

A Moment Approach to the Static Arbitrage Problem on Baskets

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Introduction

- classic Black & Scholes (1973) option pricing based on:
 - a *dynamic hedging* argument
 - a *model* for the asset dynamics (geometric BM)
- sensitive to liquidity, transaction costs, model risk ...
- what can we say about option prices with much weaker assumptions?

Static Arbitrage

Here, we rely on a *minimal set of assumptions*:

- no assumption on the asset distribution
- one period model

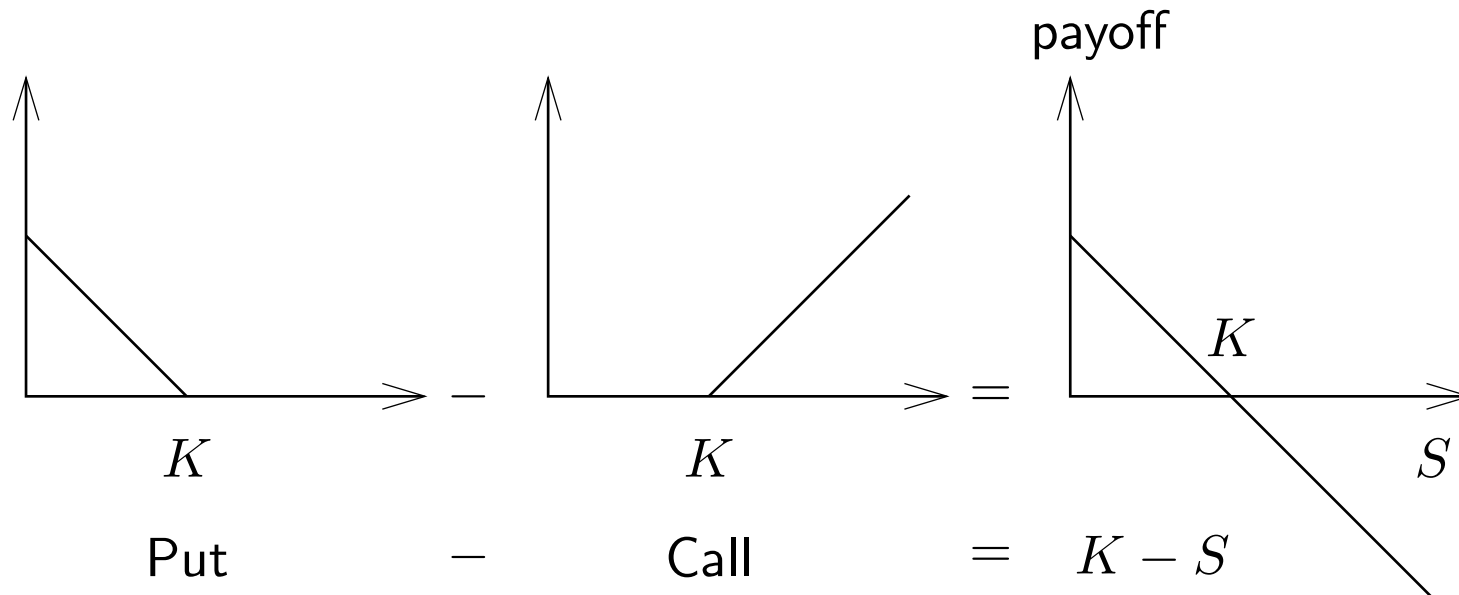
An arbitrage in this simple setting is a *buy and hold* strategy:

- form a portfolio at no cost today with a strictly positive payoff at maturity
- no trading involved between today and the option's maturity

What for?

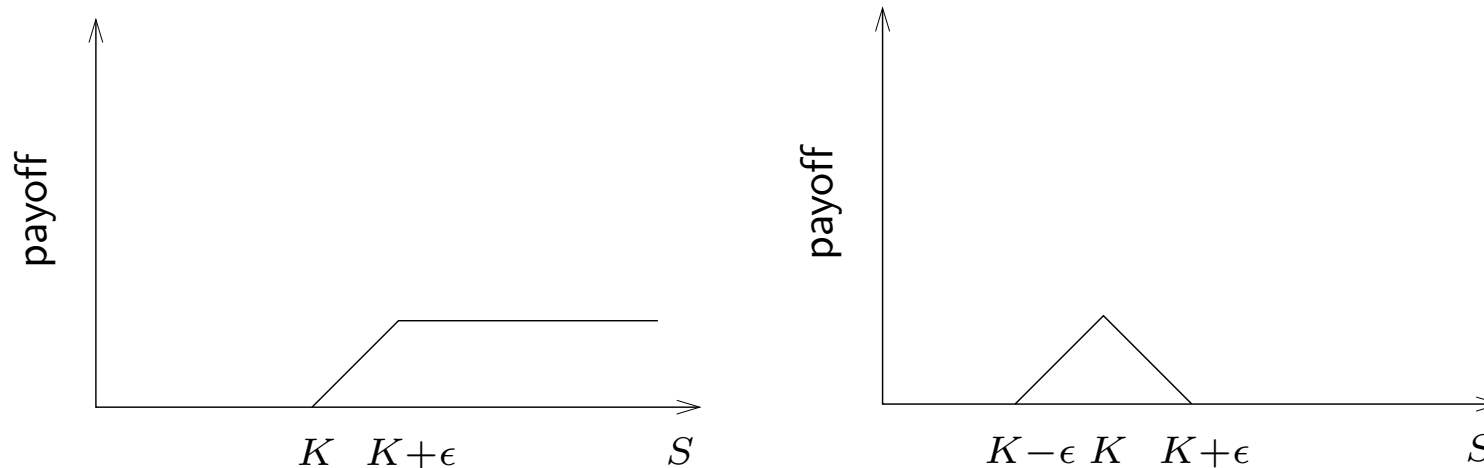
- arbitrage free data stripping before calibration
- test extrapolation formulas
- in illiquid markets, find optimal static hedge or bound risk at little cost

Simplest Example: Put Call Parity



We denote by $C(K)$ the price of the call with payoff $(S - K)^+$. If we know the forward prices, then we can deduce call prices from puts, ...

