Max-Plus Decomposition of Supermartingale Application to Portfolio Insurance

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Motivations

Insurance Portfolio

- The problem was first motivated by **portfolio insurance** which is designed to give the investor the ability to limit downside risk while allowing some participation in upside markets.
- The aim of the portfolio manager is to keep portfolio value from falling below a minimum wealth, commonly termed the **floor** at any time.
- Classical optimization problems generally try to maximize an expected utility criterion, **related to individual preferences**, through concave increasing utility function.

Related works

- In a complete market : Huang& alii(1991), NEK-M.Jeanblanc(FS1998), Cuoco& alii (1999),NEK-M.Jeanblanc-V.Lacoste (JEDC2004)
- In Incomplete Market, Duffie-Fleming-Soner-Zariphopoulo(1998), Bouchard&Pham (2003)

Martingale optimization w.r.t. to convex order

- For power utility functions, the problem may be transformed into optimization program for **martingales** with the **same** initial value

  $\Rightarrow$ subjected to the constraint to dominate a **floor** X.

  $\Rightarrow$ The optimality has to hold for any utility functions.

  $\Rightarrow$ This last point is related to **the convex order**

Max-Plus decomposition of supermartingale

- The solution is given through the decomposition of the **Snell envelope** Z (American option) of the floor in terms of an adapted increasing process Λ_t and martingale M such that $\mathbf{M_t} = \sup(\mathbf{Z_t}, \mathbf{\Lambda_t})$
- $\mathbf{\Lambda_t} = \sup_{\mathbf{0 \leq u \leq t}} \mathbf{L_u}$, $L_u \in [-\infty, +\infty]$
- Study strongly related to the work of H.Foellmer and P.Bank .

Outline of the presentation

These different steps in the **reverse order**

Assumptions

- Fix some **horizon** date T (finite or not).
- The uncertainty is modelled by a some **filtred probability space** $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$, satisfying the usual conditions of right-continuity and augmentation by $\mathbb{P}$-negligible sets.
- All processes X that we consider are adapted, right continuous with left limits **(CadLag)**, and of **class** (D),

  the family of random variables $(X(S), S \leq T, \quad S \in \mathcal{T})$ where $\mathcal{T}$ is the family of **stopping times** less than T is **uniformly integrable**, or equivalently dominated by an **u.i.martingale**.

- If the process X has not accessible jumps, it is said to be **quasi-left continuous.** It is equivalent to said that $\mathbb{E}(X_{S_n}) \rightarrow \mathbb{E}(X_S)$ if the sequence of stopping times $S_n \rightarrow S$.

Supermartingale Decompositions

Let Z be a **supermartingale** satisfying the previous assumptions

- **Doob-Meyer decomposition**

  There exist a **previsible** non-decreasing process A_t $(A_0 = 0)$ and a u.i. martingale N^{DM}, such that $Z_t = N_t^{DM} - A_t$ or equivalently

  $$\mathbf{N_t^{DM} = Z_t + A_t}.$$

  If Z is quasi-left continuous, A is continuous.

- **Multiplicative Decomposition of positive surmartingale**

  There exist a **previsible** non-decreasing process B_t $(B_0 = 1)$ and a u.i. martingale N^{multi}, such that $\mathbf{N_t^{multi} = Z_t \times B_t}$

- **Max-Plus decomposition**

  There exist a non-decreasing adapted process Λ_t $(\Lambda_{-0} = -\infty)$ and a u.i. martingale M, such that $\mathbf{M_t = Z_t \vee \Lambda_t}$.

- In all these decompositions, the martingale is **unique**, with the additional assumption in the Max-Plus representation that Λ only increases if $\mathbf{M = Z}$. Λ may be maximally chosed .

Max-plus algebra

Definition : The “exotic” algebraic structure $\mathbf{R}_{\max}$

- The symbol $R_{\max}$ denotes the set $R \cup \{-\infty\}$ with max and $+$ as the two binary operations $\oplus$ and $\otimes$, respectively.
- We call this structure **the max-plus algebra**. Sometimes this is also called an ordered group.
- We remark that the natural order on $R_{\max}$ may be defined using the $\oplus$ operation $a \leq b \quad \text{if} \quad \mathrm{a} \oplus \mathrm{b} = \mathrm{b}$.

