$$dS_s = \mu S_s\,ds + \sigma S_s\,dW_s$$

$$\bar{U}\left(X_T, \bar{Y}_s; t \le s \le T\right) = \left(-e^{-\gamma X_T}\right)\exp\left(-\int_t^T \frac{1}{2}\left(\lambda^2(Y_s) - \frac{\mu^2}{\sigma^2}\right)ds\right)$$

$$d\bar{Y}_s = \bar{b}(Y_s)\,ds + \bar{a}(Y_s)\,dW_s$$

If the factors Y, $\bar{Y}$ have diffusion dynamics

$$a(y,t) \equiv \bar{a}(y,t)$$

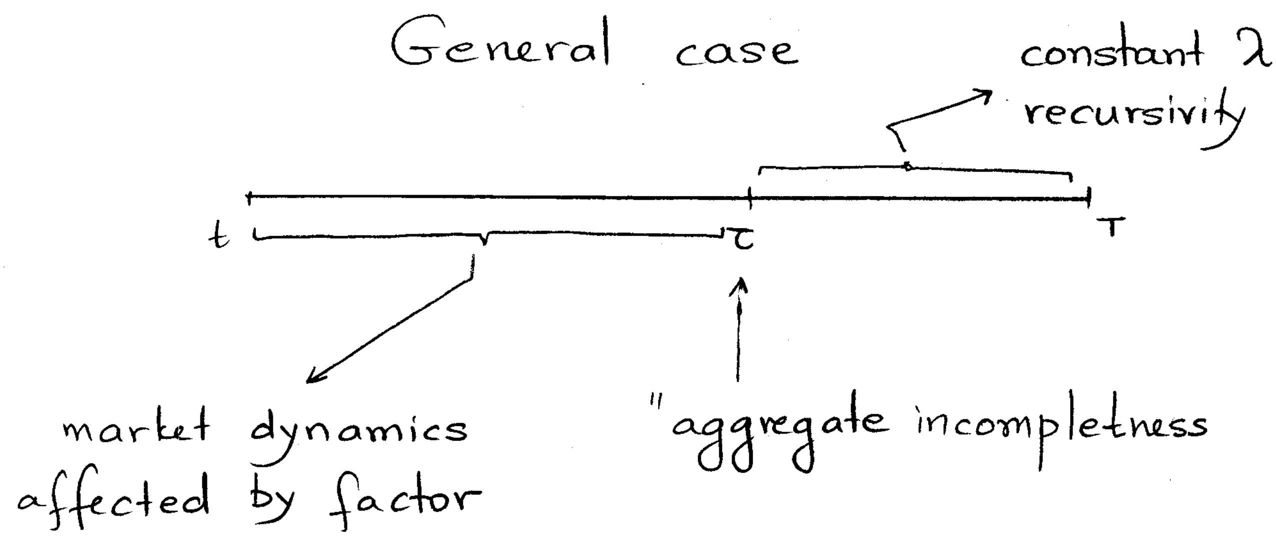
$$b(y,t) - \rho\lambda(y,t)\,a(y,t) \equiv \bar{b}(y,t) - \rho\bar{\lambda}\,a(y,t),$$

then

$$V(x,y,t) \equiv \bar{V}(x,y,t)$$

(I) : stochastic factor affects the dynamics of the traded asset

(II) : stochastic factor affects the local risk preferences.



$\downarrow$

model mispecification

$\downarrow$

possible inconsistencies when we price by indifference.