Call Spread - Butterfly Spread



Here, the absence of arbitrage implies that the price of a call spread be positive, hence call prices must be *decreasing* with strike

$$C(K + \epsilon) - C(K) \leq 0$$

it also implies that the price of a butterfly spread be positive, and call prices must then be *convex* with strike

$$C(K + \epsilon) - 2C(K) + C(K - \epsilon) \geq 0$$

Price Constraints

The absence of arbitrage implies that if $C(K)$ is a function giving the price of an option of strike K , then $C(K)$ must satisfy:

- $C(K)$ positive
- $C(K)$ decreasing
- $C(K)$ convex

With $C(0) = S$, we have a set of *necessary* conditions for the absence of arbitrage

Sufficient Conditions

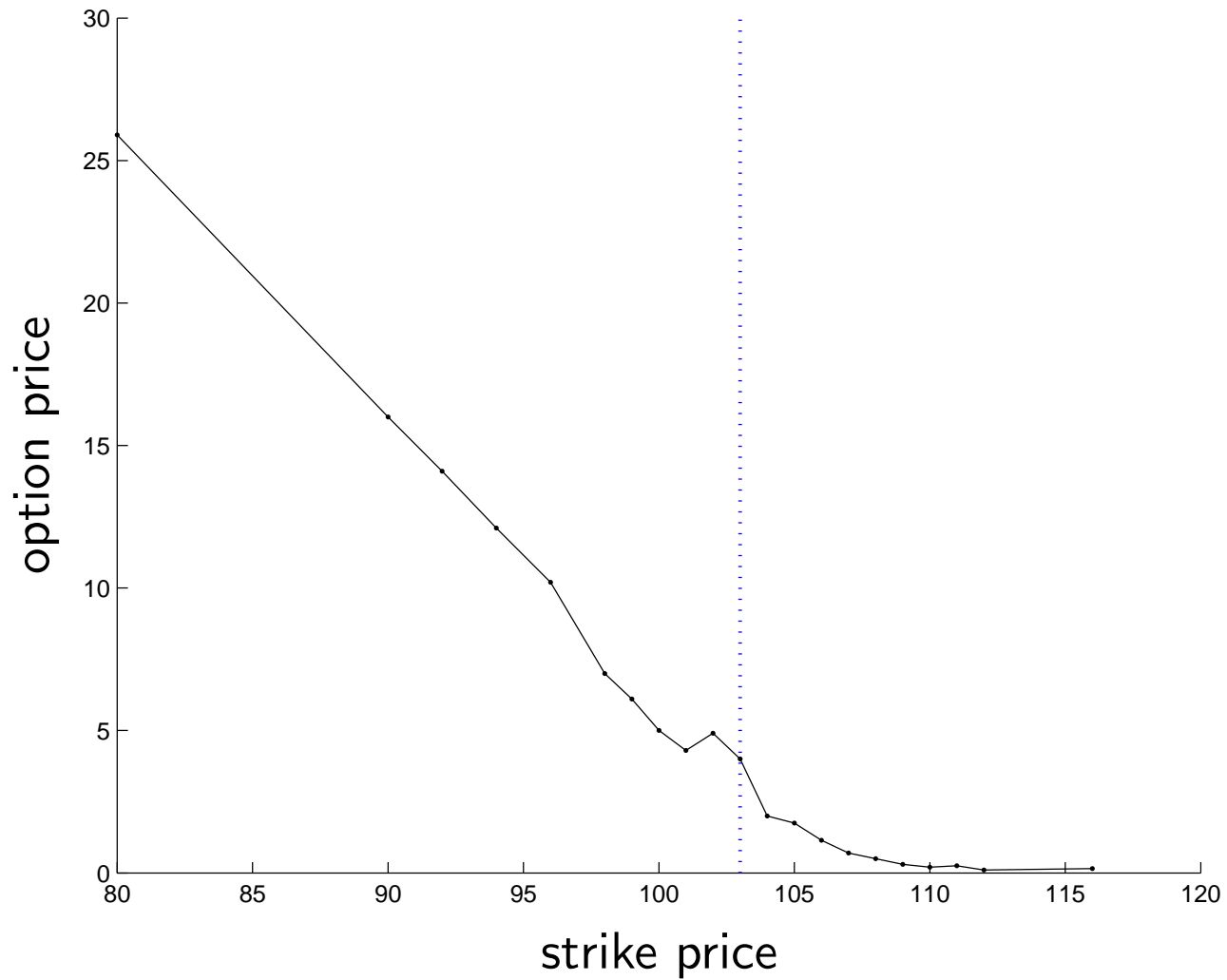
In fact, these conditions are also *sufficient*, see Laurent & Leisen (2000) and Bertsimas & Popescu (2002) among others. . .

Suppose we have a set of market prices for calls $C(K_i) = p_i$, then there is no arbitrage iff there is a function $C(K)$:

- $C(K)$ positive
- $C(K)$ decreasing
- $C(K)$ convex
- $C(K_i) = p_i$ and $C(0) = S$

This is *very easy* to test. . .

Dow Jones index call option prices on Mar. 17 2004, maturity Apr. 16 2004



Source: Reuters.

Why?

data quality...

- all the prices are last quotes (not simultaneous)
- low volume
- some transaction costs

Problem: this data is used to calibrate models and price other derivatives...

Dimension n: Basket Options

- a basket call payoff is

$$\left(\sum_{i=1}^k w_i S_i - K \right)_+$$

where w_1, \dots, w_k are the basket's weights and K is the option's strike price

- examples include: Index options, spread options, swaptions...
- basket option prices are used to gather information on *correlation*

We denote by $C(w, K)$ the price of such an option, can we get conditions to test basket price data?

Necessary Conditions

Similar to dimension one...

Suppose we have a set of market prices for calls $C(w_i, K_i) = p_i$, and there is no arbitrage, then the function $C(w, K)$ satisfies:

- $C(w, K)$ positive
- $C(w, K)$ decreasing in K , increasing in w
- $C(w, K)$ jointly convex in (w, K)
- $C(w_i, K_i) = p_i$ and $C(0) = S$

Is this still *tractable* (in dimension n)?

Tractable?

The problem can be formulated as:

$$\begin{aligned} &\text{find} && z \\ &\text{subject to} && Az \leq b, \quad Cz = d \\ &&& z = [f(x_1), \dots, f(x_k), g_1^T, \dots, g_k^T]^T \\ &&& g_i \text{ subgradient of } f \text{ at } x_i \quad i = 1, \dots, k \\ &&& f \text{ monotone, convex} \end{aligned}$$

in the variables $f \in C(\mathbf{R}^n)$, $z \in \mathbf{R}^{(n+1)k}$, $g_1, \dots, g_k \in \mathbf{R}^n$

- *discretize* and sample the convexity constraints to get a polynomial size LP feasibility problem

- enforce the convexity and subgradient constraints at the points $(x_i)_{i=1,\dots,k}$ (monotonicity is a simple inequality on g) to get:

$$\begin{aligned} &\text{find} && z \\ &\text{subject to} && Cz = d, \quad Az \leq b \\ &&& z = [f(x_1), \dots, f(x_k), g_1^T, \dots, g_k^T]^T \\ &&& \langle g_i, x_j - x_i \rangle \leq f(x_j) - f(x_i) \quad i, j = 1, \dots, k \end{aligned}$$

in the variables $f(x_i)_{i=1,\dots,k}$ and g in $\mathbf{R}^k \times \mathbf{R}^{(n+1) \times k}$

- we note $z^{\text{opt}} = [f^{\text{opt}}(x_1), \dots, f^{\text{opt}}(x_k), (g_1^{\text{opt}})^T, \dots, (g_k^{\text{opt}})^T]^T$ a solution to this problem

- from z^{opt} , we define:

$$C(x) = \max_{i=1,\dots,k} \{f^{\text{opt}}(x_i) + \langle g_i^{\text{opt}}, x - x_i \rangle\}$$

- by construction, $C(x_i)$ solves the finite LP with:

$$C(x_i) = f^{\text{opt}}(x_i), \quad i = 1, \dots, k$$

- $C(x)$ is convex and monotone as the pointwise maximum of monotone affine functions
- so $C(x)$ is also a feasible point of the original problem

This means that $C(x)$ is a *solution* for the original (infinite dimensional) problem.