Theorem : The algebraic structure $R_{\max}$ is an idempotent commutative **semifield.**

- the operation $\oplus$ is associative, commutative and has a zero element $\epsilon = -\infty$. As $a \otimes a = a$, $R_{\max}$ is idempotent.
- the operation $\otimes$ defines a commutative group on $R_{\max} - \{-\infty\}$, it is distributive with respect to $\oplus$ and its identity element $e = 0$ satisfies $\epsilon \otimes e = e \otimes \epsilon = \epsilon$.

Why Max-Plus ?

Comparison $R_{\max}$ and R

$\Rightarrow$ $\oplus$ is idempotent in place of invertible for $+$

$\Rightarrow$ there are no zero divisors in $R_{\max}$ ($a \oplus b = -\infty \Rightarrow a = -\infty$ or $b = -\infty$)

$\Rightarrow$ Algebric computations are efficient

A new old object

This idempotent semiring has been reinvented may times these late fifties

- $\star$ discret event system theory
- $\star$ graph theory (path algebra)
- $\star$ performance evaluation of manufacturing systems, Markov decision theory
- $\star$ Hamilton-Jacobi theory (McEneaney, Fleming...)
- $\star$ Asymptotic analysis, large deviations...(Quadrat, Akian,...

Uniqueness in the Max-Plus decomposition

Let Z be a **supermartingale** satisfying the previous assumptions, and suppose

1. there exist two non-decreasing adapted process Λ^1_t and Λ^2_t $(\Lambda^i_{-0} = -\infty)$ and two u.i. martingale M^1 and M^2 such that $M^i_T = \Lambda^i_T$ and $M^i_0 = Z_0$
2. Λ^i only increases at times when the martingale M^i hits the supermartingale Z
3. (M^i, Λ^i) are two (max-+) decompositions of Z ($\oplus = \vee = \max$)

$$\mathbf{M^1_t = Z_t \oplus \Lambda^1_t}, \quad \mathbf{M^2_t = Z_t \oplus \Lambda^2_t}.$$

Then, M^1 and M^2 are **indistinguishable** martingales. There exists a maximal increasing process Λ.

Remark Let us observe that the equality $\mathbf{M_t = Z_t \oplus \Lambda^1_t = Z_t \oplus \Lambda^2_t}$ does not imply that $\Lambda^1_t = \Lambda^2_t$, due to the non uniqueness of the linear equation $\mathbf{a \oplus x = b}$ in the max-plus algebra. In particular, $\Lambda^1_t \oplus \Lambda^2_t$ is also solution.

Sketch of the proof when Z and Λ are bounded by below

Recall the assumption $\int_0^T |M_s^i - Z_s| d\Lambda_s^i = 0$.
Then, for any regular convex function ($\mathcal{C}^2$ with linear growth)g, $\mathbf{g(0) = 0}$.

$$g(\mathbf{M_T^1} - \mathbf{M_T^2}) \leq g'(\mathbf{M_T^1} - \mathbf{M_T^2})(M_T^1 - M_T^2) = g'(\mathbf{\Lambda_T^1} - \mathbf{\Lambda_T^2})(M_T^1 - M_T^2)$$

$$\mathbb{E}\big(g(\mathbf{M_T^1} - \mathbf{M_T^2})\big) \leq$$

$$\mathbb{E}\big(g'(\Lambda_{\mathbf{0}}^1 - \Lambda_{\mathbf{0}}^2)(\mathbf{M_T^1} - \mathbf{M_T^2})\big) + \mathbb{E}\Big((M_T^1 - M_T^2) \int_0^T g''(\Lambda_t^1 - \Lambda_t^2)(d\Lambda_t^1 - \Lambda_t^2)\Big)$$

$$= \mathbb{E}\Big(\int_0^T (M_{\mathbf{t}}^1 - M_{\mathbf{t}}^2) g''(\Lambda_t^1 - \Lambda_t^2)(d\Lambda_t^1 - \Lambda_t^2)\Big)$$

$$= \mathbb{E}\Big(\int_0^T (\mathbf{Z}_t - M_t^2) g''(\Lambda_t^1 - \Lambda_t^2) d\Lambda_t^1 - \int_0^T (M_t^1 - \mathbf{Z}_t) g''(\Lambda_t^1 - \Lambda_t^2) d\Lambda_t^2\Big) \leq \mathbf{0}$$

by the **flat condition and the convexity of** g.
In particular, $\mathbb{E}\big(g(\mathbf{M_T^1} - \mathbf{M_T^2})\big) = 0$ for $\mathbf{g(x) = x^+}$

