

# Strategic Buyers in a Sum Bid Game for Flat Networks

Bruce Hajek and Sichao Yang

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## Abstract

This note addresses scenarios of buyers purchasing rate along paths in a nonhierarchical communication network. A sum bid game is proposed in which each buyer submits a bid, also equal to a payment, and the network allocates rate in a weighted proportionally fair fashion, with the bids as the weights. This is in contrast to an itemized bid game, considered recently by Johari and Tsitsiklis, under which a buyer separately bids for rate on each link along a path. Johari and Tsitsiklis show that the worst case efficiency for allocation by strategic buyers is 75% for the itemized bid game. It is argued in this note that the sum bid game captures more fully the interactions among strategic buyers. It is shown in a simple example that the worst case efficiency for the sum bid game is zero, meaning that the cost of strategic behavior on the part of the buyers can approach 100%. It is also shown that Nash equilibria can fail to exist for the sum bid game, and the payoff function of a buyer need not be concave.

## 1 Introduction

This note addresses scenarios of buyers purchasing rate along paths in a flat (i.e. nonhierarchical) communication network. A sum bid game is proposed in which each buyer submits a bid, also equal to a payment, and the network allocates rate in a weighted proportionally fair fashion, with the bids as the weights. This is in contrast to an itemized bid game, considered recently by Johari and Tsitsiklis [1], under which a buyer separately bids for rate on each link along a path. In either game, buyers in Nash equilibrium are considered.

This note is organized as follows. An itemized bid game, actually a slight variation of the extended network game defined by Johari and Tsitsiklis [1], is defined in Section 2. A sum bid game is defined in Section 3, and it is argued that the sum bid game captures more fully the interactions among strategic buyers than does the itemized bid game. The efficiencies for the two games are compared in Section 4 for a specific network consisting of links in series. The efficiency of allocation for the itemized bid game in the example is at least 75%, the worst case bound of [1]. However, the worst case efficiency for the sum bid game is zero, meaning that the cost of strategic behavior on the part of the buyers in the sum bid game can approach 100%. The sum bid game is explored further for a network of two links in Sections 5 and 6. The example of Section 6 shows that Nash equilibria can fail to exist for the sum bid game, and the

payoff function of a buyer need not be concave. The basic network notation is given in the remainder of this section.

Consider a network with a set of links  $L$ , set of buyers  $R = \{r_1, \dots, r_K\}$ , and vector of positive link capacities  $C = (C_l : l \in L)$ . Corresponding to each buyer  $r$  is a single path in the network, which is a subset of  $L$ . By a slight abuse of notation we write  $l \in r$  if link  $l$  is in the path of buyer  $r$ . A feasible rate vector is a vector  $(x_r : r \in R)$  with nonnegative coordinates such that  $\sum_{r:l \in r} x_r \leq C_l$  for each  $l$ . For each buyer  $r$  there is given a valuation function<sup>1</sup> $U_r$ , assumed to be a nonnegative, strictly increasing, differentiable, concave function on  $\mathbb{R}_+ = [0, \infty)$ . A feasible rate vector  $x$  is said to be *efficient* if it maximizes the sum of values  $\sum_r U(x_r)$  over the set of all feasible rate vectors. The *efficiency* of a rate vector is the sum of values for the rate vector, divided by the sum of values for an efficient rate vector.

## 2 An itemized bid game

The focus of this note is the sum bid game, but for comparison purposes an itemized bid game is considered in this section. The definition of itemized bid game given here is a variation of the definition given by Johari and Tsitsiklis [1] (see the remark below). Each buyer  $r$  submits a bid  $\alpha_{lr}$  for each link  $l \in r$ . Roughly speaking, each link assigns capacity to the buyers in proportion to their bids on that link. The corresponding rate assigned to a buyer is the minimum of the rates that she receives from the links in her path. A question arises, however, about how to allocate capacity on a link if all the bids for that link are zero. The following allocation will be used. Given the bids  $(\alpha_{lr})$ , let  $\mathcal{B}$  denote the set of links for which at least one bid is positive:  $\mathcal{B} = \{l : \sum_{r:l \in r} \alpha_{lr} > 0\}$ . If  $l \in \mathcal{B}$  and  $r$  is a buyer with  $l \in r$ , define  $x_{lr}^o$  to be the proportional share of capacity

