Optimal Risk Transfer under Dynamic Risk Measures

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Agenda

- New financial instruments and motivation of this study
- A "toy model" : the exponential utility framework
- Some results on risk measures : basic properties and new developments
- Optimal risk transfer problem
- Inf-convolution and BSDEs

Introduction and motivation

Development of new financial products

A new type of assets :

- Recent introduction of a new type of financial contracts with a **non-financial underlying** risk : “cat-bonds”, weather derivatives...
- **Illiquid** instruments, with an underlying asset which is **not traded** on financial markets.
- Underlying risk possibly related to a financial market risk.

Question

How to design derivatives, and price them in such a way that transaction may occur in an illiquid market ?

Interplay between finance and insurance :

⇒ Use of the knowledge of financial risk management to the management of other kinds of risk.

⇒ Use of the insurance technology to design structured products.

From now on, we will use the insurance risk transfer approach, and vocabulary.

Main question

What is the **"optimal" transfer of a non-tradable risk** between different agents having access to other possible investments ?

⇒ The notion of "optimality" requires a choice criterion. We consider first **exponential utility** and then **convex risk measures**.

Related works

- Convex risk measure : Foellmer and Schied, Artzner, Delbaen, Heath
- Indifference pricing : Musiela, Zariphopoulou, Davis, Rouge-NEK, Becherer, DGRSSS(six authors)
- Calibration and stress testing : Geman, Madan, Avellaneda.

An exponential utility “toy model”

Uncertainty is modelled via a standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$. T is the **time horizon**. All flows are capitalized till T by using the **capitalization factor** $\beta_{0,T}$.

Transaction involving two agents

$\star$ **Agent A** : at time T, agent A is exposed towards a **non-tradable** risk $\mathbf{\Theta}$ for an amount $\mathbf{X}(\mathbf{\Theta}, \omega)$ in the market scenario ω . It calls on an investor, agent B, to reduce its exposure by the **sale of a structured contract** $\mathbf{F}(\mathbf{\Theta}, \omega)$.

$\star$ **Agent B** : it pays a **premium** $\boldsymbol{\pi}$ at time 0 and receives in exchange the structure $\mathbf{F}$ at time T. Its initial wealth is denoted by $\mathbf{x}$.

Utility criterion

Both agents are **risk-averse**. Their respective attitude towards risk is modelled by an **exponential utility function** $U(z) = -\gamma \exp(-\frac{1}{\gamma} z), z \in R$ (with **risk tolerance** coefficient γ_A, resp. γ_B).

Relationship between both agents

Both agents do not have the same goals

⋆ Agent A looks for an **optimal reduction of its exposure**, that is for the structure (F, π) as to maximize its expected utility :

$$\max_{F \in \mathcal{X}, \pi} \mathbb{E}_{\mathbb{P}}(U_A(X - F + \pi\beta_{0,T}))$$

⋆ Agent B wants to **improve its expected utility** by doing the F-transaction, in the following sense

$$\mathbb{E}(U_B(F + (x - \pi)\beta_{0,T})) \geq \mathbb{E}(U_B(x\beta_{0,T}))$$

Using the properties of the exponential utility, the program becomes

$$\min_{\mathbf{F} \in \mathcal{X}, \pi} \gamma_{\mathbf{A}} \mathbb{E}\big(\exp[-\frac{\mathbf{1}}{\gamma_{\mathbf{A}}}(\mathbf{X} - \mathbf{F} + \pi\beta_{\mathbf{0},\mathbf{T}})]\big) \text{ given } \quad \mathbb{E}\big(\exp[-\frac{\mathbf{1}}{\gamma_{\mathbf{B}}}(\mathbf{F} - \pi\beta_{\mathbf{0},\mathbf{T}})]\big) \leq \mathbf{1}$$

Pricing Rule and Optimal structure

⇒ Binding the constraint leads immediately to the **optimal pricing rule**

$$\boxed{\pi^*(\mathbf{F})\beta_{0,T} = -\gamma_B \ln \mathbb{E}(\exp(-\tfrac{1}{\gamma_B}F)) = -\mathbf{e}_{\gamma_\mathbf{B}}(\mathbf{F})}$$