Remark Let $Z = N - A$ the Doob Meyer decomposition of Z.
By Itô's formula for $Z \vee \Lambda$, and the martingale property (**in the continuous case**)

$$dM_t = \mathbf{1}_{\{Z_t > \Lambda_t\}}(dN_t^{DM} - dA_t) + \mathbf{1}_{\{Z_t \leq \Lambda_t\}} d\Lambda_t + \frac{1}{2} dL_t^{\mathrm{loc}}$$

$$\mathbf{1}_{\{Z_t = \Lambda_t\}} d\Lambda_t + \frac{1}{2} dL_t^{\mathrm{loc}} = \mathbf{0}, \quad \mathbf{1}_{\{Z_t > \Lambda_t\}} dA_t = 0.$$

The **flat-off condition** on Λ is in fact a **necessary condition**.

Existence via A Convex family of supermartingales

Quite similar to the construction given in papers of NEK-Bank, NEK-Foellmer.

Z is to be assumed left quasi-continuous

The supermartingale convex family

- Introduce the **Snell envelope** of the convex family of processes $(\mathbf{Z_t} \vee \mathbf{m})_{\mathbf{t} \geq \mathbf{0}}$ indexed by a real parameter m and defined by

$$Z_t(m) = \text{esssup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\left(Z_\tau \vee m | \mathcal{F}_t\right),$$

- Given Snell envelope properties, $Z_.(m)$ is the smallest Cadlag supermartingale dominating $Z \vee m$, and $T_t(m)$ is the smallest of the optimal stopping times $\mathbf{T_t(m)} = \inf\{\mathbf{s} \in [\mathbf{t}, \mathbf{T}], \mathbf{Z_s(m)} = \mathbf{Z_s} \vee \mathbf{m}\}$.
- Each martingale dominating $Z_t \vee m$ necessarily dominates $Z_t(m)$.
- If $(Z_t)_{t \geq 0}$ is a **martingale**, then $(Z_t \vee m)_{t \geq 0}$ is a sub-martingale and $\mathbf{Z_t(m)} = \mathbb{E}\left(\mathbf{Z_T} \vee \mathbf{m} | \mathcal{F}_\mathbf{t}\right)$.

Convex Analysis

Properties

1. For every $t \in [0,T]$, $m \mapsto Z_t(m)$ is **convex** and non-decreasing.
2. $m \mapsto Z_t(m) - m$ is non-negative, convex and non-increasing, and $(Z_t(m) - m)_{t\geq 0}$ is the **Snell envelope** of $(Z_t - m)^+_{t\geq 0}$
3. The family of optimal stopping times $T_t(m)$ is a **non-decreasing, left-continuous** $\mathbf{Z_t(m) = \mathbb{E}\left(Z_{T_t(m)} \vee m | \mathcal{F}_t\right)}$.

Main theorem

By the **envelop theorem**, $m \mapsto Z_t(m)$ has **left-hand derivatives** given by

$$\frac{\partial^-}{\partial m} Z_t(m) = \mathbb{E}\left(\mathbf{1}_{\{m > Z_{T_t(m)}\}} | \mathcal{F}_t\right) = \mathbb{E}\left(\mathbf{1}_{\{T_t(m)=T\}\cap\{m>Z_T\}} | \mathcal{F}_t\right).$$

since $m > Z_{T_t(m)} \Leftrightarrow T_t(m) = T$ and $m > X_T$.

