Optimal investments in markets with stochastic opportunity sets

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Stochastic Optimization

- Equilibrium asset pricing
- Derivative valuation
- Investments
- Optimal resource allocation

Structure

$$\begin{tikzcd}
\text{Market} \arrow[rr, leftrightarrow] & & \text{Preferences} \arrow[dl, leftrightarrow] \\
& \text{Objective} \arrow[ul] &
\end{tikzcd}$$

Model specification

Market dynamics

$(\Omega, \mathcal{F}, \mathbb{P})$

- $dS_s = \mu(S_s, s) S_s \, ds + \sigma(S_s, s) S_s \, dW_s^1$

- $dS_s = \mu(Y_s, s) S_s \, ds + \sigma(Y_s, s) S_s \, dW_s^1$

  $dY_s = b(Y_s, s) ds + a(Y_s, s) dW_s$

  $\rho = \mathrm{cor}(W^1, W)$

$$\vdots$$

- $dS_s = \mu(\alpha_s, \pi_s, S_s) S_s \, ds + \sigma(\alpha_s, \pi_s, S_s) S_s \, dW_s^1$

  α_s = policy of player I (fund manager)

  π_s = policy of player II (investor)

  Non-zero sum stochastic differential games
  Partial SDEs

- $$dS_s = \mu(Y_s, s) S_s \, ds + \sigma(Y_s, s) S_s \, dW_s^1$$

$$dY_s = b(Y_s, s) \, ds + a(Y_s, s) \, dW_s^2$$

$$dZ_s = d(Z_s, s) \, ds + e(Z_s, s) \, dW_s^3$$

Y_s : stochastic factor

Z_s : contract / project

Endogeneous / Exogeneous Incompleteness

$$\begin{tikzcd}
 & W^1 \text{ (traded asset)} \arrow[dl, leftrightarrow, "\rho^{12}"'] \arrow[dr, leftrightarrow, "\rho^{13}"] & \\
\underset{\text{(stochastic factor)}}{W^2} \arrow[rr, leftrightarrow, "\rho^{23}"'] & & \underset{\text{(non-traded asset)}}{W^3}
\end{tikzcd}$$

Preferences

- Time additivity
- Recursivity
- "Mental accounting"

Objectives / Criteria

- Terminal wealth
- Intermediate consumption
- Exogeneous (labor) income

The analysis of risks and identification of prices relies heavily on the proper understanding of optimal behavior

Questions

How the optimal investment / consumption behaves in terms of

- Preferences
- Trading horizon
- Sharpe ratio of traded assets
- Market price of risk

Broader perspective

$$\begin{array}{ccc} & \text{Optimal behavior} & \\ \swarrow & & \searrow \\ \text{Preferences} & & \text{Market parameters} \end{array}$$

Campbell-Viceira, Wachter, Liu, Barberis, Brandt, Kim-Omberg, Hodrick

Optimal behavior

Objectives

- Universal representation
- Qualitative analysis
- Sensitivity
- Robustness
- Inverse problems

Representative model

- $dS_s = \mu(Y_s, s) S_s \, ds + \sigma(Y_s, s) S_s \, dW_s^1$ (stock)

  $dY_s = b(Y_s, s) \, ds + a(Y_s, s) \, dW_s$ (stochastic factor)

  $\rho = cor(W^1, W)$

- $dB_s = r B_s \, ds$ (bond)

- $dX_s = r X_s \, ds + (\mu(Y_s, s) - r) \pi_s \, ds - C_s \, ds + \sigma(Y_s, s) \pi_s \, dW_s^1$

  (wealth)

- Criteria

  "$\sup_{\mathcal{A}} E\left[U(c, x) \right]$"

  $\sup_{\mathcal{A}} E\left[-e^{-\gamma X_T} / X_t = x, Y_t = y \right]$

  $\sup_{\mathcal{A}} E\left[\frac{X_T^\gamma}{\gamma} / X_t = x, Y_t = y \right]$

  $\sup_{\mathcal{A}} E\left[\int_t^T C_T^\gamma \, ds + \frac{X_T^\gamma}{\gamma} / X_t = x, Y_t = y \right]$

