

Efficiency Loss in Resource Allocation and Supplier Selection Games

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Agenda

Allocation of divisible resources
(e.g., bandwidth in a network)

Simple “mechanisms”

Study efficiency loss, w.r.t. social optimum

Outline

1. Single resource, fixed supply
2. Multiple resources, fixed supplies
3. Competing suppliers, fixed demand

Price Mechanisms

1. Single price, market clearing

2. Efficiency metric:

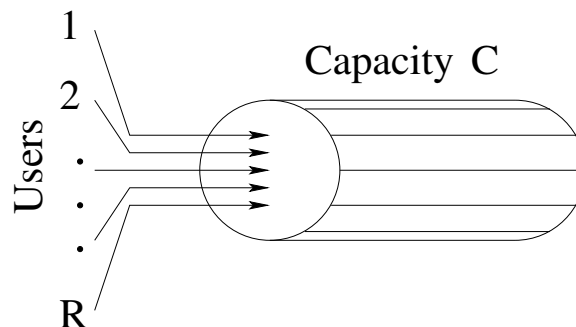
Social Welfare = Utility of consumers – Cost of resources

3. Price taking \implies full efficiency

Our main question:

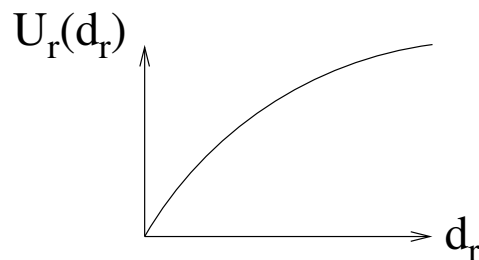
What happens if selfish market participants are price anticipating?

PART I: Single Resource, Fixed Supply



Amount d_r of resource to user $r \implies$ monetary utility $U_r(d_r)$

U_r : concave, strictly increasing, nonnegative



The Social Optimum

$$\begin{array}{ll} \text{maximize} & \sum_r U_r(d_r) \\ \text{subject to} & \sum_r d_r \leq C \\ & \mathbf{d} \geq 0 \end{array}$$

A Pricing Mechanism

User r submits a bid w_r .

Amount of resource received: $\frac{w_r}{w_1 + \dots + w_R} C$

Example: $w_1 = 2$ $d_1 = 2C/5$
 $w_2 = 3$ $d_2 = 3C/5$

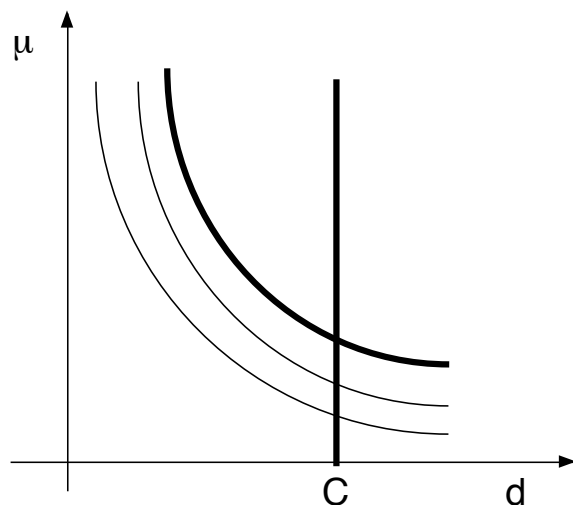
All of the resource is allocated

Unit price of resource: $\mu = \frac{w_1 + \dots + w_R}{C}$

$$d_r = \frac{w_r}{\mu}$$

“Supply = Demand” Interpretation

User r submits w_r : same as submitting “demand curve”: $d_r = \frac{w_r}{\mu}$



Mechanism clears the market:

$$\frac{w_1}{\mu} + \dots + \frac{w_R}{\mu} = \text{total demand} = \text{supply} = C$$

Users as Price Takers

Given price μ , user r solves:

$$\max_{w_r \geq 0} U_r \left(\frac{w_r}{\mu} \right) - w_r$$

Theorem 1 (Existence of Competitive Equilibrium; Kelly, 1997)

There exist w and μ such that:

(a) *w_r is optimal for user r given the price μ .*

(b) *$(w_1 + \dots + w_R)/\mu = C$.*

The resulting allocation is socially optimal.