Partial representation result

Theorem :

Define for $\alpha \in (t, T]$, the left-continuous inverse of $T_t(.)$ w.r.t. m by

$$\mathbf{L_t^*}(\alpha) := \sup\{m; T_t(m) < \alpha\} \Leftrightarrow \{L_t^*(\alpha) \leq m\} = \{T_t(m) \geq \alpha\}$$

Put $\Lambda_{t,T} = L_t^*(T) \vee Z_T$, then

$$\mathbf{Z_t(m)} = \mathbb{E}\left(\mathbf{\Lambda_{t,T} \vee m} | \mathcal{F}_\mathbf{t}\right), \quad \mathbf{Z_t} = Z_t(-\infty) = \mathbb{E}\left(\mathbf{\Lambda_{t,T}} | \mathcal{F}_\mathbf{t}\right))$$

Proof : That is because for a.e. m $\quad \frac{\partial^-}{\partial \mathbf{m}} \mathbf{Z_t(m)} = \mathbb{E}\left(\mathbf{1}_{\mathbf{L_{t,T}^{*,+} \leq m}} | \mathcal{F}_\mathbf{t}\right).$

Since $\lim_{m \to +\infty} Z_t(m) - m = 0$.

$$\Rightarrow Z_t(m) - m = \int_m^{+\infty} -\frac{\partial}{\partial \alpha}\left(Z_t(\alpha) - \alpha\right) \, d\alpha = \int_m^{+\infty} \mathbf{1}_{L_{t,T}^{*,+} > \alpha} \, d\alpha$$

Max-plus density of $\Lambda_{t,T}$

Max-plus density

Let L_t be $\mathbf{L_t} := \sup\{\mathbf{m}, \mathbf{Z_t}(\mathbf{m}) = \mathbf{Z_t}\}$ the right-point of the closed interval $\{m \in R | Z_t(m) = Z_t\}$, with the convention $L_T := Z_T$.
The left-inverse of $T_t(m)$ is also given by

$$L_t^*(\alpha) = \sup_{t \leq s \leq \alpha} L_s = \oplus_t^\alpha L_s \quad \mathbf{\Lambda_{t,T}} = \sup_{t \leq s \leq T} L_s = \oplus_t^T L_s$$

Max-plus decomposition

Since $\mathbf{\Lambda_{0,t}} \oplus \mathbf{\Lambda_{t,T}} = \mathbf{\Lambda_{0,T}}$, we obtain the Z **max-plus decomposition** via the increasing process $\mathbf{\Lambda_{0,t}}$.

$$Z_t = \mathbb{E}(\Lambda_{t,T} | \mathcal{F}_t) = \mathbb{E}(\sup_{t \leq s \leq T} L_s | \mathcal{F}_t), \quad \mathbf{M_t^\oplus} = \mathbb{E}\left(\mathbf{\Lambda_{0,T}} | \mathcal{F}_\mathbf{t}\right) = \mathbf{Z_t} \vee \mathbf{\Lambda_{0,t}}$$

Decomposition of the Snell Envelope of process X

$\mathbf{L_t} := \sup\{\mathbf{m}, \mathbf{Z_t}(\mathbf{m}) = \mathbf{Z_t}\}$ and $\Lambda_{0,t}$ only increases on $Z = M^\oplus$

Example

Martingale case

$\mathbf{L_t} := \sup\{\mathbf{m}, \mathbb{E}(\mathbf{M_T} \vee \mathbf{m}|\mathcal{F}_\mathbf{t}) = \mathbf{M_t} = \mathbf{E}(\mathbf{M_T}|\mathcal{F}_\mathbf{t})\}$

So, $M_T \vee m = M_T, \quad \mathcal{F}_t - \mathbb{P}, a.s.$, and L_t is the **conditionnal ess inf** of M_T. In this case $\mathbf{L_t}$ is an **non decreasing process**.

Monotone case

Suppose Z to be a **decreasing** process. The $\mathbf{L_t = Z_0}$.

Stochastic order and Max-plus decomposition

Stochastic orders

Definition :

Let X_1 and X_2 be two random variables. Then we say that X_1 is **less variable** than X_2 in the **convex stochastic order**, and we write $\mathbf{X_1 \leq_{cx} X_2}$ if for all **convex** functions g (if that makes sense)

$$\mathbb{E}\left[\mathbf{g(X_1)}\right] \leq \mathbb{E}\left[\mathbf{g(X_2)}\right]$$

- If the inequality holds only for all **decreasing** convex functions, then X_1 is said to be smaller than X_2 in the **decreasing convex order** (denoted by $X_1 \leq_{dcx} X_2$).