Relaxation

The previous result means that the price conditions remain tractable on basket options... They are equivalent to the following *feasibility problem*:

$$\begin{aligned} & \text{find} && g_i \\ & \text{subject to} && \langle g_i, (w_j, K_j) - (w_i, K_i) \rangle \leq p_j - p_i \\ & && g_{i,j} \geq 0, \quad j = 1, \dots, n \\ & && -1 \leq g_{i,n+1} \leq 0 \\ & && \langle g_i, (w_i, K_i) \rangle = p_i, \quad i = 1, \dots, m, \end{aligned}$$

where the variables $g_i \in \mathbf{R}^n$ are the subgradients of $C(w, K)$ at the points (w_i, K_i) .

Sufficient?

A key difference with dimension one: Bertsimas & Popescu (2002) show that the exact problem is NP-Hard.

- the conditions are *only necessary*...
- however, numerical cost is minimal (small LP)
- we can show *sufficiency* in some particular cases
- how tight are these conditions in general?

Numerical Example

- two assets: x_1, x_2 , we look for upper and lower bounds on the price of a particular basket $(x_1 + x_2 - K)^+$
- simple discrete model for the assets:

$$x = \{(0, 0), (0, .8), (.8, .3), (.6, .6), (.1, .4), (1, 1)\}$$

with probability

$$\pi = (.2, .2, .2, .1, .1, .2)$$

- the forward prices are given, together with the following call prices:

$$\begin{aligned} & (.2x_1 + x_2 - .1)^+, (.5x_1 + .8x_2 - .8)^+, (.5x_1 + .3x_2 - .4)^+, \\ & (x_1 + .3x_2 - .5)^+, (x_1 + .5x_2 - .5)^+, (x_1 + .4x_2 - 1)^+, (x_1 + .6x_2 - 1.2)^+ \end{aligned}$$

- we compare the (outer) price bounds given by the previous relaxation with inner bounds computed by discretizing.

Numerical Example

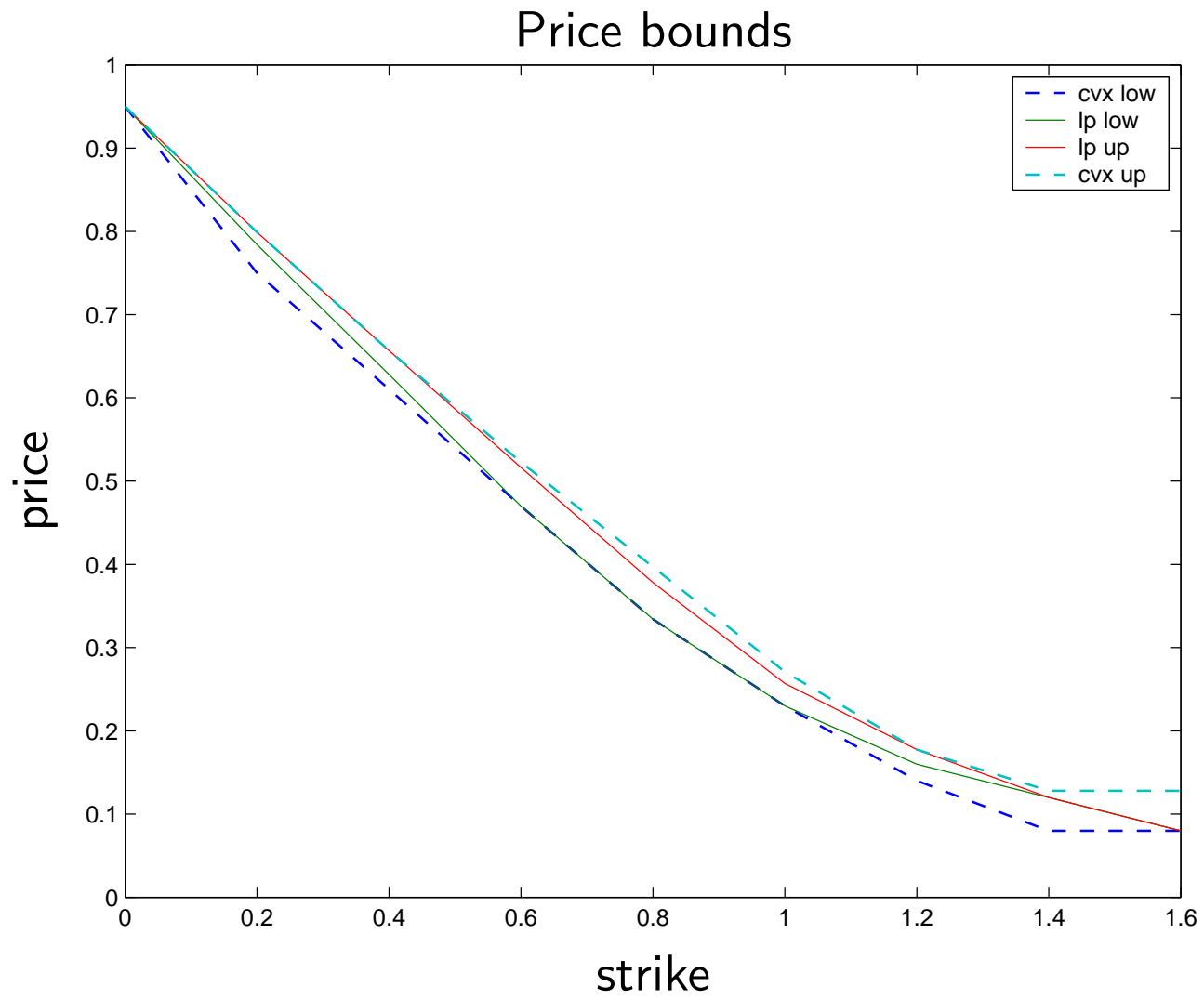
We compare the *outer bounds* on the price p_0 of the $(x_1 + x_2 - K)^+$ basket obtained by solving:

$$\begin{aligned} \text{max./min.} \quad & p_0 \\ \text{subject to} \quad & \langle g_i, (w_j, K_j) - (w_i, K_i) \rangle \leq p_j - p_i \\ & g_{i,j} \geq 0, \quad j = 1, \dots, n \\ & -1 \leq g_{i,n+1} \leq 0 \\ & \langle g_i, (w_i, K_i) \rangle = p_i, \quad i = 1, \dots, m, \end{aligned}$$