⇒ Using Lagrangian multiplier, the optimal structure is minimizing for

$$\mathbb{E}\big(\gamma_A \exp[-\frac{1}{\gamma_A}(X - F + \pi\beta_{0,T})] + \lambda \exp[-\frac{1}{\gamma_B}(F - \pi\beta_{0,T})]\big)$$

It is obtained as : $\boxed{F^* = \frac{\gamma_B}{\gamma_A+\gamma_B}X \qquad \mathbb{P}\ a.s.}$ + constant

Put $\gamma_C = \gamma_A + \gamma_B$. Then

$$\boxed{e_{\gamma_C}(X) = e_{\gamma_A}\big(X - (F^* - \pi^*_B\beta_{0,T})\big) = \inf_F \big(e_{\gamma_A}(X - F) + e_{\gamma_B}(F)\big) = E_{AB}(X)}$$

Remark : The optimal structure is Pareto optimal.

Risk measures : basic properties

Let $(\Omega, \mathcal{F})$ be a standard measurable space and $\mathcal{X}$ the linear space of bounded functions (including constant functions).

Definition : The functional ρ is said to be a **convex risk measure** in the sense of Fōllmer and Schied (2002) if for any X and Y in $\mathcal{X}$

◇ **Convexity** ; $\rho(\lambda X + (1-\lambda)Y) \leq \lambda\rho(X) + (1-\lambda)\rho(Y)$

◇ **Decreasing** monotonicity ; $\rho(X) \geq \rho(Y)$ if $X \leq Y$

◇ **Translation invariance** : $\forall m \in \mathbb{R}, \quad \rho(\mathbf{X}+\mathbf{m}) = \rho(\mathbf{X}) - \mathbf{m}.$

The following set plays a key role

- The **acceptance set** associated with ρ is defined as :

$$\mathcal{A}_\rho = \{\Psi \in \mathcal{X}, \quad \rho(\Psi) \leq 0\}$$

- ρ may be defined from the acceptance set by

$$\rho(\mathbf{X}) = \inf\{\mathbf{m}, \ \mathbf{m} + \mathbf{X} \in \mathcal{A}_\rho\}$$

Penalty function

The dual formulation of this convex functional is a key point for our study

Theorem : (FS) There exists a **penalty function** α taking values in $\mathbb{R} \cup \{+\infty\}$ s.t.

$$\forall \Psi \in \mathcal{X}, \quad \rho(\mathbf{\Psi}) = \sup_{\mathbb{Q}\in\mathcal{M}_{\mathbf{1},\mathbf{f}}}\{\mathbb{E}_{\mathbb{Q}}(-\mathbf{\Psi}) - \alpha(\mathbb{Q})\}$$

$$\forall \mathbb{Q} \in \mathcal{M}_{1,f}, \quad \alpha(\mathbb{Q}) = \sup_{\mathbf{\Psi}\in\mathcal{X}}\{\mathbb{E}_{\mathbb{Q}}(-\mathbf{\Psi}) - \rho(\mathbf{\Psi})\} = \sup_{\mathbf{\Psi}\in\mathcal{A}_\rho}\{\mathbb{E}_{\mathbb{Q}}(-\mathbf{\Psi})\}$$

where $\mathcal{M}_{1,f}$ is the set of all additive measures on $(\Omega, \mathcal{F})$, dual set of $\mathcal{X}$

Example : Entropic risk measure

$$\mathbf{e}_\gamma(\mathbf{X}) = \gamma\mathbb{E}\big(\exp(-\frac{1}{\gamma}X)\big) = \sup_{\mathbb{Q}\in\mathcal{M}_1} \big(\mathbb{E}_{\mathbb{Q}}(-X) - \gamma\mathbf{h}(\mathbb{Q}|\mathbb{P})\big)$$

where h is the **entropic function**

$$h(\mathbb{Q}|\mathbb{P}) = \begin{cases} \mathbb{E}_{\mathbb{P}}(\frac{d\mathbb{Q}}{d\mathbb{P}} \ln \frac{d\mathbb{Q}}{d\mathbb{P}}) & if\ \mathbb{Q} \ll \mathbb{P} \\ +\infty & otherwise \end{cases}$$

Risk measure generated by a convex subset $\mathcal{H}$

Let $\mathcal{H}$ be a convex set such that $\inf\{\mathbf{m} \in \mathbb{R},\ \exists \xi_{\mathbf{T}} \in \mathcal{H}\ \mathbf{s.t.}\ \mathbf{m} \geq \xi_{\mathbf{T}}\} > -\infty$.