Users as Price Anticipators

Suppose users *know* the price setting procedure.

Given $(w_s, s \neq r)$, user r solves:

$$\max_{w_r \geq 0} U_r \left(\frac{w_r}{w_r + \sum_{s \neq r} w_s} C \right) - w_r$$

This is now a **game**, where the strategy of user r is the bid w_r .

Interested in Nash equilibria

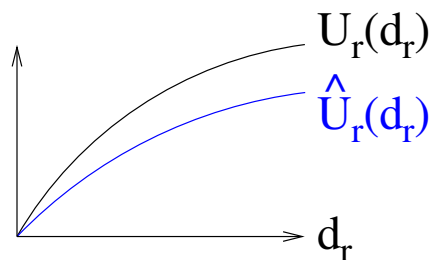
Nash Equilibrium

Theorem 2 (Hajek & Gopal, 2002) Assume $R > 1$.

There exists a *unique* Nash equilibrium w .

The resulting allocations d_r are the unique socially optimal solution for *modified utilities*:

$$\hat{U}_r(d_r) = \left(1 - \frac{d_r}{C}\right) U_r(d_r) + \left(\frac{d_r}{C}\right) \left(\frac{1}{d_r} \int_0^{d_r} U_r(z) dz\right)$$



Efficiency Loss

Theorem 3 *The efficiency loss is no more than 25%:*

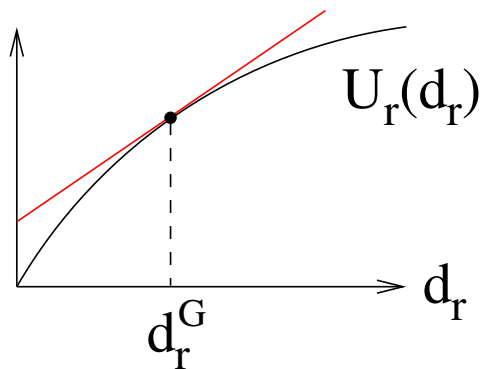
$$(\text{Nash eq. utility}) \geq \frac{3}{4} \times (\text{socially optimal utility})$$

Furthermore, this bound is tight.

Efficiency Loss: Proof

$$? \leq \frac{\sum_r U_r(d_r^G)}{\sum_r U_r(d_r^S)} \leq 1$$

Linearize:



Worst case: $U_r(d_r) = \alpha_r d_r$ (linear utilities)

Efficiency Loss: Proof

Optimize over all games with linear utility functions:

$$U_r(d_r) = \alpha_r d_r$$

Assume (w.l.o.g.): $C = 1$, $\max_r \alpha_r = \alpha_1 = 1$.

$$\begin{array}{ll} \text{minimize} & d_1^G + \sum_r \alpha_r d_r^G \\ \text{subject to} & \mathbf{d}^G \text{ is a Nash equilibrium} \\ & \text{when utilities are } U_r(d_r) = \alpha_r d_r; \\ & \text{and } \max_r \alpha_r = \alpha_1 = 1 \end{array}$$

The Worst Case

$$C = 1$$

$$U_1(d_1) = d_1$$

$$U_r(d_r) \approx d_r/2$$

$$R \rightarrow \infty$$

$$d_1^G \rightarrow 1/2$$

$$\sum_{r>1} d_r^G \rightarrow 1/2$$

$$\mu \rightarrow 1/2$$

$$\sum_r U_r(d_r^S) = 1$$

$$\sum_r U_r(d_r^G) \rightarrow 3/4$$

PART II: Multiple Resources, Fixed Supplies

C_j units of resource j available, to be allocated

R users with concave utility functions U_r
(concave, nondecreasing, continuous, nonnegative)