$\Rightarrow$ $X_1 \leq_{cx} X_2 \Rightarrow \mathbb{E}(X_1) = \mathbb{E}(X_2)$ provided the expectations exist and $\mathrm{var}(X_1) \leq \mathrm{var}(X_2)$, whenever $\mathrm{var}(X_2)$ is finite.

$\Rightarrow$ $X_1 \leq_{dcx} X_2 \Rightarrow \mathbb{E}(X_1) \geq \mathbb{E}(X_2)$ provided the expectations exist.

Formulation of the martingale optimization problem

X-Envelope de Snell

In that follows, the previous results are applied to the Snell envelope Z^X of X.

The optimization problem

Set $\mathcal{M}(x) = \Big\{ (M_t)_{t\geq 0}\, \text{u.i.martingale} | M_0 = x \text{ and } \mathbf{M_t \geq X_t} \;\; \forall t \in [0,T] \Big\}$

- We aim at finding a martingale (M_t^*) in $\mathcal{M}(x)$ such that for all martingales (M_t) in $\mathcal{M}(x)$

$$\mathbf{M^*_T \leq_{cx} M_T}$$

- The initial value of any martingale dominating X must be **at least** equal to the one of the Snell envelope $Z_0^X = \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}\left[X_\tau\right],$

Necessary and Sufficient condition of optimality

Theorem

- Let us consider a martingale $(M_t)^*_{t\geq 0}$ in $\mathcal{M}(Z_0^X)$, satisfying the terminal condition $\mathbf{M^*_T} = \mathbf{K_T}$, where $(K_t)_{t\geq 0}$ is an adapted increasing process, which **only increases** when the martingale hits the floor X.

$\Rightarrow$ Then $(M^*_t)_{t\geq 0}$ is the **"smallest martingale"** in $\mathcal{M}(Z_0^X)$ dominating the floor, **with respect to the convex stochastic ordering** .

$\Rightarrow$ Suppose $x = Z_0^X$. The martingale $\mathbf{M}^{\oplus}$ of the max-plus Z-decomposition is an **optimale solution**.

- In particular, $\mathbf{M}^{\oplus}_{\mathbf{T}}$ is less variable than $\mathbf{M}^{\mathbf{DM}}_{\mathbf{T}}$ where $M^{DM} \in \mathcal{M}(Z_0^X)$ is the martingale of the Doob's decomposition of Z.
- If $\mathbf{x} \geq \mathbf{Z_0^X}$, the same results holds, in terms of increasing process $\mathbf{\Lambda_{0,T}} \vee \mathbf{x}$

Proof

- Let $(\mathbf{M_t})_{\mathbf{t}\geq \mathbf{0}}$ be an arbitrary element of $\mathcal{M}(x)$ and $\mathbf{g}$ be a real convex function, for which $\mathbb{E}\left[g(M_T)\right]$ and $\mathbb{E}\left[g(M_T^*)\right]$ are well defined.

$$\Rightarrow \qquad g \text{ convex} \Rightarrow g(M_T) - g({M_T}^*) \geq g'({M_T}^*)(M_T - {M_T}^*) \Rightarrow$$
$$\mathbb{E}\left[g(M_T)\right] - \mathbb{E}\left[g({M_T}^*)\right] \geq \mathbb{E}\left[g'({M_T}^*)(M_T - {M_T}^*)\right].$$

- Remark that $(g'(K_t), t \geq 0)$ is a nondecreasing process and that $g'(K_T) = g'(K_0) + \int_0^T dg'(K_t)$.

$$\Rightarrow \quad \mathbb{E}\left[g'({M_T}^*)(M_T - {M_T}^*)\right] = \mathbb{E}\left[g'(\mathbf{K_T}^*)(M_T - {M_T}^*)\right]$$
$$= \mathbb{E}\left[\mathbf{g}'(\mathbf{K_0})(M_T - {M_T}^*)\right] + \mathbb{E}\Big[\int_0^T (M_T - {M_T}^*)\, dg'(K_t)\Big].$$
$$\Rightarrow \quad \mathbb{E}[g'(K_0)(M_T - {M_T}^*)] = g'(K_0)(M_0 - {M_0}^*) = 0$$
$$\Rightarrow \quad \mathbb{E}\Big(\int_0^T (M_T - {M_T}^*)\, dg'(K_t)\Big) = \mathbb{E}\Big(\int_0^T (\mathbf{M_t} - \mathbf{M_t}^*)\, dg'(K_t)\Big)$$
$$= \mathbb{E}\Big[\int_0^T (M_t - \mathbf{X_t})\, dg'(K_t)\Big] \geq \mathbf{0}$$