- A convex risk measure $\nu^{\mathcal{H}}$ is defined by :

$$\forall \mathbf{\Psi} \in \mathcal{X} \qquad \nu^{\mathcal{H}}(\mathbf{\Psi}) = \inf\{\mathbf{m} \in \mathbb{R}, \exists \xi_{\mathbf{T}} \in \mathcal{H}\ \mathbf{s.t.} \quad \mathbf{m} + \mathbf{\Psi} \geq \xi_{\mathbf{T}}\}$$

- The associated penalty function is

$$\forall \mathbb{Q} \in \mathcal{M}_{1,f}, \quad \kappa^{\mathcal{H}}(\mathbb{Q}) = \sup_{\Psi \in \mathcal{H}} \{\mathbb{E}_{\mathbb{Q}}(-\Psi)\}$$

- Moreover,$\mathcal{H}$ is a cone, $\nu^{\mathcal{H}}$ is a **coherent risk measure** in the sense of Artzner and alii, and the penalty function is

$$\boxed{l^{\mathcal{H}}(\mathbf{Q}) = 0 \quad \text{if } \mathbf{Q} \in \mathcal{M}_{\mathcal{H}} \quad ; \quad +\infty \quad \text{otherwise}}$$

where $\mathcal{M}_{\mathcal{H}}$ is the set of all additive measures such that $\forall \xi \in \mathcal{H}$, $\mathbb{E}_{\mathbf{Q}}(\xi) \geq \mathbf{0}$.

Inf-convolution of risk measures

Main result

Theorem : Let ρ_1 and ρ_2 be two convex risk measures with respective penalty functions α_1 and α_2. Let $\boldsymbol{\rho_{1,2} = \rho_1 \Box \rho_2}$ be the **inf-convolution** of ρ_1 and ρ_2 defined by

$$\boxed{\Psi \to \rho_{1,2}(\Psi) \stackrel{def}{=} \inf_H [\rho_1(\Psi - H) + \rho_2(H)]}$$

and assume that $\rho_{1,2}(0) > -\infty$.
Then $\rho_{1,2}$ is a finite convex risk measure. The associated penalty function is given by

$$\boxed{\forall \mathbb{Q} \in \mathcal{M}_{1,f} \quad \alpha_{1,2}(\mathbb{Q}) = \alpha_1(\mathbb{Q}) + \alpha_2(\mathbb{Q})}$$

The related acceptance set $\mathcal{A}_{\rho_{1,2}}$ is the "pseudo-closure" of $\mathcal{A}_{\rho_1} + \mathcal{A}_{\rho_2}$.

Remark : Note also that

$$\rho_{1,2}(\Psi) = \inf \left\{ \rho_1(\Psi - H), \quad H \in \mathcal{A}_{\rho_2} \right\}$$

$\mathcal{H}$-reduced risk measure

Let $\mathcal{H}$ be a **cone** of $\mathcal{X}$ and ρ be a convex risk measure with penalty function α such that $\inf\{\rho(-H), H \in \mathcal{H}\} > -\infty$.
The inf-convolution $\rho^{\mathcal{H}}$ of ρ and $\nu^{\mathcal{H}}$ is the $\mathcal{H}$-reduced ρ defined as

$$\boxed{\rho^{\mathcal{H}}(\boldsymbol{\Psi}) \overset{def}{=} \rho \square \nu^{\mathcal{H}}(\Psi) = \inf\{\rho(\boldsymbol{\Psi} - \mathbf{H}), \mathbf{H} \in \mathcal{H}\} = \sup_{\mathbf{Q} \in \mathcal{M}^{\mathcal{H}}} \{\mathbb{E}_{\mathbf{Q}}(-\Psi) - \alpha(\mathbf{Q})\}}$$

is a convex risk measure with penalty function

$$\boxed{\alpha^{\mathcal{H}}(\mathbf{Q}) = \alpha(\mathbf{Q}) \quad \text{if} \quad \mathbb{E}_{\mathbf{Q}}(\mathbf{H}) \geq \mathbf{0}, \forall H \in \mathcal{H}\,; \quad +\infty \quad \text{otherwise}}$$

Comments :

⋆ A typical example of a cone is the set $\mathcal{V}_T$ of the gain processes associated with financial investments (most liquid part of the market).
$\rho^{\mathcal{V}_T} \overset{def}{=} \rho^m$ is the **market modified risk measure**.