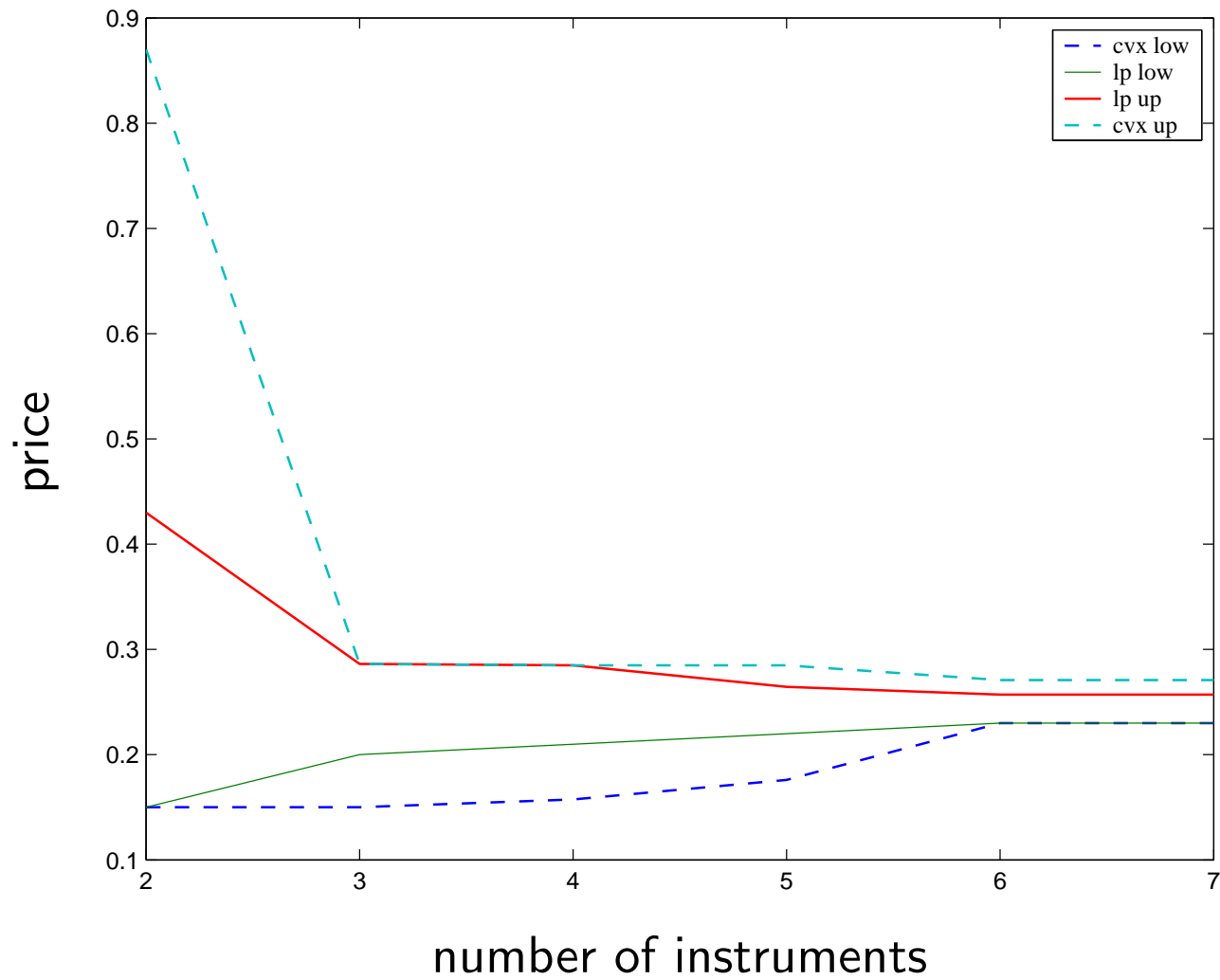
with the *inner bounds* obtained by solving:

$$\begin{aligned} \text{max./min.} \quad & \mathbf{E}_\pi \left((w_0^T x - K_0)^+ \right) \\ \text{subject to} \quad & \mathbf{E}_\pi \left((w_i^T x - K_i)^+ \right) = p_i, \quad i = 1, \dots, m, \end{aligned}$$

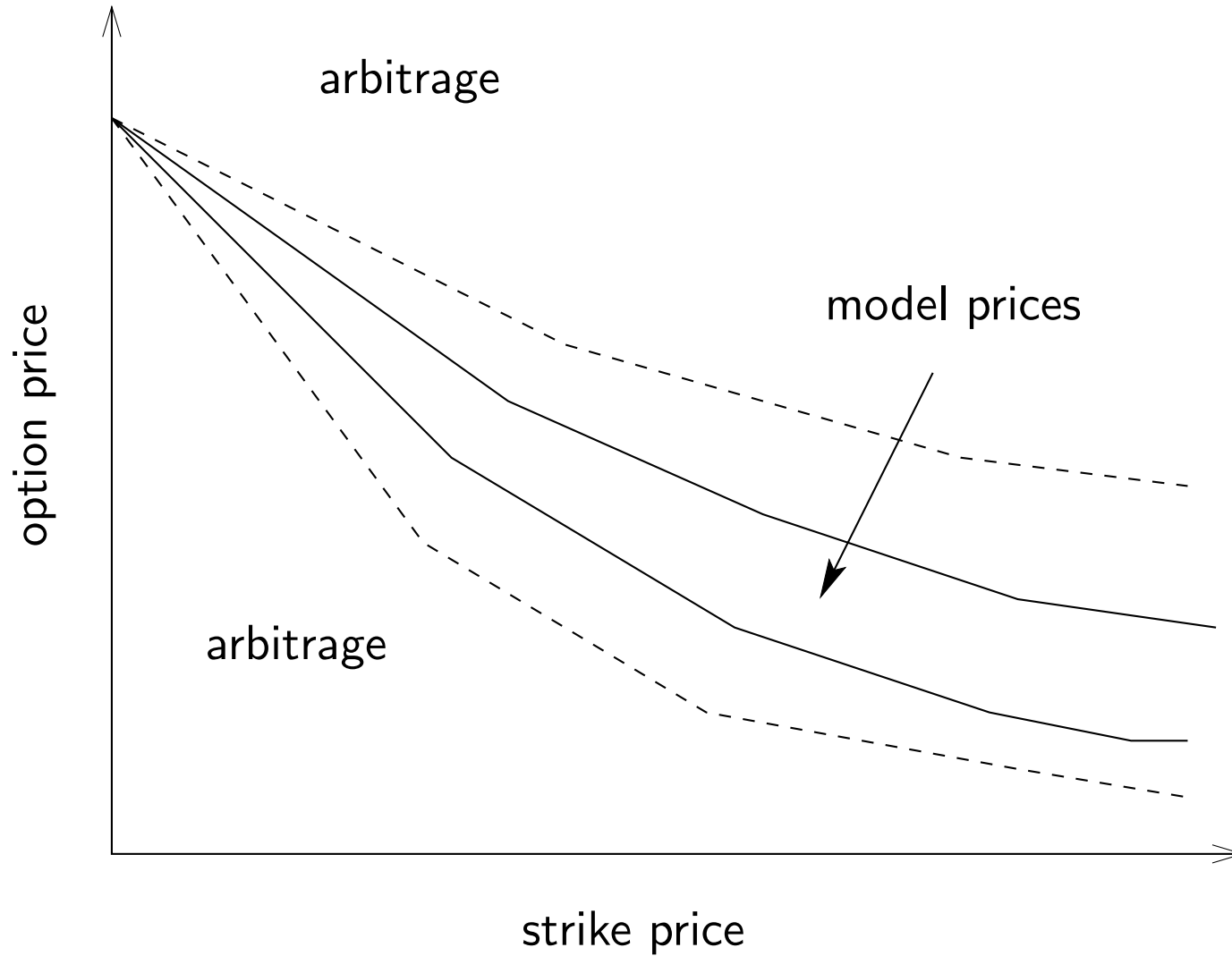
which becomes a simple (albeit large) linear program after we discretize π .