Related measures

$\mathbb{P}$: historical measure

$\mathbb{Q} \in \mathcal{Q}^e$

$$\min_{\mathbb{Q}} E_{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}} \ln \frac{d\mathbb{Q}}{d\mathbb{P}} \right) \quad ; \quad \tilde{\mathbb{Q}} \quad \text{MEM}$$

$$\min_{\mathbb{Q}} E_{\mathbb{P}}\left(- \ln \frac{d\mathbb{Q}}{d\mathbb{P}} \right) \quad ; \quad \hat{\mathbb{Q}} \quad \text{MMM}$$

$$\min_{\mathbb{Q}} E_{\mathbb{P}}\left(\left(\frac{d\mathbb{Q}}{d\mathbb{P}} \right)^{q(\gamma)} \right) \quad ; \quad \mathbb{Q}^{q(\gamma)} \quad q\text{-MM}$$

MEM : Frittelli, Delbaen et al., Rouge-El Karoui, Kabanov-Stricker

MMM : Föllmer-Schweizer

γ-MM : Hobson

Exponential preferences

- $dX_s = \mu(Y_s,s)\pi_s\,ds + \sigma(Y_s,s)\pi_s\,dW_s^1$

  $dY_s = b(Y_s,s)\,ds + a(Y_s,s)\,dW_s$

  ($r=0$; $\pi_s = \$$ in stock)

  $V(x,y,t) = \sup_{\pi} E\left[-e^{-\gamma X_T} / X_t = x, Y_t = y\right]$

- (HJB) : $V_t + \max_{\pi}\Big(\frac{1}{2}\sigma^2(y,t)\pi^2 V_{xx} +$

  $+ \pi\big(\rho a(y,t)\sigma(y,t)V_{xy} + \mu(y,t)V_x\big)\Big)$

  $+ \frac{1}{2}a^2(y,t)V_{yy} + b(y,t)V_y = 0$

- $\pi^*(x,y,t) = -\frac{\mu(y,t)}{\sigma^2(y,t)}\frac{V_x(x,y,t)}{V_{xx}(x,y,t)} - \rho\frac{a(y,t)}{\sigma(y,t)}\frac{V_{xy}(x,y,t)}{V_{xx}(x,y,t)}$

  $\downarrow$ myopic

  $\downarrow$ excess demand for risky asset.

Two representations

- $V(x,y,t) = -\, e^{-\gamma x - \tilde{u}(y,t)}$

  $\tilde{u}$: relative entropy distance ; $\tilde{Q}$: MEM

$$u(y,t) = E_{\tilde{Q}}\left(\int_t^T \frac{1}{2}\left(\lambda^2(Y_s,s) + \lambda_\perp^2(Y_s,s)\right) ds \,/\, Y_t = y \right)$$

Delbaen et al., Kabanov-Stricker, Frittelli, Rouge - El Karoui ...

- $V(x,y,t) = -\, e^{-\gamma x}\, \hat{u}(y,t)^{1/1-\rho^2}$

$$\hat{u}(y,t) = E_{\hat{Q}}\left(e^{-(1-\rho^2)\int_t^T \frac{1}{2}\lambda^2(Y_s,s)\,ds} \,/\, Y_t = y \right)$$

- $\hat{Q}$: MMM

$$\begin{cases} \hat{u}_t + \frac{1}{2}a^2(y,t)\,\hat{u}_{yy} + \left(b(y,t) - \rho\lambda(y,t)a(y,t)\right)\hat{u}_y = \frac{1}{2}(1-\rho^2)\lambda^2\hat{u} \\ \hat{u}(y,T) = 0 \end{cases}$$

(Stoikov-Z., distortion method)

MEM $\longleftrightarrow$ MMM

Reconciling the two formulae yields

$$H(\tilde{\mathbb{Q}}/\mathbb{P}) = -\frac{1}{1-\rho^2} \ln E_{\hat{\mathbb{Q}}}\left(e^{-(1-\rho^2)\int_0^T \frac{1}{2}\lambda^2(Y_s,s)\,ds} \right)$$

Optimal investment

- $$\pi^*(x,y,t) = \frac{1}{\gamma}\frac{\mu(y,t)}{\sigma^2(y,t)} + \rho\,\frac{a(y,t)}{\sigma(y,t)}\,\frac{1}{\gamma(1-\rho^2)}\,\frac{\hat{u}_y(y,t)}{\hat{u}(y,t)}$$
- $$\pi_s^* = \pi_s^m + H_s$$

$$\pi_s^m = \frac{1}{\gamma}\,\frac{\mu(Y_s,s)}{\sigma^2(Y_s,s)}$$

$$H_s = \rho\,\frac{a(Y_s,s)}{\sigma(Y_s,s)}\,\frac{1}{\gamma(1-\rho^2)}\,\frac{\hat{u}_y(Y_s,s)}{\hat{u}(Y_s,s)}$$