⋆ The financial market plays there the same role as an intermediate agent having a risk measure $\nu^{\mathcal{V}_T}$.

Dilatation and semi-group property

Let ρ be a convex risk measure with penalty function α.

Definition : The associated **dilated risk measure** ρ_γ is defined by

$$\forall \Psi \in \mathcal{X} \qquad \rho_\gamma(\Psi) \stackrel{def}{=} \gamma \rho\left(\frac{1}{\gamma}\Psi\right) \qquad \text{with} \quad \alpha_{\rho_\gamma}(\mathbf{Q}) = \gamma \alpha(\mathbf{Q})$$

where $\gamma > 0$ is the **risk tolerance coefficient.**

Semi-group property w.r.to inf convolution

If ρ and ρ' are two risk measures

$$\rho_\gamma \Box \rho_{\gamma'} = \rho_{\gamma+\gamma'}, \quad \rho_\gamma \Box \rho'_\gamma = (\rho \Box \rho')_\gamma$$

Typical example : entropic risk measure.

Properties

1. $\rho_\gamma \Box \rho_{\gamma'}(X) = \inf_F\{\rho_\gamma(X-F)+\rho_{\gamma'}(F)\} = \rho_\gamma(X - \frac{\gamma'}{\gamma+\gamma'}\mathbf{X}) + \rho_{\gamma'}(\frac{\gamma'}{\gamma+\gamma'}\mathbf{X})$
   $\frac{\gamma'}{\gamma+\gamma'}\mathbf{X}$ is optimal, as in the entropic case.
2. ρ is a **coherent** risk measure if and only if, for any γ strictly positive, $\rho_\gamma \equiv \rho$. Moreover, $\forall \mathbf{n} \geq \mathbf{1} \qquad \rho^{\Box \mathbf{n}} = \rho$
3. Assume $\rho(0) = 0$ Then ρ_γ is a **decreasing function** of γ, with asymptotic behavior
   $\Rightarrow \ \rho_\infty \stackrel{def}{=} \lim_{\gamma\to\infty}\rho_\gamma$ is a **coherent** risk measure and

$$\rho_\infty(\Psi) = \sup\{\mathbb{E}_{\mathbf{Q}}(-\Psi) | \mathbf{Q} \in \mathcal{M}_{1,f},\ \alpha(\mathbf{Q}) = \mathbf{0}\}$$

$\Rightarrow \ \rho_0$ is simply the **super-pricing** rule of $-\Psi$:

$$\rho_0(\Psi) = \sup\{\mathbb{E}_{\mathbf{Q}}(-\Psi) | \mathbf{Q} \in \mathcal{M}_{1,f},\ \alpha(\mathbf{Q}) < \infty\}$$

Optimal Risk Transfer : the framework

Transaction involving two agents

⋆ **Agent A** : at a future time T, agent A is exposed towards a non-tradable risk $\mathbf{\Theta}$ for an amount $\mathbf{X}(\mathbf{\Theta},\omega)$ in the market scenario ω. It calls on an investor, agent B, to reduce its exposure by the **sale of a structured contract** $\mathbf{F}(\mathbf{\Theta},\omega)$.

⋆ **Agent B** : it pays a premium $\boldsymbol{\pi}$ at time 0 and receives in exchange the structure F at time T. Its initial wealth is denoted by x.

⋆ Both agents can also invest in **financial markets** via two **cones** of bounded terminal gains associated with self-financing investment strategies, $\boldsymbol{\mathcal{V}_T^{(A)}}$ and $\boldsymbol{\mathcal{V}_T^{(B)}}$.

⋆ Both agents assess their risk exposure using two **convex risk measures**, $\boldsymbol{\rho_A}$ and $\boldsymbol{\rho_B}$, with respective **penalty functions** α_A and α_B.