Price bounds



Price Bounds



Multivariate Black-Scholes Model

Here, we compare the *outer bounds* on the price p_0 of a basket obtained by solving the relaxation:

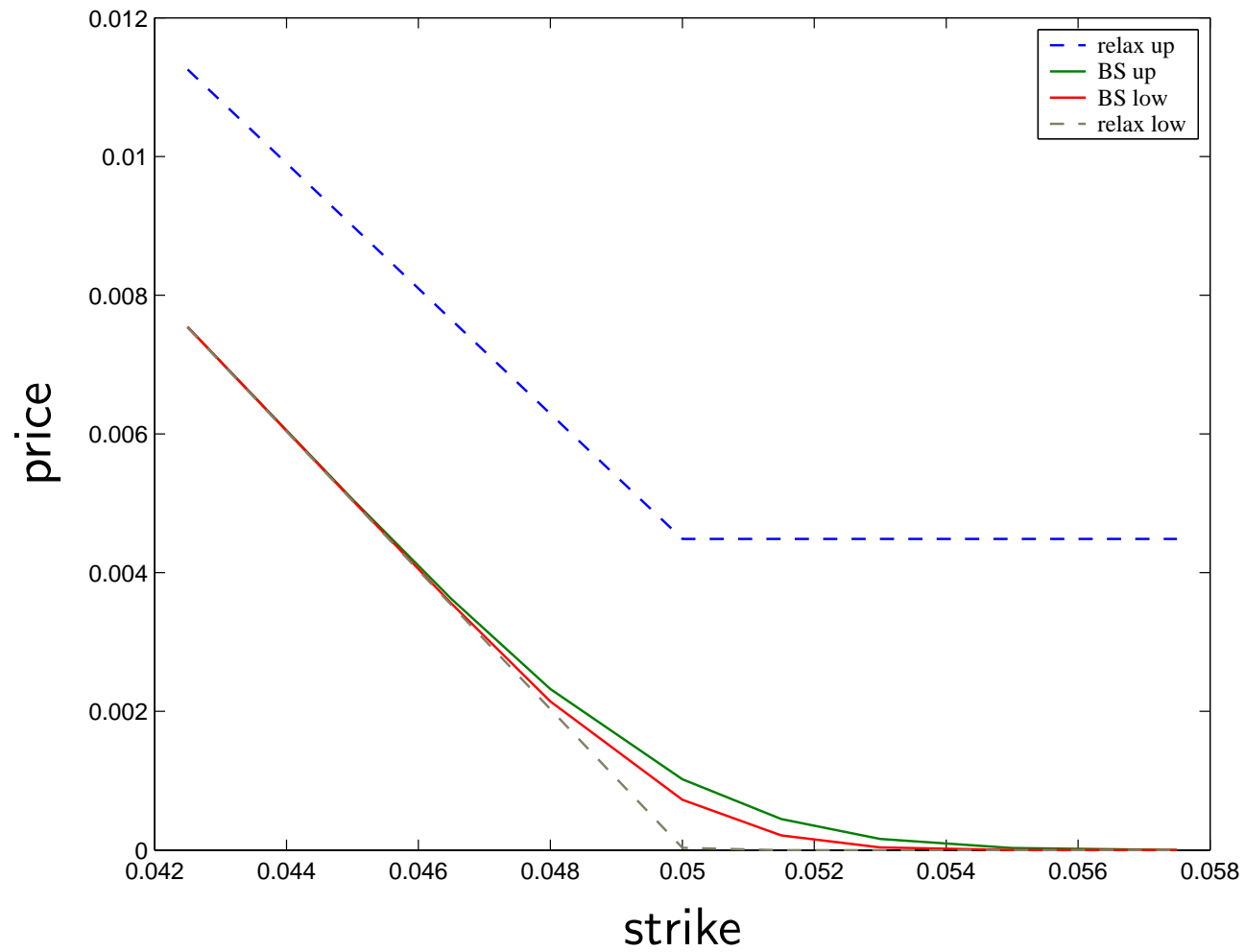
$$\begin{aligned} \text{max./min.} \quad & p_0 \\ \text{subject to} \quad & \langle g_i, (w_j, K_j) - (w_i, K_i) \rangle \leq p_j - p_i \\ & g_{i,j} \geq 0, \quad j = 1, \dots, n \\ & -1 \leq g_{i,n+1} \leq 0 \\ & \langle g_i, (w_i, K_i) \rangle = p_i, \quad i = 1, \dots, m, \end{aligned}$$

with the *inner bounds* computed as:

$$\begin{aligned} \text{max./min.} \quad & BS(T, w_0, K_0, V) \\ \text{subject to} \quad & BS(T, w_i, K_i, V) = p_i, \quad i = 1, \dots, m, \end{aligned}$$

in the variable $V \in \mathbf{S}^n$, corresponding to extreme prices on a basket option in a multivariate Black-Scholes model, given prices p_i of other basket options with weights w_i .

Multivariate Black-Scholes Model



Close the Gap

The gap is surprisingly large. . .

- ATM prices are not supposed to be very sensitive to the smile
- approx. lognormal model calibrate easily to swaption data in practice

How can we improve the static bounds (so we know when to blame the model)?

Integral Transform Solution

- we can write the set off call prices as:

$$\begin{aligned} C(w, K) &= \mathbf{E}_\pi(w^T x - K)_+ \\ &= \int_{\mathbf{R}_+^n} (w^T x - K)_+ d\pi(x), \end{aligned}$$

and think of $C_\pi(w, K)$ as a particular integral transform of the measure π

- at least formally, we have:

$$\frac{\partial^2 C(w, K)}{\partial K^2} = \int_{\mathbf{R}_+^n} \delta(w^T x - K) \pi(x) dx$$

- this means that $\partial^2 C(w, K)/\partial K^2$ is the *Radon transform* (see Helgason (1999) or Ramm & Katsevich (1996)) of the measure π

A Range Characterization Problem...