Portfolio Insurance with American Constraint

Portfolio management with guarantee

We are concerned with the portfolio problem where the **goal** of the manager is

> **to exceed the performance of a given benchmark process at any time** *during the life of the fund.*

Example : A fund guarantees

1. a part of an index performance, I_t
2. at any time the investor could receive 90% of his initial investment, without capitalization, that is

$$V_t \geq G_t = \sup(\alpha V_0, \beta \frac{I_t}{I_0})$$

Similar problems appear when the manager is submitted to **legal constraints** .

The guarantee may be called the **floor**.

Investment funds strategies

In pratice, the first step in the management of investment fund or pension fund is to define a **Strategic allocation**

According to the **investor's risk aversion**, the manager decides the proportion of Indexs, securities, coupon bonds, in a well-diversified portfolio with present value S_t.
In mathematical framework S_t is the **optimal portfolio** for a non constrained problem associated with given utility function

Tactic Allocation

or How to manage the **strategic portfolio**(underlying) in such a way that they are

- high performance of the fund (optimality ?)
- no large **losses**

This last condition may be required by the **regulator** as a legal constraint
As we would see, by doing that **the manager has an optimal behaviour**.

Links with the classical optimization problem

Let us introduce a financial market, where interest rate and risk premium are given trough the **state price density** H_t, in such way that for any self-financing portfolio V_t, $\mathbf{H_t V_t}$ is a local **martingale**.

- Consider the following **"non-constrained"** problem :

$$\max_{V_t} \mathbb{E}\{u(V_T), (V_t H_t)_{t\geq 0} \,\text{martingale and}\, V_0 = x\}$$

⇒ $(V^{*x}_t)_{t\geq 0}$ is **optimal** iff $\mathbb{E}\left[u'(V^{*x}_T)(V^x_T - V^{*x}_T)\right] = 0$, for any self-financing portfolio $(V^x_t)_{t\geq 0}$, where x stands for the initial capital.

⇒ In a **complete** market $u'(V^{*x}_T) = \lambda H_T$, where λ is a Lagrange multiplier, and H the state prices process.

⇒ For **CRRA utility function** : $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$, for all $x \in R^+$, with $\gamma \in]0,1[$, $\mathbf{V^{*x}_T = xS_T}$, where $S_T = V^{*1}_T$ denotes the optimal portfolio with initial capital $S_0 = 1$.

Change of numeraire and martingale problem

- **Constrained** decision problem : $\max_{\mathbf{V_t}} \mathbb{E}(\mathbf{u}(\mathbf{V_T}))$, subject to $(V_t)_{t\geq 0}$ self-financing portfolio, $\mathbf{V_t} \geq \mathbf{G_t}\ \ \forall \mathbf{t} \in [\mathbf{0}, \mathbf{T}]$ and $V_0 = x$.

$\Rightarrow$ Let $M_t^S = H_t S_t$ be the S-martingale and define Q^S the risk-neutral probability w.r.to S

$$\frac{d\mathbb{Q}^S}{d\mathbb{P}} = \frac{M_T^S}{M_0^S} = H_T S_T.$$

- In the case of a **CRRA utility function**, since S is the optimal portfolio, $(S_T)^{-\gamma} = \lambda H_T$

$$\mathbb{E}(\mathbf{u}(\mathbf{V_T})) = \frac{1}{1-\gamma}\mathbb{E}\big(S_T^{1-\gamma}(\frac{V_T}{S_T})^{1-\gamma}\big) = \frac{1}{1-\gamma}\lambda\mathbb{E}\big(H_T S_T(\frac{V_T}{S_T})^{1-\gamma}\big)$$

$$= \lambda\mathbb{E}_{\mathbb{Q}^\mathbf{S}}\big(\mathbf{u}(\mathbf{V_T^S})\big).$$