Optimal transfer : agents goals

Relationship between both agents :

1. Agent A looks for a **hedge of its exposure**. It wants to determine the structure (F, π, ξ_A) as to minimize the risk measure of its wealth :

$$\min_{F,\pi,\xi_A \in \mathcal{V}_T^{(A)}} \rho_A(X - F + \pi + \xi_A) = \min_{F,\pi} \min_{\xi_\mathbf{A} \in \mathcal{V}_\mathbf{T}^{(\mathbf{A})}} \rho_\mathbf{A}(\mathbf{X} - \mathbf{F} + \pi + \xi_\mathbf{A})$$

$$= \boxed{\min_{\mathbf{F},\pi} \rho_\mathbf{A}^\mathbf{m}(\mathbf{X} - \mathbf{F} + \pi)}$$

2. Agent B wants to **improve its risk measure** by doing the F-transaction. Its interest in doing the transaction may be written as :

$$\min_{\xi_B \in \mathcal{V}_T^{(B)}} \rho_B(F - \pi + x + \xi_B) \leq \min_{\xi_B \in \mathcal{V}_T^{(B)}} \rho_B(x + \xi_B)$$

or equivalently, using the cash invariance translation property and the market modified risk measure ρ_B^m :

$$\boxed{\rho_B^m(F - \pi) \leq \rho_B^m(0)}$$

Optimal pricing rule and residual risk measure

⋆ The optimal pricing rule is simply obtained by **binding the constraint** of Agent B and using the cash invariance property :

$$\boxed{\pi^*(F) = \rho_B^m(0) - \rho_B^m(F)}$$

⋆ The program $\min_F \rho_A^m(X - F + \pi)$ subject to $\rho_B^m(F - \pi) \leq \rho_B^m(0)$ may be reduced to (to within the constant $\rho_B^m(0)$)

$$\begin{aligned} R_{AB}^m(X) &\overset{def}{=} \min_F(\rho_A^m(X-F) + \rho_B^m(F)) \\ &= \rho_A^m \square \rho_B^m(X) = \rho_A \square \nu^{\mathcal{V}_T^{(A)}} \square \rho_B \square \nu^{\mathcal{V}_T^{(B)}}(X) \end{aligned}$$

⇒ Using the previous results, R_{AB}^m is a convex risk measure with penalty function α_{AB}^m defined for any $\mathbf{Q} \in \mathcal{M}_{1,f}$ by :

$$\begin{aligned} \alpha_{AB}^m(\mathbf{Q}) &= \alpha_A^m(\mathbf{Q}) + \alpha_B^m(\mathbf{Q}) \\ &= \alpha_A(\mathbf{Q}) + \alpha_B(\mathbf{Q}) \quad \text{if } \mathbf{Q} \in \mathcal{M}^{(A)} \cap \mathcal{M}^{(B)} \quad ; \quad +\infty \quad \text{otherwise} \end{aligned}$$

Characterization of the optimal structure

The program to be solved is $\boxed{\min_F(\rho_A^m(X-F)+\rho_B^m(F))}$

The case of dilated risk measures

Both agents have **dilated risk measures**, ρ_{γ_A} and ρ_{γ_B}.

1. If the agents have acces to the **same market**, $\rho_A^m=\rho_{\gamma_A}^m$ and the final risk measure $R_{AB}^m(X)=\rho_{\gamma_A+\gamma_B}^m$.
   An optimal structure is the given by $\mathbf{F}^*=\frac{\gamma_\mathbf{B}}{\gamma_\mathbf{A}+\gamma_\mathbf{B}}\mathbf{X}$
2. When agents have acces to different markets, the final risk measure $R_{AB}^m(X)$ is the risk measure $\rho_{\gamma_A+\gamma_B}$ **reduced by** $\mathcal{H}=\mathcal{V}_T^A+\mathcal{V}_T^B$
3. Suppose $\eta_A^*+\eta_B^*$ is an optimal solution of the final hedging program :

$$\boxed{F^*=\frac{\gamma_B}{\gamma_A+\gamma_B}X+\frac{\gamma_B}{\gamma_A+\gamma_B}\eta_A^*-\frac{\gamma_A}{\gamma_A+\gamma_B}\eta_B^*}$$

   is an optimal structure.
4. In a **more general framework**, we simply obtain a **necessary and sufficient condition** for the optimality but not an explicit form of F^*.