- the general arbitrage problem can be written as the following infinite dimensional problem:

$$\begin{aligned} &\text{find} && C(w, K) \\ &\text{subject to} && C(w_i, K_i) = p_i, \quad i = 1, \dots, m \\ &&& C(w, K) \in \mathcal{R}_C, \end{aligned}$$

- here, \mathcal{R}_C is the range of the (linear) integral transform

$$\begin{aligned} C : \mathcal{K} &\rightarrow \mathcal{R}_C \\ \pi &\rightarrow C(w, K) = \int_{\mathbf{R}_+^n} (w^T x - K)_+ d\pi(x) \end{aligned}$$

Full Conditions

derived by Henkin & Shaninin (1990). A function can be written

$$C(w, K) = \int_{\mathbf{R}_+^n} (w^T x - K)^+ d\pi(x)$$

with $w \in \mathbf{R}_+^n$ and $K > 0$, if and only if:

- $C(w, K)$ is *convex* and *homogenous* of degree one;
- $\lim_{K \rightarrow \infty} C(w, K) = 0$ and $\lim_{K \rightarrow 0^+} \frac{\partial C(w, K)}{\partial K} = -1$
- $F(w) = \int_0^\infty e^{-K} d\left(\frac{\partial C(w, K)}{\partial K}\right)$ belongs to $C_0^\infty(\mathbf{R}_+^n)$
- For some $\tilde{w} \in \mathbf{R}_+^n$ the inequalities: $(-1)^{k+1} D_{\xi_1} \dots D_{\xi_k} F(\lambda \tilde{w}) \geq 0$, for all positive integers k and $\lambda \in \mathbf{R}_{++}$ and all ξ_1, \dots, ξ_k in \mathbf{R}_+^n .

Finer Conditions

- the LP relaxations are sufficient in some particular cases
- can we improve their performance in the general case?
- how do we get super/subreplicating portfolio?
- the method in Bertsimas & Popescu (2002) only gives a relaxation for the case $x \in \mathbf{R}^n$
- the last two conditions (smoothness and total positivity) in the Radon range characterization are hard to implement, yet they suggest a *moment approach*. . .

Harmonic Analysis on Semigroups

some quick definitions...

- a pair (\mathbb{S}, \cdot) is called a *semigroup* iff:
 - if $s, t \in \mathbb{S}$ then $s \cdot t$ is also in \mathbb{S}
 - there is a neutral element $e \in \mathbb{S}$ such that $e \cdot s = s$ for all $s \in \mathbb{S}$
- the *dual* \mathbb{S}^* of \mathbb{S} is the set of *semicharacters*, *i.e.* applications $\chi : \mathbb{S} \rightarrow \mathbf{R}$ such that
 - $\chi(s)\chi(t) = \chi(s \cdot t)$ for all $s, t \in \mathbb{S}$
 - $\chi(e) = 1$, where e is the neutral element in \mathbb{S}
- a function α is called an *absolute value* on \mathbb{S} iff
 - $\alpha(e) = 1$
 - $\alpha(s \cdot t) \leq \alpha(s)\alpha(t)$, for all $s, t \in \mathbb{S}$

Harmonic Analysis on Semigroups

last definitions (honest)...

- a function $f : \mathbb{S} \rightarrow \mathbf{R}$ is *positive semidefinite* iff for every family $\{s_i\} \subset \mathbb{S}$ the matrix with elements $f(s_i \cdot s_j)$ is positive semidefinite
- a function f is *bounded* with respect to the absolute value α iff there is a constant $C > 0$ such that

$$|f(s)| \leq C\alpha(s), \quad s \in \mathbb{S}$$

- f is *exponentially bounded* iff it is bounded with respect to an absolute value

Harmonic Analysis on Semigroups: Central Result

The central result, see Berg, Christensen & Ressel (1984) based on Choquet's theorem:

- the set of exponentially bounded *positive definite functions* is a *Bauer simplex* whose extreme points are the bounded semicharacters...
- this means that we have the following representation for positive definite functions on \mathbb{S} :

$$f(s) = \int_{\mathbb{S}^*} \chi(s) d\mu(\chi)$$

where μ is a Radon measure on \mathbb{S}^*

Harmonic Analysis on Semigroups: Simple Examples

- *Berstein's theorem* for the Laplace transform

$$\mathbb{S} = (\mathbf{R}_+, +), \chi_x(t) = e^{-xt} \quad \text{and} \quad f(t) = \int_{\mathbf{R}_+} e^{-xt} d\mu(x)$$

- with involution, *Bochner's theorem* for the Fourier transform

$$\mathbb{S} = (\mathbf{R}, +), \chi_x(t) = e^{2\pi ixt} \quad \text{and} \quad f(t) = \int_{\mathbf{R}} e^{2\pi ixt} d\mu(x)$$

- *Hamburger's solution* to the unidimensional moment problem

$$\mathbb{S} = (\mathbf{N}, +), \chi_x(k) = x^k \quad \text{and} \quad f(k) = \int_{\mathbf{R}} x^k d\mu(x)$$

The Option Pricing Problem Revisited

- the basket option payoffs $(w^T x - K)_+$ are not ideal in this setting
- solution, use *straddles*: $|w^T x - K|$
- as straddles are just the *sum of a call and a put*, their price can be computed from that of the corresponding call and forward by call-put parity
- the fact that $|w^T x - K|^2$ is a polynomial keeps the complexity low

Payoff Semigroup

- the fundamental semigroup \mathbb{S} is here the multiplicative *payoff semigroup* generated by the cash, the forwards and the straddles:

$$\mathbb{S} = \{1, x_1, \dots, x_n, |w_1^T x - K_1|, \dots, |w_m^T x - K_m|, x_1^2, x_1 x_2, \dots\}$$

- the *semicharacters* are the functions $\chi_x : \mathbb{S} \rightarrow \mathbf{R}$ which evaluate the payoffs at a certain point x

$$\chi_x(s) = s(x), \quad \text{for all } s \in \mathbb{S}$$

The Option Pricing Problem Revisited

- the original static arbitrage problem can be reformulated as

$$\begin{array}{ll} \text{find} & f \\ \text{subject to} & f(|w_i^T x - K_i|) = p_i, \quad i = 1, \dots, m \\ & f(s) = \mathbf{E}_\pi[s], \quad s \in \mathbb{S} \quad (\text{f moment function}) \end{array}$$

- the variable is now $f : \mathbb{S} \rightarrow \mathbf{R}$, a function that associates to each payoff s in \mathbb{S} , its price $f(s)$
- the representation result in Berg et al. (1984) shows when a (price) function $f : \mathbb{S} \rightarrow \mathbf{R}$ can be represented as

$$f(s) = \mathbf{E}_\pi[s]$$

Option Pricing: Main Theorem

If we assume that the asset distribution has a compact support included in \mathbf{R}_+^n , and note e_i for $i = 1, \dots, n + m$ the forward and option payoff functions we get:

A function $f(s) : \mathbb{S} \rightarrow \mathbf{R}$ can be represented as

$$f(s) = \mathbf{E}_\nu[s(x)], \quad \text{for all } s \in \mathbb{S},$$

for some measure ν with compact support, iff for some $\beta > 0$:

- (i) $f(s)$ is positive semidefinite*
- (ii) $f(e_i s)$ is positive semidefinite for $i = 1, \dots, n + m$*
- (iii) $\left(\beta f(s) - \sum_{i=1}^{n+m} f(e_i s) \right)$ is positive semidefinite*

this turns the basket arbitrage problem into a *semidefinite program*

Semidefinite Programming

A *semidefinite program* is written:

$$\begin{aligned} & \text{minimize} && \mathbf{Tr} CX \\ & \text{subject to} && \mathbf{Tr} A_i X = b_i, \quad i = 1, \dots, m \\ & && X \succeq 0, \end{aligned}$$

in the variable $X \in \mathbf{S}^n$, with parameters $C, A_i \in \mathbf{S}^n$ and $b_i \in \mathbf{R}$ for $i = 1, \dots, m$. Its *dual* is given by:

$$\begin{aligned} & \text{maximize} && b^T \lambda \\ & \text{subject to} && C - \sum_{i=1}^m \lambda_i A_i \succeq 0, \end{aligned}$$

in the variable $\lambda \in \mathbf{R}^m$.

A recent extension of interior point techniques for linear programming shows how to solve these convex programs *very efficiently* (see Nesterov & Nemirovskii (1994), Sturm (1999) and Boyd & Vandenberghe (2003)).

Feasibility Problems

Of course, the related feasibility problems:

$$\begin{array}{ll} \text{find} & X \\ \text{such that} & \mathbf{Tr} A_i X = b_i, \quad i = 1, \dots, m \\ & X \succeq 0, \end{array}$$

and

$$\begin{array}{ll} \text{find} & \lambda \\ \text{such that} & C - \sum_{i=1}^m \lambda_i A_i \succeq 0, \end{array}$$

can be solved as efficiently (setting for example $C = I$ or $b = \mathbf{1}$ in the previous programs).

Also, because most solvers produce both primal and dual solution, we also get a Farkas type *certificate of infeasibility* or a *proof of optimality* in the duality gap.

Option Pricing: a Semidefinite Program

we get a relaxation by only sampling the elements of \mathbb{S} up to a certain degree, the variable is then the vector $f(s)$ with

$$e = (1, x_1, \dots, x_n, |w_1^T x - K_1|, \dots, |w_m^T x - K_m|, x_1^2, x_1 x_2, \dots, |w_m^T x - K_m|^N)$$

testing for the absence of arbitrage is then a *semidefinite program*:

$$\begin{array}{ll} \text{find} & f \\ \text{subject to} & M_N(f(s)) \succeq 0 \\ & M_N(f(e_j s)) \succeq 0, \quad \text{for } j = 1, \dots, n, \\ & M_N\left(f\left(\left(\beta - \sum_{k=1}^{n+m} e_k\right)s\right)\right) \succeq 0 \\ & f(e_j) = p_j, \quad \text{for } j = 1, \dots, n+m \text{ and } s \in \mathbb{S} \end{array}$$

where $M_N(f(s))_{ij} = f(s_i s_j)$ and $M_N(f(e_k s))_{ij} = f(e_k s_i s_j)$

Price Bounds

We can also consider the related problem of finding *bounds* on the price of a straddle, given prices of other similar options:

$$\begin{array}{ll} \text{max./min.} & \mathbf{E}_\pi(|w_0^T x - K_0|) \\ \text{subject to} & \mathbf{E}_\pi(|w_i^T x - K_i|) = p_i, \quad i = 1, \dots, m, \end{array}$$

which, using the previous result becomes the following semidefinite program:

$$\begin{array}{ll} \text{max./min.} & f(e_0) \\ \text{subject to} & M_N(f(s)) \succeq 0 \\ & M_N(f(e_j s)) \succeq 0, \quad \text{for } j = 1, \dots, n, \\ & M_N\left(f\left(\left(\beta - \sum_{k=1}^{n+m} e_k\right)s\right)\right) \succeq 0 \\ & f(e_j) = p_j, \quad \text{for } j = 1, \dots, n+m \text{ and } s \in \mathbb{S} \end{array}$$

where $M_N(f(s))_{ij} = f(s_i s_j)$ and $M_N(f(e_k s))_{ij} = f(e_k s_i s_j)$.

Duality

- the price maximization program is:

$$\begin{aligned} & \text{maximize} && \int_{\mathbf{R}_+^n} (w_0^T x - K_0)^+ \pi(x) dx \\ & \text{subject to} && \int_{\mathbf{R}_+^n} (w_i^T x - K_i)^+ \pi(x) dx = p_i, \quad i = 1, \dots, m \\ & && \int_{\mathbf{R}_+^n} \pi(x) dx = 1, \end{aligned}$$

in the variable $\pi \in \mathcal{K}$.

- the dual is a *portfolio problem*:

$$\begin{aligned} & \text{minimize} && \lambda^T p + \lambda_0 \\ & \text{subject to} && \sum_{i=1}^m \lambda_i (w_i^T x - K_i)^+ + \lambda_0 \geq \psi(x) \text{ for every } x \in \mathbf{R}_+^n \end{aligned}$$

in the variable $\lambda \in \mathbf{R}^{m+1}$.

very intuitive, but completely intractable. . .

Conic Duality

let $\Sigma \subset \mathcal{A}(\mathbb{S})$ be the set of polynomials that are sums of squares of polynomials in $\mathcal{A}(\mathbb{S})$, and \mathcal{P} the set of positive semidefinite sequences on \mathbb{S}

- instead of the conic duality between probability measures and positive portfolios

$$p(x) \geq 0 \Leftrightarrow \int p(x) d\nu \geq 0, \quad \text{for all measures } \nu$$

- we use the duality between positive semidefinite sequences \mathcal{P} and sums of squares polynomials Σ

$$p \in \Sigma \Leftrightarrow \langle f, p \rangle \geq 0 \text{ for all } f \in \mathcal{P}$$

with $p = \sum_i q_i \chi_{s_i}$ and $f : \mathbb{S} \rightarrow \mathbf{R}$, where $\langle f, p \rangle = \sum_i q_i f(s_i)$

Option Pricing: Dual

- the dual of the price maximization problem

$$\begin{aligned}
 &\text{maximize} && f(e_0) \\
 &\text{subject to} && M_N(f(s)) \succeq 0 \\
 &&& M_N(f(e_j s)) \succeq 0, \quad \text{for } j = 1, \dots, n, \\
 &&& M_N\left(f\left(\left(\beta - \sum_{k=1}^{n+m} e_k\right)s\right)\right) \succeq 0 \\
 &&& f(e_j) = p_j, \quad \text{for } j = 1, \dots, n+m \text{ and } s \in \mathbb{S}
 \end{aligned}$$

- now becomes...