$$x_{lr}^o = \frac{\alpha_{lr} C_l}{\sum_{r':l \in r'} \alpha_{lr'}}.$$

Let  $\mathcal{Y}^o$  denote the set of feasible rate vectors  $y = (y_r : r \in R)$  satisfying the two constraints:

$$\begin{aligned} y_r &\leq x_{lr}^o && \text{whenever } l \in r \cap \mathcal{B} \\ \sum_{r:l \in r} y_r &\leq C_l && \text{for } l \in L \cap \mathcal{B}^c \end{aligned}$$

Let  $\mathcal{Y}$  denote the set of maximal elements of  $\mathcal{Y}^o$ . Assignments in  $\mathcal{Y}$  correspond to the network allocating capacity on the links in  $\mathcal{B}^c$  in a “best effort” fashion. Define a pair of vectors  $\bar{y} = (\bar{y}_r)$  and  $\underline{y} = (\underline{y}_r)$  as follows:

$$\begin{aligned} \bar{y}_r &= \max\{y_r : y \in \mathcal{Y}\} \\ \underline{y}_r &= \min\{y_r : y \in \mathcal{Y}\} \end{aligned}$$

The actual rate allocation vector  $x = (x_r : r \in R)$  is given by  $x = \underline{y}$ . To emphasize that the sets  $\mathcal{Y}^o$  and  $\mathcal{Y}$  and vectors  $x$ ,  $\bar{y}$ , and  $\underline{y}$  are functions of  $\alpha$ , we sometimes write

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<sup>1</sup>In the literature on communication systems and networks, such functions are usually called utility functions. However, in the economics literature, a buyer’s utility includes the payment.

them as  $\mathcal{Y}^o(\alpha)$ ,  $\mathcal{Y}(\alpha)$ ,  $x(\alpha)$ ,  $\bar{y}(\alpha)$ , and  $\underline{y}(\alpha)$ , respectively. The philosophy of using the allocation  $x = \underline{y}$  is that  $\underline{y}_r$  is the rate guaranteed to buyer  $r$  no matter which rate allocation vector  $y$  is selected by the network from the best effort set  $\mathcal{Y}$ . It is not hard to see that the following three conditions are equivalent: (i)  $\bar{y} \in \mathcal{Y}$ , (ii)  $\mathcal{Y} = \{\bar{y}\}$ , (iii)  $\bar{y} = \underline{y}$ .

We seek a Nash equilibrium of the game in which the payoff to buyer  $r$  is given by  $P_r(\alpha) = U_r(x_r(\alpha)) - \sum_l \alpha_{lr}$ . The following proposition is proved in the appendix.

**Proposition 2.1** *A Nash equilibrium point (NEP) exists. A vector of bids  $(\alpha_{lr})$  is a Nash equilibrium point if and only if it satisfies the following conditions:*

$$x_r = x_{lr}^o \quad \text{whenever } l \in r \cap \mathcal{B} \quad (1)$$

$$\text{For each } l \in L, \text{ the cardinality of } \{r : l \in r \text{ and } \alpha_{lr} > 0\} \text{ is not one} \quad (2)$$

$$U'_r(x_r) - \sum_{l \in r \cap \mathcal{B}} \frac{\sum_{r'} \alpha_{lr'}}{C_l - x_r} \begin{cases} \leq 0 & \text{if } x_r = 0 \\ = 0 & \text{if } 0 < x_r < C_l \text{ for all } l \in r \\ \geq 0 & \text{if } x_r = C_l \text{ for some } l \in r \end{cases} \quad (3)$$