- **Under** Q^S, $\Big(V_t^S = \frac{V_t}{S_t}\Big)_{t\geq 0}$ is a **martingale** and the problem becomes :

$$\max_{V_t^S}\Big\{\mathbb{E}_{\mathbb{Q}^S}\big(u(V_T^S)\big), \text{subject to } \mathbf{V_t^S} \geq \frac{\mathbf{G_t}}{\mathbf{S_t}} = \mathbf{X_t} \text{ and } V_0^S \text{ given}\Big\}.$$

Characterization of the optimal solution

Suppose $x = Z_0$

The solution is given by the martingale of the **max-plus decomposition** of the Snell envelope Z^X of X in the $\mathbb{Q}_\mathbf{S}$-market s.t.

$$Z_t^X(m) = \text{esssup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_{\mathsf{Q}^S}\left[(Z_\tau^X \vee m)|\mathcal{F}_t\right] = \text{esssup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_{\mathsf{Q}^S}\left[(X_\tau \vee m)|\mathcal{F}_t\right].$$

$\mathbf{L_t^S} = \sup\{m, Z_t^X(m) = Z_t^X = X_t\}$ is the boundary in strike of the $\mathbb{Q}_\mathbf{S}$-**American Call option** with pay-off $(X_t - m)^+$.

Then, the **optimal** $\mathbb{Q}_\mathbf{S}$**-martingale** is

$$\mathbf{M_t^{S,\star}} = \mathbb{E}_{\mathsf{Q}^S}\left[\Lambda_{0,T}^S|\mathcal{F}_t\right] = \mathbb{E}_{\mathsf{Q}^S}\left[\Lambda_{0,t}^S \vee \Lambda_{t,T}^S|\mathcal{F}_t\right],$$

if $\Lambda_{0,t}^S = \sup_{0<u\leq t} \mathbf{L_u^S}$

American Option in BS Framework

We assume the underlying to be a **geometrical Brownian motion**.

- The **American Put** with strike K is a function, $U^a(t,x)$ satisfying the variational inequality

$$\partial_t U^a + \frac{1}{2}\sigma^2 x^2 \partial^2_{xx} U^a + rx\partial_x U^a - rU^a = 0$$

on the **continuation region** $\{x > b(t)\} = \{U^a(t,x) > K - x\}$ ($b(t)$ is the **exercise boundary**) with the **smooth-fit condition** at the boundary $\partial_{\mathbf{x}}\mathbf{U^a(t, b(t)) = -1}$.

- Then, for $S^x_t = xS_t$, $\quad U^a(t, xS_t) = \text{esssup}_{U \geq t}\mathbb{E}\big(e^{-r(U-t)}(K - xS_U)^+ | \mathcal{F}_t\big)$

By **change of numeraire**, under the probability $\mathbb{Q}_S$

$$\frac{U^a(t, xS_t)}{S_t} = \mathbf{C^a(t, KS_t^{-1}, x)} = \text{esssup}_{\mathbf{U \geq t}}\mathbb{E}_{\mathbf{Q_S}}\big(\mathbf{KS_U^{-1} - x})^+ | \mathcal{F}_\mathbf{t}\big)$$

Closed form of the American Call option

Using previous notation

$$\mathbf{L_t^S} = \sup\{x, C^a(t, KS_t^{-1}, x) = (KS_t^{-1} - x)^+\} = \mathbf{b(t)/S_t}$$

$$\mathbf{\Lambda_{t,s}^S} = \sup_{t \le u \le s} b(u)/S_u$$

- The strategy $\mathbf{M_t^{S,\star} = \Lambda_{0,t}^S + C^a(t, KS_t^{-1}, \Lambda_{0,t}^S)} \geq KS_t^{-1}$ is the **optimal** $\mathbb{Q}_S$ **self-financing martingale strategy** dominating $X_t = KS_t^{-1}$ with terminal value

$$M_T^{S,\star} = (K/S_T) \vee \sup_{0 \le u \le T} \mathbf{(b(u)/S_u)}$$

- The price of the American Call option is given by

$$C^a(t, KS_t^{-1}, x) = \mathbb{E}_{\mathbb{Q}_S}\left(\mathbf{KS_T^{-1} - x \vee \Lambda_{t,T}^S})^+ | \mathcal{F}_t\right)$$