$$\begin{aligned}
 &\text{minimize} && \sum_{j=1}^{n+m} p_j \lambda_j + \lambda_{n+m+1} \\
 &\text{subject to} && \sum_{j=1}^{n+m} \lambda_j e_j(x) + \lambda_{n+m+1} - |w_0^T x - K_0| \\
 &&& = q_0(x) + \sum_{j=1}^{n+m} q_j(x) e_j(x) + \left(\beta - \sum_{k=0}^{n+m} e_k(x)\right) q_{n+1}(x)
 \end{aligned}$$

in the variables $\lambda \in \mathbf{R}^{n+m+1}$ and $q_j \in \Sigma$ for $j = 0, \dots, (n+1)$

Option Pricing: Numerical Example

- two assets: x_1, x_2 , we look for bounds on the price of $|x_1 + x_2 - K|$
- simple discrete model for the assets:

$$x = \{(0, 0), (0, 3), (3, 0), (1, 2), (5, 4)\}$$

with probability

$$p = (.2, .2, .2, .3, .1)$$

- the forward prices are given, together with the following straddles:

$$|x_1 - .9|, |x_1 - 1|, |x_2 - 1.9|, |x_2 - 2|, |x_2 - 2.1|$$

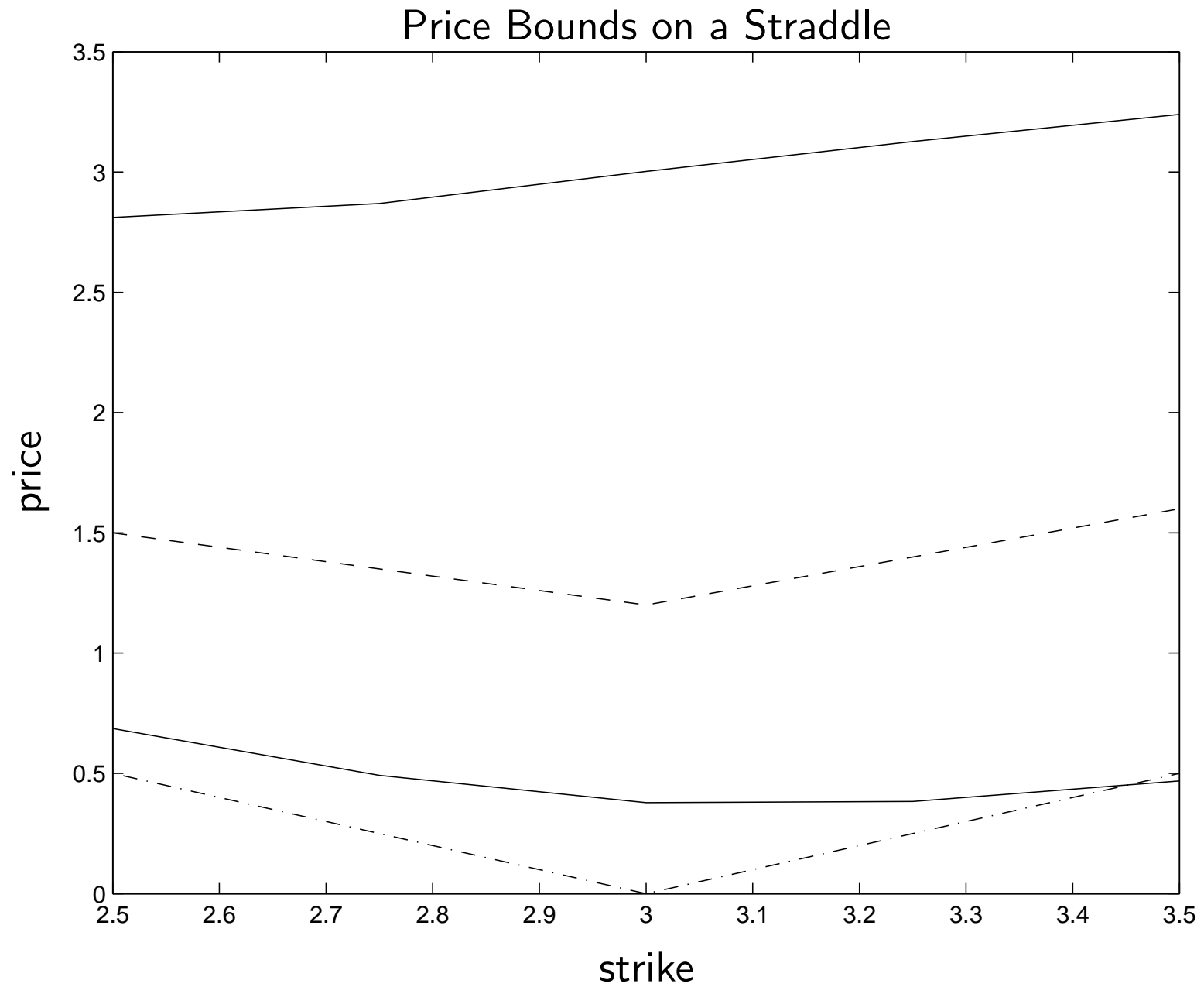


Figure 1: Upper and lower price bounds on a straddle.

Option Pricing: Caveats

- *size*: grows exponentially with the number of assets: no free lunch, even in numerical complexity. . .
- some numerical difficulties

Conclusion

- testing for static arbitrage in option price data is easy in dimension one
- the extension on basket options (swaptions, etc) is NP-hard but good relaxations can be found
- we get a computationally friendly set of conditions for the absence of arbitrage
- small scale problems are tractable in practice as semidefinite programs

References

- Berg, C., Christensen, J. P. R. & Ressel, P. (1984), *Harmonic analysis on semigroups : theory of positive definite and related functions*, Vol. 100 of *Graduate texts in mathematics*, Springer-Verlag, New York.
- Bertsimas, D. & Popescu, I. (2002), 'On the relation between option and stock prices: a convex optimization approach', *Operations Research* **50**(2), 358–374.
- Black, F. & Scholes, M. (1973), 'The pricing of options and corporate liabilities', *Journal of Political Economy* **81**, 637–659.
- Boyd, S. & Vandenberghe, L. (2003), *Convex Optimization*, Cambridge University Press.
- Helgason, S. (1999), *The Radon transform*, 2nd edn, Birkhauser, Boston. Progress in mathematics (Boston, Mass.) ; vol. 5.

- Henkin, G. & Shananin, A. (1990), 'Bernstein theorems and Radon transform, application to the theory of production functions', *American Mathematical Society: Translation of mathematical monographs* **81**, 189–223.
- Laurent, J. & Leisen, D. (2000), Building a consistent pricing model from observed option prices, *in* M. Avellaneda, ed., 'Quantitative Analysis in Financial Markets', World Scientific Publishing.
- Nesterov, I. & Nemirovskii, A. (1994), *Interior-point polynomial algorithms in convex programming*, Society for Industrial and Applied Mathematics, Philadelphia.
- Ramm, A. G. & Katsevich, A. I. (1996), *The radon transform and local tomography*, CRC Press, Boca Raton.
- Sturm, J. F. (1999), 'Using sedumi 1.0x, a matlab toolbox for optimization over symmetric cones.', *Working paper, Department of Quantitative Economics, Maastricht University, The Netherlands.* .