**Remark** A simpler allocation mechanism would be for the network to allocate zero capacity on the links in  $\mathcal{B}^c$ . The paper [1] gives three simple examples for which Nash equilibrium points do not exist for such a mechanism. The paper [1] defines a related extended game such that Nash equilibrium points do exist. The extended game of [1] includes routing, and it can handle the case that the valuation functions are not strictly increasing. Here the allocation mechanism is changed but the space of bids is not extended. It is not clear how to incorporate routing or to include valuation functions that are not strictly increasing into the itemized bid game defined above. A similar philosophy is used to define a sum bid game in the next section.

### 3 A sum bid game

A sum bid game is defined in this section. Recall that the system optimization problem is to find a feasible assignment vector to maximize the sum of the values,  $\sum_r U_r(x_r)$ . Kelly and his co-workers (see [2]) noticed that the social optimization problem decouples into a buyer problem and a network problem. The buyers submit bids  $w_r$  to the network, and given the vector of bids  $(w_r : r \in R)$ , the network determines a feasible rate allocation vector that maximizes

$$\sum_{r \in R} w_r \log x_r$$

Again there is some ambiguity if some of the bids are zero. The specific rate allocation rule for the sum bid game is defined as follows. Let  $\mathcal{Y}$  denote the set of maximal elements of the set of solutions to the problem of maximizing

$$\sum_{r \in R} w_r \log y_r$$

over the set of feasible vectors  $y$ . Define  $\underline{y}$  and  $\bar{y}$  as before, and, finally, let  $x = \underline{y}$ .

*Remarks:* (1) If there is only one link in the network, this game reduces to the same one link game as before.

(2) Let us briefly compare the itemized bid game to the sum bid game. Under the itemized bid game, each buyer is responsible for deciding how to spread her money among the links in her path. In the sum bid game, a buyer  $r$  simply specifies a total amount  $w_r$ , and it is up to the network to distribute the money among the links in an efficient manner. Therefore, strategic buyers of the itemized bid game take into account their own impact on the prices of each of the links they use, but they do not account for the natural shifting of bids that will result by the other buyers in response. Intuitively, the sum bid game requires more sophisticated anticipation on the part of the buyers than the itemized bid game.

## 4 Comparison of efficiencies for a series network

The sum of values is found for a simple network with links in series for three rate allocations: the efficient allocation, the allocation arising from the itemized bid game, and allocations arising from the sum bid game. The network has  $L$  links, numbered 1 through  $L$ , each with unit capacity, and there are  $L + 1$  buyers, numbered 0 through  $L$ . The path of buyer zero contains all  $L$  links, and  $U_0(x) = \gamma x$  for some constant  $\gamma > 0$ . The path of buyer  $l$  for  $1 \leq l \leq L$  contains just the single link  $l$ , and  $U_l(x) = x$ .

*Efficient allocation* To obtain the efficient allocation, set  $x_l = 1 - x_0$  for  $1 \leq l \leq L$ , and to select  $x_0$  to maximize the sum of values:  $\gamma x_0 + L(1 - x_0)$ . The efficient allocation is to grant all the capacity to buyer 0 if  $\gamma > L$ , and grant all the capacity to the other buyers if  $\gamma < L$ . The maximum sum of values, then, is  $\max\{\gamma, L\}$ .