Social optimum:

$$\begin{aligned} & \text{maximize} && \sum_r U_r(\mathbf{d}_r) \\ & \text{subject to} && \sum_r d_r^j \leq C_j, \text{ for all resources } j \\ & && \mathbf{d} \geq 0 \end{aligned}$$

A Pricing Mechanism

Independent markets for each resource

User r submits a bid w_r^j , for each resource j

Amount of resource received: $d_r^j(w) = \frac{w_r^j}{w_1^j + \dots + w_R^j} C_j$

User r payoff: $U_r(d_r(w)) - \sum_j w_r^j$ **Concave!**

Existence of Nash equilibrium if $\frac{\partial U_r}{\partial d_r^j} = \infty$ when $d_r^j = 0$

(else, can guarantee existence in an extended game)

Multiple Resources: Efficiency Loss

Theorem 4 *The efficiency loss is no more than 25%:*

$$(\text{Nash eq. utility}) \geq \frac{3}{4} \times (\text{socially optimal utility})$$

Proof outline:

- Replace utility functions by linear ones
 chosen to have the “right” value and slope at the Nash equilibrium
- Nash equilibrium and its social welfare is preserved
- Social welfare same or higher (Nash efficiency loss worsens)
- Linear utilities \implies decoupled single-resource problems
- Worst-case efficiency loss limited to 25%

PART III: Competing Suppliers, Fixed Demand

Motivation: resource allocation in electricity networks.

n suppliers bid to meet a fixed demand D .

$C_i(s_i)$: Monetary cost to supplier i of producing s_i units

Assume: C_i is convex, strictly increasing, nonnegative.

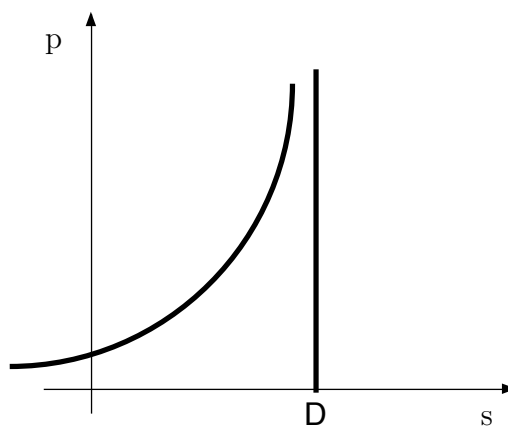
Social optimization:

$$\begin{array}{ll} \text{minimize} & \sum_i C_i(s_i) \\ \text{subject to} & \sum_i s_i = D \\ & \mathbf{s} \geq 0 \end{array}$$

A Pricing Mechanism

Supplier i chooses w_i and submits a *supply function*:

$$S(p; w_i) = D - \frac{w_i}{p}$$



System operator chooses p so that $\sum_i S(p; w_i) = D$

Suppliers as Price Takers

Suppliers maximize profit (revenue minus cost)

Theorem 5 *There exist w and p such that:*

(a) w_i is optimal for supplier i , given the price p .

(b) $\sum_i S(p; w_i) = D$

Furthermore, the resulting supplies s_1, \dots, s_n are socially optimal.

Suppliers as Price Anticipators

Suppose suppliers anticipate the effect of their bid on the price

Theorem 6 *If $n > 2$, there exists a Nash equilibrium w , and the resulting allocation is unique.*

*Furthermore, the resulting production vector s is the unique **socially optimal** solution for **modified costs**:*

$$\hat{C}_i(s_i) = C_i(s_i) \left(1 + \frac{s_i}{(n-2)D} \right) - \frac{1}{(n-2)D} \int_0^{s_i} C_i(z) dz$$

Efficiency Loss

Theorem 7 *Assume $n > 2$. Then:*

$$\text{(Nash eq. prod. cost)} \leq \left(1 + \frac{1}{n-2}\right) \text{(socially opt. prod. cost)}$$

Furthermore, this bound is tight.

Result also holds with uncertain (but still inelastic) demand.

Worst case: Piecewise linear cost functions.