Complete market case

$\rho^2 \longrightarrow 1$

$$\hat{u}(y,t) = \left(E_{\hat{\mathbb{Q}}} \, e^{-(1-\rho^2)\int_t^T \lambda^2(Y_s,s)\,ds} \Big/ Y_t = y \right)^{1/1-\rho^2}$$

$$\longrightarrow \quad e^{-E_{\mathbb{P}^*}\left(\int_t^T \lambda^2(S_s,s)\,ds \,/ S_t = S\right)}$$

- $dS_s = \mu(S_s,s)S_s\,ds + \sigma(S_s,s)S_s\,dW_s^1$

  $S \equiv Y \; ; \; a \equiv \sigma S$

- $\rho \dfrac{a(y,t)}{\sigma(y,t)} \dfrac{1}{\gamma(1-\rho^2)} \dfrac{\hat{u}_y(y,t)}{\hat{u}(y,t)} \longrightarrow S \, \dfrac{(-u_s)}{\gamma}$

$$\begin{cases} u_t + \frac{1}{2}\sigma^2 S^2 u_{ss} = \frac{1}{2} \dfrac{\lambda^2(S,t)}{\gamma} \\ u(S,T) = 0 \end{cases}$$

- $\pi_s^* = \text{myopic} + \text{hedge}$

Supporting claim: $\Lambda(S) = -\displaystyle\int_t^T \frac{1}{2} \frac{\lambda^2(S_s,s)}{\gamma}\,ds$

Path-dependent claim maturing at T.

Incomplete market case

Analogous optimal investment decomposition

$$\pi_s^* = \pi_s^m + H_s$$

Need to specify

- Supporting claim
- Appropriate valuation method

- Path-dependent claim $\left(G(\lambda(Y_s ; t \le s \le T)\right)$
- Valuation by indifference.

Indifference prices of path-dependent claims

(joint with M. Musiela)

$$C(Y) = \int_t^T c_1(Y_s)\,ds + c_2(Y_T)$$

Buyer's price.

$$V^C(x,y,t) = \sup_{\pi} E\left[-e^{-\gamma\left(X_T + C(Y_s\,;\,t\le s\le T)\right)} \,/\, X_t = x,\, Y_t = y\right]$$

Indifference price : $h_t(C)$

$$V^0(x,y,t) = V^C(x - h_t(C), y, t)$$

(Recall that the claim matures at expiration T).

Indifference price

- Non-linear pricing functional

$$\mathcal{E}_Q(G/\cdot) = -\frac{1}{\gamma(1-\rho^2)} \ln E_Q\left(e^{-\gamma(1-\rho^2)G}/\cdot\right)$$

- $$h_t\left(c(Y_s\,;\,t \le s \le T)\right) = h(Y_t, t)$$

$$h(y,t) = \mathcal{E}_{\tilde{Q}}\left(c(Y_s\,;\,t \le s \le T)/Y_t = y\right)$$

$$= \mathcal{E}_{\hat{Q}}\left(c(Y) + \int_t^T \frac{1}{2}\frac{\lambda^2(Y_s,s)}{\gamma}\,ds \,/\, Y_t = y\right)$$

$$- \mathcal{E}_{\hat{Q}}\left(\int_t^T \frac{1}{2}\frac{\lambda^2(Y_s,s)}{\gamma}\,ds \,/\, Y_t = y\right)$$

$\tilde{Q}$: MEM

$\hat{Q}$: MMM

- Indifference price quasilinear pde.

$$h_t + \tilde{\mathcal{L}} h - \frac{1}{2}\gamma(1-\rho^2)a^2 h_y^2 + c_1(y,t) = 0$$

$$h(y,T) = c_2(y)$$

$$\tilde{\mathcal{L}} = \frac{1}{2}a^2\frac{\partial^2}{\partial y^2} + \left(b - \rho\lambda a + a^2\frac{\hat{u}_y}{\hat{u}}\right)\frac{\partial}{\partial y}$$

(Pham, Benth-Karlsen, Sircar-Z., Stoikov-Z.)