*Allocation for the itemized bid game* Suppose that the buyers are strategic using itemized bids. We shall identify the bid vector  $(\alpha_{10}, \alpha_{20}, \dots, \alpha_{L0}, \alpha_{11}, \dots, \alpha_{LL})$  that is an NEP. The vector of all zeros is not an NEP, because any buyer could increase her payoff by switching to a very small positive bid. Hence, at least one bid is positive. In view of (2), all bids are positive. By (1), buyer 0 purchases the same amount of capacity  $x_0$  on each link, which means that buyers 1 through  $L$  each purchase capacity  $1 - x_0$  on their respective links. Consequently, by condition (3), the bids of buyers 1 through  $L$  are identical. Therefore, the  $L$  link bids of buyer 0 are equal to each other. Let  $a$  denote the bid of buyer 0 on each link, and let  $b$  denote the bid of each single link buyer. It remains to find  $a$  and  $b$  for this to be an NEP. The payoff to buyer zero is  $P_0 = \gamma \left( \frac{a}{a+b} \right) - La$ , and the payoff to buyer  $l$  for  $1 \leq l \leq L$  is  $P_1 = \left( \frac{b}{a+b} \right) - b$ . Maximizing  $P_0$  with respect to  $a$  for  $b$  fixed, and  $P_1$  with respect to  $b$  for  $a$  fixed, yields that there is a unique NEP and it satisfies  $\gamma b = La$ . Therefore, the rate allocation under the NEP vector is given by  $x_0 = \frac{\gamma}{\gamma+L}$  and  $x_l = \frac{L}{\gamma+L}$  for  $1 \leq l \leq L$ . Thus, the sum of values for the unique NEP allocation of the itemized bid game is given by  $\frac{\gamma^2+L^2}{\gamma+L}$ . Assuming that  $L \geq \gamma$ , the efficiency for the itemized bid game is  $\frac{\gamma^2+L^2}{L\gamma+L^2}$ . The efficiency is minimized over  $\gamma$  and  $L$  by taking  $L/\gamma = \sqrt{2} + 1$ , yielding the minimum efficiency  $2 - \sqrt{2} \approx 0.8284$ . Thus, for the network of this example, the loss in efficiency due to the use of strategic buyers with itemized bid allocations is always less than 18%. The

loss is moderate, and is smaller than the maximum efficiency loss for the itemized bid game for arbitrary networks, identified by Johari and Tsitsiklis [1] to be 25%.

*Allocation for the sum bid game* Suppose, next, that the buyers are strategic using sum bids. For notational convenience we write  $w_0$  for the bid of buyer 0 and  $b_l$  for the bid of buyer  $l$ , for  $1 \leq l \leq L$ . We begin by finding conditions on the sum bid vector  $(w_0, b_1, \dots, b_L)$  that are necessary for being an NEP. The vector of all zeros is not an NEP, because any buyer could increase her payoff by switching to a very small positive bid. An NEP must have  $w_0 > 0$ , or else a single link buyer with a positive bid could decrease her bid slightly, and still get rate 1. Also, an NEP must have  $\sum_l b_l > 0$ , or else buyer 0 could decrease her bid slightly and still get rate 1. In summary, both  $w_0$  and  $\sum_l b_l$  are positive for any NEP in the sum bid game. Given the bid vector, the network allocates rate  $x_0$  to buyer 0 and rate  $x_l = 1 - x_0$  to each single link buyer, where  $x_0$  is selected to minimize the NETWORK cost function:

$$w_0 \log(x_0) + \left( \sum_l b_l \right) \log(1 - x_0).$$

This yields the allocation  $x_0 = \frac{w_0}{w_0 + \sum_l b_l}$  and  $x_l = 1 - x_0$  for  $1 \leq l \leq L$ . The payoff functions and their partial derivatives are thus given by

$$\begin{aligned} P_0(w_0, b_1, \dots, b_L) &= \frac{\gamma w_0}{w_0 + \sum_l b_l} - w_0 \\ P_l(w_0, b_1, \dots, b_L) &= \frac{\sum_l b_l}{w_0 + \sum_l b_l} - b_l \quad 1 \leq l \leq L \\ \frac{\partial P_0}{\partial w_0}(w_0, b_1, \dots, b_L) &= \frac{\gamma \sum_l b_l}{(w_0 + \sum_l b_l)^2} - 1 \\ \frac{\partial P_l}{\partial b_l}(w_0, b_1, \dots, b_L) &= \frac{w_0}{(w_0 + \sum_l b_l)^2} - 1 \quad 1 \leq l \leq L \end{aligned}$$