Comments

One market

⋆ This result that is optimal to transfer the same ration of the initial risk as in the problem without market is very strong as it does **not require** any **specific underlying modelling** either for the non-tradable risk or the financial market.

⋆ The optimal structure F^* **does not depend on the financial market**. The impact of the financial market is simply visible through the pricing rule.

⋆ The underlying logic is **non-speculative** as the issuer has an interest to sell a structure if and only if it is initially exposed.

Two markets

- Even if the issuer A is **not initially exposed**, a transaction may **take place**. It is an opportunity for the agent B to buy "derivatives product" in the A-market, which is unaccessible for trading.

Dynamic Risk measures and Backward Stochastic Differential Equations

Motivation

We want to study risk measures defined by their **local specifications** and propose a method to characterize the optimal solution of the inf-convolution problem.

BSDEs

We introduce a Brownian filtration $(\Omega, \mathcal{F}_t = \sigma(W_s; 0 \le s \le t), \mathbb{P})$

$$-dY_t = f(t, Y_t, Z_t)dt - \langle Z_t, dW_t \rangle, \quad Y_T = \Psi$$

1. Given standard Lipschitz assumptions, there exists a unique solution $(\mathbf{Y_t}, \mathbf{Z_t})$ with square integrability properties.
   Moreover a **comparison theorem** holds.
2. For any bounded Ψ, if f has **quadratic growth w.r.t. to z** , there exists a **unique solution** (Y,Z) in $\mathbb{L}_\infty \otimes \mathbb{H}^2 \otimes \mathbb{H}^2$ (Kobylanski 2000), Lepeltier-San Martin,(1999)
3. BSDEs and PDE's :Suppose that $\Psi = g(X_T)$, $f(t, ., y, z) = f(t, X_t, y, z)$ where X is a R^d-valued diffusion process with **elliptic generator L**. Then, with some regularity assumptions, the process Y is given by $u(t, X_t)$ where u is viscosity solution of the non linear PDE
   $(\partial_\mathbf{t} + \mathbf{L})\mathbf{u}(\mathbf{t}, \mathbf{x}) + \mathbf{f}(\mathbf{t}, \mathbf{x}, \mathbf{u}(\mathbf{t}, \mathbf{x}), \nabla \mathbf{u}(\mathbf{t}, \mathbf{x})) = \mathbf{0}, \quad \mathbf{u}(\mathbf{T}, \mathbf{x}) = \mathbf{g}(\mathbf{x})$

The comparison theorem is nothing than the maximum principle for PDEs.

Localization of convex risk measures

We extend the notion of static entropic risk measure to a more localized dynamic one :

Dynamic Entropic risk measure :

$$\boxed{e_{\gamma,t}(X) = \gamma \ln \mathbb{E}_{\mathbb{P}}\big(\exp\big(-\tfrac{1}{\gamma}X\big)|\mathcal{F}_t\big), \quad e_{\gamma,T}(X) = -X}$$

The dynamics of the process $(e_{\gamma,t}(X); t \in [0,T])$ is given by the BSDE with the **quadratic driver** $f(t,z) = \frac{1}{2\gamma}\|z\|^2$:

$$\boxed{-de_{\gamma,t}(X) = \frac{\mathbf{1}}{\mathbf{2}\gamma}\|\mathbf{Z_t}\|^{\mathbf{2}} dt - \langle Z_t, dW_t\rangle \qquad e_{\gamma,T}(X) = -X}$$

Given Kobylansky, it is the **only solution** of this BSDE.

Extension to a family of convex risk measures :

The idea is to generalize the notion of static convex risk measure to a more dynamic notion by considering BSDEs :

Theorem : Suppose that the regular driver $f(t,z)$ is **convex** w.r. to z.
The solution $(\rho_t(X); t \leq T)$ of the BSDE with terminal condition $\mathbf{-X}$.

$$-d\rho_t(X) = f(t, Z_t)dt - \langle Z_t, dW_t \rangle, \qquad \rho_T\big(X\big) = -X$$

is, for any time $t \leq T$, a **convex risk measures**.
Comment : This convexity result has already been proved in the study of pricing functionals with constraints (for instance, El Karoui-Quenez (1996) or Peng (1997,2003)).