- Density of the MEM.

$$\gamma \to 0 \qquad h^{\gamma} \longrightarrow h^{0}$$

$$h^0(y,t) = E_{\tilde{Q}}(C(y)); \qquad v = -\frac{1}{1-\rho^2}\ln\hat{u}$$

$$\frac{d\tilde{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \lambda(Y_s,s)\,dW_s^1 - \int_0^T \sqrt{1-\rho^2}\,a(Y_s,s)\,v_y(Y_s,s)\,dW_s^{\perp} - \frac{1}{2}\int_0^T\left(\lambda^2(Y_s,s) + (1-\rho^2)a^2(Y_s,s)\,v_y^2(Y_s,s)\right)ds\right)$$

Valuation algorithm

Representation of indifference prices via a time-consistent pricing iterative scheme.

$$h_t(C) = \mathcal{E}^{(t,T)}_{\tilde{Q}}(C)$$

- Key properties.

$$h_t(C) = \mathcal{E}^{(t,s)}_{\tilde{Q}}\left(h_s(C)\right) \qquad \text{(semigroup)}$$

$$h_t\left(C_1(Y,S) + C_2(S)\right) = \mathcal{E}^{(t,T)}_{\tilde{Q}}\left(C_1(Y,S)\right)$$

$$+ E_{\tilde{Q}}\left(C_2(S)\right) \qquad \text{(translation invariance w.r.t. hedgeable risks)}$$

(Musiela - Z.)

- In general, closed form slns cannot (should not!) exist.

- Constant Sharpe ratio $\lambda = \frac{\mu}{\sigma}$

  $\tilde{Q} \equiv \hat{Q}$

  $c = c(y, S)$

- Indifference price pde:

  $$h_t + \tilde{\mathcal{L}} h - \frac{1}{2}\gamma (1-\rho^2) a^2(y,t) h_y^2 + c_1(y) = 0$$

  $$\tilde{\mathcal{L}} = \frac{1}{2}\sigma^2 S^2 h_{SS} + \rho \sigma a(y,t) S h_{Sy} + \frac{1}{2} a^2(y,t) h_{yy} + \left(b(y,t) - \rho \lambda a(y,t)\right) h_y$$

- Iterative scheme.

  $\tilde{Q}$: MEM

  "Infinitesimal" nonlinear pricing functional

$$\mathcal{E}_{\tilde{Q}}^{(t,t+dt)}(Z) = E_{\tilde{Q}}\left(-\frac{1}{\gamma} \ln E_{\tilde{Q}}\left(e^{-\gamma Z} / \mathcal{F}_t \vee \mathcal{F}^S_{t,t+\Delta t} \right) / \mathcal{F}_t \right)$$

Indifference price

$$h_t(C) = \mathcal{E}^{(t,t+dt)}\left(h_{t+dt} + \int_t^{t+dt} c_1(Y_s,s)\,ds \right)$$

$$h_t(C) = \mathcal{E}^{(t,t+dt)}\left(\mathcal{E}^{t+dt,t+2dt}\left(\cdots + \int_{t+dt}^{t+2dt} c_1(Y_s,s)\,ds \right) + \int_t^{t+dt} c_1(Y_s,s)\,ds \right)$$

$$h_{T-dt}(C) = \mathcal{E}^{(T-dt,T)}\left(c_2(Y_T) + \int_{T-dt}^{T} c_1(Y_s)\,ds \right)$$

$$\downarrow$$

$$h_t(C) = \mathcal{E}_{\tilde{Q}}^{(t,T)}\left(c_2(Y_T) + \int_t^T c_1(Y_s)\,ds \,/\, Y_t \right)$$

- Constitutive analogue to Black-Scholes for indifference prices
- A "Feyman-Kac" type representation for solutions to quasilinear pdes of quadratic nonlinearities

- State-dependent Sharpe ratio

Two effects

i) Compilation of entropy

ii) Dynamic risk-preferences
(term-structure of utility)

$$\downarrow$$

"Spot-utility" infinitesimal nonlinear pricing functional

$$\mathcal{E}^{(t,t+dt)}_{\tilde{Q}}(Z) = E_{\tilde{Q}}\left(F^{-1}\left(E_{\tilde{Q}}\left(F(Z, t+dt)/\mathcal{F}_t \vee \mathcal{F}^{S}_{t,t+dt}\right), t\right)/\mathcal{F}_t\right)$$

$\tilde{Q}$: MEM.