Note that  $P_0(w_0, b_1, \dots, b_L)$  is a differentiable concave function of  $w_0$  for the bids  $b_l$  fixed with  $\sum_l b_l > 0$ . Since any NEP must have  $w_0 > 0$ , any NEP must have  $\frac{\partial P_0}{\partial w_0} = 0$ . Similarly, for  $1 \leq l \leq L$ ,  $P_l(w_0, b_1, \dots, b_L)$  is a differentiable function of  $b_l$ , for the other bids fixed with  $w_0 > 0$ . Since any NEP must have  $b_l > 0$  for some  $l$ , any NEP must have  $\frac{\partial P_l}{\partial b_l} = 0$  (note that this equation doesn't really depend on  $l$ ). Solving these two equations yields that every NEP for the sum bid game satisfies the following conditions.

$$\sum_l b_l = \frac{\gamma}{(\gamma + 1)^2} \quad 1 \leq l \leq L \quad (4)$$

$$w_0 = \left( \frac{\gamma}{\gamma + 1} \right)^2 \quad (5)$$

$$x_0 = \frac{\gamma}{\gamma + 1} \quad x_l = \frac{1}{\gamma + 1} \quad 1 \leq l \leq L \quad (6)$$

Conversely, if a bid vector satisfies (4) and (5), then if any one buyer switches her bid, the payoff of that buyer is a concave function of her new bid, and the payoff is maximized by the original bid because the derivative of the payoff function is zero at the original bid value. Hence, any bid vector satisfying (4) and (5) is an NEP. In

summary, an allocation vector is an NEP for the sum bid game if and only if it satisfies (4) and (5). While the NEP is not unique, all NEPs have the same rate allocation vector, given by (6). Thus, the sum of utilities and the efficiency for the sum bid game are given as follows.

$$\begin{aligned} \text{Sum of utilities for NEP for sum bid game: } & \frac{\gamma^2 + L}{\gamma + 1} \\ \text{Efficiency (for sum bid game assuming } L \geq \gamma): & \frac{\gamma^2 + L}{L(\gamma + 1)} \end{aligned}$$

As  $L \rightarrow \infty$  the efficiency ratio converges to  $1/(\gamma + 1)$ . Also, taking  $\gamma$  to be an integer and  $L = \gamma^2$  yields the efficiency ratio  $2/(1 + \gamma)$ . The ratio converges to zero as  $\gamma \rightarrow \infty$ . Thus, the loss of efficiency due to anarchy can be made arbitrarily close to 100% by selecting  $\gamma$  to be a large enough integer, and  $L = \gamma^2$ .

*Discussion and comparison* Among the NEPs for the sum bid game for this network, there is a unique one such that all single link buyers have the same bid. This NEP is given by  $b_l = \gamma/(L(\gamma + 1)^2)$  and  $w_0 = \frac{\gamma}{\gamma+1}$ . If buyer 0 were to distribute her bid  $w_0$  equally among the  $L$  links, so that she bid  $\alpha_{l0} = \frac{\gamma}{L(\gamma+1)}$  on each link, then an itemized bid vector results. The rate allocation for this itemized bid vector for the itemized bid game is identical to the rate allocation for the sum bid game. Why isn't this bid vector an NEP for the itemized bid game? Suppose a single link buyer  $l$  unilaterally increases her bid. In the itemized bid game the bid only competes against the bid  $\alpha_{l0}$  made by buyer 0 for the same link. Let  $\delta$  denote the resulting increase in rate allocation to buyer  $l$ . The rate allocation increase for buyer  $l$  results in a decrease of rate allocation to buyer 0 on that link, and hence to buyer 0 overall, of size  $\delta$ . However, it also means that buyer 0 is purchasing  $\delta$  units more rate on the other links than she can use. Naturally, buyer 0 would like to rectify this situation by transferring some of the money she is spending on other links to link  $l$ . Buyers in the itemized bid game do not take this reaction into account, whereas buyers in the sum bid game do take this into account, because in the sum bid game the network automatically rebalances the bids of buyer 0.