Dual Representation

Fenchel representation of the driver and penalty function

Let us denote by $\alpha_{\mathbf{t}}(\nu) \in]\mathbf{0}, +\infty]$ the Fenchel transform of the convex driver f, $(\nu \in R^d)$

$$f(t,z) = \sup_{\nu}\{< z, \nu > -\alpha(t,\nu)\} =: \sup_{\nu} f^{\nu}(t,z)$$

Suppose that the domain $\mathcal{D}_{\sqcup} = \{\nu, \quad \alpha(t,\nu) < +\infty\}$ is a bounded subset of R^d and define the linear BSDE by

$$-d\rho^{\nu}_{\mathbf{t}}(\mathbf{X}) = f^{\nu}(t, Z_t)dt - \langle Z_t, dW_t\rangle, \qquad \rho_T\big(X\big) = -X$$

Then, by the comparison theorem

$$f(t,z) = \sup_{\nu} f^{\nu}(t,z) \quad \Rightarrow \quad \rho(X) = \operatorname{ess\,sup} \rho^{\nu}_t(X)$$

Linear BSDE

Let us introduce

- the exponential martingale $\mathbf{H_t^\nu = \exp\big(\int_0^t < \nu_s, dW_s > - \frac{1}{2} \int_0^t |\nu_s|^2 ds \big)}$
- the $\mathbb{P}$-equivalent martingale measure $\mathbb{Q}^\nu$ with density H_T^ν.
- the $\mathbb{Q}^\nu$-Brownian motion $\mathbf{dW^\nu = dW_t - \nu_t dt}$

The risk-measure

$$-d\rho_t^\nu(X) = (< Z_t, \nu_t > -\alpha(t, \nu_t))dt - \langle Z_t, dW_t \rangle = -\alpha(t, \nu_t)dt - \langle Z_t, \mathbf{dW_t^\nu} \rangle$$

is given by

$$\rho_t^\nu(X) = \mathbb{E}_{\mathbb{Q}^\nu}(-X) - \mathbb{E}_{\mathbb{Q}^\nu}\left(\int_t^T \alpha(s, \nu_s) ds | \mathcal{F}_t \right)$$

Remark : The penalty functional $\alpha(\mathbf{t}, \mathbb{Q}^\nu)$ (in the sens of Foellmer and Schied) is given by $\alpha(\mathbf{t}, \mathbb{Q}^\nu) = \mathbb{E}_{\mathbb{Q}^\nu}\big(\int_\mathbf{t}^\mathbf{T} \alpha(\mathbf{s}, \nu_\mathbf{s}) \mathbf{ds} | \mathcal{F}_\mathbf{t} \big)$

Inf-convolution problem

Notation

Let us assume the both agents risk measures to be given by BSDEs

- ⋆ Their dynamic versions are denoted by ρ_t^A and ρ_t^B.
- • Their respective BSDEs are associated with the convex drivers $f^A(t,z)$ and $f^B(t,z)$.
- ⋆ Let $(f^A \square f^B)(t,z) = \inf_u (f^A(t,z-u) + f^B(t,u))$ be classical inf convolution in R^d of f^A and f^B.

Theorem : Suppose $(\mathbf{f^A \square f^B})(\mathbf{t},\mathbf{z})$ to be a regular driver. Let $(\rho_{\mathbf{t}}^{\mathbf{A},\mathbf{B}}(\mathbf{X}),\mathbf{Z_t})$ the associated BSDE with terminal value X. Then

1. For any $F \in \mathcal{X}$, $\quad \rho_t^{A,B}(X) \leq \rho_t^A(X-F) + \rho_t^B(F) \quad \mathbb{P}\ a.s.$
2. If there exists an admissible $\widehat{\mathbf{Z}}_{\mathbf{t}}^{\mathbf{B}}$ such that

$$\forall t \geq 0 \qquad f^A \square f^B(t,Z_t) = f^A(t, Z_t - \widehat{\mathbf{Z}}_{\mathbf{t}}^{\mathbf{B}}) + f^B(t, \widehat{\mathbf{Z}}_{\mathbf{t}}^{\mathbf{B}})$$

   then at any time t, $\rho_t^{A,B}(X)$ is the **inf-convolution** of ρ_t^A and ρ_t^B