Constant Relative Risk Aversion

(CRRA) preferences

- $dX_s = \mu(Y_s, s)\pi_s \, ds + \sigma(Y_s, s)\pi_s \, dW_s^1$

  $dY_s = b(Y_s, s)\, ds + \alpha(Y_s, s)\, dW_s$

- $$V(x,y,t) = \sup_{\pi} E\left(\frac{X_T^{\gamma}}{\gamma} \Big/ X_t = x, Y_t = y \right) \quad (\gamma < 0)$$

$$\Big\downarrow \text{ distortion}$$

$$V(x,y,t) = \frac{x^{\gamma}}{\gamma} \left(E_{\mathbb{Q}^{\gamma}} \left(e^{\frac{1}{\delta}\int_t^T - \frac{|\gamma|}{1+|\gamma|} \lambda_s^2 \, ds} \right) \Big/ Y_t = y \right)^{\delta}$$

$$\delta = \frac{1-\gamma}{1-\gamma+\rho^2\gamma}$$

- $\mathbb{Q}^{\gamma}$ minimizes $E_{\mathbb{P}}\left(\left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right)^{\frac{|\gamma|}{1+|\gamma|}} \right)$

- The problem is isomorphic to the one of exponential preferences.

Supporting claim: $\Lambda(Y_s ; t \le s \le T) = \int_t^T -\frac{1}{2\bar{\gamma}} \bar{\lambda}^2(Y_s, s) ds$

$$\bar{\lambda}(Y_s, s) = \lambda(Y_s, s) \sqrt{\frac{|\gamma|}{1+|\gamma|}}$$

$$\bar{\rho} = \rho \sqrt{\frac{|\gamma|}{1+|\gamma|}}$$

$$\bar{\gamma} = 1 + |\gamma|$$

$$\downarrow$$

Optimal portfolio process π_s^*

$$\pi_s^* = \pi_s^m + H_s$$

$$\pi_s^m = \frac{1}{1+|\gamma|} \frac{\mu(Y_s, s)}{\sigma^2(Y_s, s)} , \quad H_s = \rho \frac{a(Y_s, s)}{\sigma(Y_s, s)} h_y(Y_s, s)$$

- Indifference price of the supporting claim

$$h_t + \frac{1}{2}a^2 h_{yy} + (b - \bar{\rho}\bar{\lambda}a)h_y + \frac{1}{2}\bar{\gamma}(1-\bar{\rho}^2)a^2 h_y^2 = \frac{1}{2\bar{\gamma}}\bar{\lambda}^2$$

$$h(y,T) = 0$$

$$\downarrow$$

$$h(y,t) = \mathcal{E}_{\mathbb{Q}^\gamma}\left(-\int_t^T \frac{1}{2\bar{\gamma}}\bar{\lambda}^2(Y_s, s)\,ds \,/\, Y_t = y\right)$$

- Traded versus non-traded market risk

$$\frac{d\mathbb{Q}^\gamma}{d\mathbb{P}} = \exp\left(-\int_t^T \lambda_s\,dW_s^1 - \int_t^T \lambda_s^\perp\,dW_s^{1,\perp} - \frac{1}{2}\int_t^T \left(\lambda_s^2 + (\lambda_s^\perp)^2\right)ds\right)$$

$$\downarrow$$

$$\pi_s^* = \frac{1}{(1-\gamma)\sigma(Y_s, s)}\left(\lambda_s(Y_s, s) - \frac{\rho}{\bar{\rho}}\lambda^\perp\right)$$

Summary

$$\pi_s^* = \pi_s^m + H_s$$

- H_s : hedge of a supporting claim priced by indifference

- $$\pi_s^* = \frac{1}{\alpha\,\sigma(Y_s, s)}\left(\lambda(Y_s, s) - \beta\,\lambda^{\perp}\right)$$

α, β constants depending on risk preferences and correlation

$\lambda, \lambda^{\perp}$ components of the density $\frac{d\mathbb{Q}^*}{d\mathbb{P}}$

$\mathbb{Q}^* \longrightarrow$ MEM, $q(\gamma)$-MM.

How far can we go?

Open problems

- Intermediate consumption

$$V(x,y,t) = \sup_{(\pi,c)} E\left[\int_t^T C_s^\gamma \, ds + \frac{X_T^\gamma}{\gamma} \Big/ X_t = x, Y_t = y \right]$$

- Limited results: Campbell-Viceira, Wachter

Distortion method leads to a reaction-diffusion eqn

- Labor income

$$V(x,y,t) = \sup E\left[\left(X_T + g_2(Y_T)\right)^\gamma \Big/ X_t = x, Y_t = y \right]$$

$$dX_s = \mu \pi_s \, ds + \sigma \pi_s \, dW_s^1 + g_1(Y_s) \, ds$$

$$dY_s = b(Y_s, s) \, ds + a(Y_s, s) \, dW_s$$

(Koo, Duffie-Z., Werner, El Karoui-Jeanblanc...)

Optimal investment behavior with intermediate consumption

(Stoikov - Z.)

$$V(x,y,t) = \sup_{(\pi,c)} E\left[\int_t^T U_1(C_s)\,ds + U_2(X_T) \Big/ X_t = x, Y_t = y \right]$$

$$\downarrow$$

$$V_t + \underbrace{\max_{c \geq 0} \left(-cV_x + U(c) \right)}_{F_1(DV)} \qquad \text{(HJB)}$$

$$+ \underbrace{\max_{\pi} \left(\frac{1}{2}\sigma^2\pi^2 V_{xx} + \pi\left(\rho a \sigma V_{xy} + \mu V_x\right) + \frac{1}{2}a^2 V_{yy} + b V_y \right)}_{F_2(DV,\, D^2V)}$$

$$\text{(HJB)} : \quad V_t + \left(F_1 + F_2\right)\left(DV, D^2V\right) = 0$$

Trotter-Kato product.

- Solution operators : $S_{F_1+F_2}$, S_{F_1} , S_{F_2}

$\mathcal{P} = \{ \ t = t_1 < t_2 < \cdots < t_n = T \ \}$ partition

- Approximations

$$V^{\mathcal{P}}(x,y,t) = \begin{cases} \left[S_{F_2}(t-t_i) S_{F_1}(t-t_i) V^{\mathcal{P}}(x,y,t_{i+1}) \right](x,y,t) & t \in [t_i, t_{i+1}) \\ U(x) & t = T \end{cases}$$

$S_{F_1+F_2} \longrightarrow$ investment / consumption

$S_{F_1} \longrightarrow$ (deterministic) pure consumption

$S_{F_2} \longrightarrow$ pure investment / stochastic factor

Convergence of $V^{\mathcal{P}}$ to the value function

$$V^{\mathcal{P}}(x,y,t) = S^{(\Delta t)}(V^{\mathcal{P}})$$

Properties of the scheme

i) Monotonicity

$$S^{(\Delta t)}(u) \geq S^{(\Delta t)}(v) \qquad u \geq v$$

ii) Translation invariance

$$S^{(\Delta t)}(u+k) = S^{(\Delta t)}(u) + k \qquad k \in \mathbb{R}$$

iii) Consistency

$$\lim_{\Delta t \to 0} \frac{\phi - S^{(\Delta t)}(\phi)}{\Delta t} = \phi_t + (F_1 + F_2)\phi$$

(difficult due to noncompactness of portfolio set)

$$U(x) = \frac{x^\gamma}{\gamma}$$

- 1^{st} update:

$$\begin{cases} V^1_t + \max\limits_{c \geq 0} \left[-c V^1_x + U_1(c) \right] = 0 & \text{(H-J eqn)} \\ V^1(x, y, T) = U_1(x) \end{cases}$$

$$\Big\downarrow \text{ Lax formula}$$

$$V^1(x, y, t\,; T)$$

- 2^{nd} update:

$$V_s + \max_{\pi} \left(\frac{1}{2} \sigma^2 \pi^2 V_{xx} + \pi \left(\rho a \sigma V_{xy} + \mu V_x \right) \right)$$
$$+ \frac{1}{2} a^2 V_{yy} + b V_y = 0 \qquad s \in (t, T)$$

$$V(x, y, T) = V^1(x, y, t\,; T)$$

CRRA Preferences

$T-2\Delta t \qquad\qquad T-\Delta t \qquad\qquad T$

- $t \in (T-\Delta t, T]$

1st update

$$\left.\begin{aligned} & V_t^1 + \max_{c \geq 0}\left(-cV_x + c^\gamma\right) = 0 \\ & V(x,y,T) = \frac{x^\gamma}{\gamma} \end{aligned}\right\} \Longrightarrow V^1(x,y,t) = \frac{x^\gamma}{\gamma} e^{k(T-t)}$$