   $\forall t \geq 0 \qquad \rho_t^{A,B}(X) = \left(\rho^A \square \rho^B\right)_t(X) \quad \mathbb{P}\ a.s$

3. Under this assumption, let $\mathbf{F}^*$ the structure defined by the **forward equation**

$$\mathbf{F}^* = \int_0^T f^B(t, \widehat{\mathbf{Z}}_\mathbf{t}^\mathbf{B})dt - \int_0^T \langle \widehat{\mathbf{Z}}_\mathbf{t}^\mathbf{B}, dW_t \rangle$$

Then F^* is an **optimal solution** for the inf-convolution problem

$$\left(\rho^A \square \rho^B\right)_t(X) = \inf_F \left\{\rho_t^A(X - F) + \rho_t^B(F)\right\}$$

Usual regularity assumptions are made on the different convex drivers.
The proof uses intensively the comparison theorem.

Entropic Indifference buyer price

Let us introduce the hedging space $\mathcal{V}_T = \{\xi = \int_0^T < \phi_s, dW_s + \lambda_s ds, \quad \phi_s \in \mathcal{K}_\mathbf{t}\}$.
The **coherent dynamic** risk measure associated is the opposite of the price

$$-d\rho_t^{\mathcal{V}}(\xi) = - < Z_s, \lambda_s > ds, - < Z_s, dW_s >, \quad \rho_T^{\mathcal{V}}(\xi) = -\xi$$

if $\forall t,\ Z_t \in \mathcal{K}_\mathbf{t}$, $-\infty$ **if not**.
The modify entropic risk measure $e^{\mathcal{V}}_{\gamma,t}(X) = e_{\gamma,t} \square \rho_t^{\mathcal{V}}(X)$ is the unique solution of the Quadratic BSDE

$$\begin{aligned} -de^{\mathcal{V}}_{\gamma,t}(X) &= \inf_{\{\phi_t \in \mathcal{K}_t\}} \big(\frac{1}{2\gamma}\big\|Z_t - \phi_s\big\|^2 - < \phi_t, \lambda_t > \big)dt - \big\langle Z_t, dW_t\big\rangle \\ e_{\gamma,T}(X) &= -X \end{aligned}$$

Non-speculative logic

We now suppose that the Agent A does not support any risk $(X = 0)$

Corollary :

Assume that $f^A(t,0) = f^B(t,0) = 0$ and

$$\partial_{\mathbf{z}} \mathbf{f}^{\mathbf{A}}(\mathbf{t},\mathbf{0}) = \partial_{\mathbf{z}} \mathbf{f}^{\mathbf{B}}(\mathbf{t},\mathbf{0}) = \mathbf{0}$$

1. The inf-convolution $\big(f^A \Box f^B\big)(t,0)$ and the associated risk measure $\big(\rho_A \Box \rho_B\big)(t,0)$ are **identically null**.
2. Moreover, $\mathbf{F}^* \equiv \mathbf{0}$ is an optimal solution for the inf-convolution problem

$$\big(\rho^A \Box \rho^B\big)_t(0) = \inf_F \left\{\rho_t^A\big(-F\big) + \rho_t^B\big(F\big)\right\}$$

3. If both drivers f^A and f^B are **strictly convex**, then $F^* \equiv 0$ is the **unique** optimal solution for the inf-convolution problem.

In this case the logic of the transaction is **non-speculative** since the issuer has an interest to sell a structure if and only if it is initially exposed.

Concluding remarks

⋆ *Standard diversification* occurs in exchange economies when agents have dilated risk measures.

⇒ The regulator has to impose very different risk measures (or penalty functions) to the different agents in the market to increase the diversification.

⋆ During *trade talks* preceding the transaction, both agents will reveal some information on their risk measure. This step is crucial, because the structure depends only upon this information.

⋆ Both agents take part in the negotiation process for the transaction. *Not only the price is at stake but also the structure (or the amount).* Consequently, this increases the probability to reach an agreement.

⋆ In the general framework of convex risk measures, *modifications in the investment opportunities* of the agent is simply translated by an explicit *modification of the risk measure.*

⋆ Extensions on *localized version* of convex risk measures.