2nd update

$$V_s^2 + \max_{\pi}\left(\frac{1}{2}\sigma^2\pi^2 V_{xx}^2 + \pi\left(\rho a \sigma V_{xy}^2 + \mu V_x^2\right)\right)$$
$$+ \frac{1}{2}a^2 V_{yy}^2 + b V_y^2 = 0$$

$$V^2(x,y,T) = \frac{x^\gamma}{\gamma} e^{k(T-t)}$$

$$\downarrow$$

$$V^{\mathcal{P}}_{(T-\Delta t, T)}(x,y,t) = \frac{x^\gamma}{\gamma} e^{k(T-t)} \left(E_{\mathbb{Q}^\gamma}\left(e^{\frac{1}{\delta}\int_t^T \frac{\gamma}{1-\gamma}\lambda^2(Y_s,s)\,ds} \Big/ Y_t = y \right)\right)^\delta$$

- $t \in (T-2\Delta t, T-\Delta t)$

1^{st} update

$$V_t^1 + \max_{c \geq 0}\left(-cV_x + c^\gamma\right) = 0$$

$$V(x,y,T-\Delta t) = \frac{x^\gamma}{\gamma} e^{\Delta t}\left(E_{Q^\gamma}\left(e^{\frac{1}{\delta}\int_{T-\Delta t}^{T} \frac{\gamma}{1-\gamma}\lambda^2(Y_s,s)\,ds} \Big/ Y_{T-\Delta t} = y\right)\right)^\delta$$

$$= \frac{x^\gamma}{\gamma} e^{\Delta t} G(Y;(T-\Delta t, T))$$

2^{nd} update

$$V_s + \max_{\pi}\left(\frac{1}{2}\sigma^2\pi^2 V_{xx} + \pi\left(\rho a \sigma V_{xy} + \mu V_x\right)\right)$$

$$+ \frac{1}{2}a^2 V_{yy} + bV_y = 0$$

$$V(x,y,T-\Delta t) = \frac{x^\gamma}{\gamma} e^{\Delta t} G(Y;(T-\Delta t, T))$$

$$\downarrow$$

$$V(x,y,t) = \frac{x^\gamma}{\gamma} e^{\Delta t} E_{Q^\gamma}\left(G(Y;(T-\Delta t, T))\, e^{\frac{1}{\delta}\int_t^{T-\Delta t} \frac{\gamma}{1-\gamma}\lambda^2(Y_s,s)\,ds} \Big/ Y_t = y\right)^\delta$$

Approximation $V^{\mathcal{P}}(x,y,t)$

$$V^{\mathcal{P}}(x,y,t) = \frac{x^\gamma}{\gamma} e^{k(T-t)} \left(\mathbb{E}_{\mathbb{Q}^\gamma} \left(\cdots \left(\frac{1}{\delta} \int_{T-(n-1)\Delta t}^{T-n\Delta t} \frac{1}{2} \lambda^2(Y_s)\, ds \Big/ \mathcal{F}_{T-(n-1)\Delta t} \right) \cdots \right) \right)^\delta$$

$\mathbb{E}_{Q^\gamma}\left(z\, e^{\frac{1}{\delta}\int \frac{\lambda^2}{\delta}\cdots} \right)$: nested conditional distorted expectations of the aggregate (traded) market price of risk.

Work in progress

- Convergence of $\mathcal{P}$-optimal policies to the ones of the original problem
- Identification of supporting claims
- Connection between $\mathcal{E}_{Q^\gamma}$ and $\mathbb{E}_{Q^\gamma}$ through supply-demand curves.