## 5 The sum bid game in a two link network

The sum link game for the two link network shown in Figure 5 is investigated in this section, in order to illustrate some of the properties of the sum bid game. The links have capacity  $C_1$  and  $C_2$  respectively, and there are three buyers. Buyer 0 uses both of the links, buyer 1 uses link 1, and buyer 2 uses link 2.

We shall begin by identifying the mapping from the bid vector  $w = (w_0, w_1, w_2)$  to the rate vector  $x = \underline{y}$ . First, if  $w_0 = w_1 = w_2 = 0$ , then  $\mathcal{Y}^o = \{y : y_0 + y_1 \leq C_1, y_0 + y_2 \leq C_2\}$  and  $\mathcal{Y} = \{y_0 + y_1 = C_1, y_0 + y_2 = C_2\}$ , which implies  $x = \underline{y} = (0, 0, 0)$ .

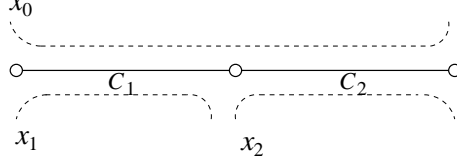


Figure 1: The 2-link network topology

If  $w_0 + w_1 + w_2 > 0$ , then  $x = (x_0, C_1 - x_0, C_2 - x_0)$ , where

$$x_0 = \begin{cases} 0 & \text{if } (w_0 = 0, w_1 + w_2 > 0) \\ C_1 \wedge C_2 & \text{if } (w_0 > 0, w_1 + w_2 = 0) \\ & \text{or } (\frac{w_0}{w_0+w_1} \geq \frac{C_2}{C_1}, w_0 > 0, w_1 > 0, w_2 = 0) \\ & \text{or } (\frac{w_0}{w_0+w_2} \geq \frac{C_1}{C_2}, w_0 > 0, w_1 = 0, w_2 > 0) \\ \text{solution to } \frac{w_0}{x_0} = \frac{w_1}{x_1} + \frac{w_2}{x_2} & \text{if } w \in W \end{cases} \quad (7)$$

where the set  $W$  is defined as follows:

$$\begin{aligned} W &= \{w_0 > 0, w_1 > 0, w_2 > 0\} \\ &\cup \left\{ \frac{w_0}{w_0+w_1} < \frac{C_2}{C_1}, w_0 > 0, w_1 > 0, w_2 = 0 \right\} \\ &\cup \left\{ \frac{w_0}{w_0+w_2} < \frac{C_1}{C_2}, w_0 > 0, w_1 = 0, w_2 > 0 \right\} \end{aligned}$$

Note that  $W = \{w : x_i(w) > 0, i = 0, 1, 2\}$ . The mapping from  $w$  to  $x$  has now been completely identified.

Our goal now is to narrow the search for NEPs. First, it is proved that  $W$  contains all the NEPs, if there are any. Obviously,  $w = (0, 0, 0)$  is not an NEP. If  $(w_0 = 0, w_1 + w_2 > 0)$ , then buyer 1 or buyer 2 can decrease her bid by a small  $\epsilon$  and still get the same rate allocation, so that such  $w$  is not an NEP. If  $(w_0 > 0, w_1 + w_2 = 0)$ , then buyer 0 can decrease her bid by a small  $\epsilon$  and keep the same rate allocation, so that such  $w$  is not an NEP. If  $(\frac{w_0}{w_0+w_1} \geq \frac{C_2}{C_1}, w_0 > 0, w_1 > 0, w_2 = 0)$ , then  $x_1 = C_1 - C_2$  as long as  $w_1 > 0$  because  $x_0$  is constrained by the capacity  $C_2$ . Thus buyer 1 can decrease her bid by a small  $\epsilon$  and still get the same rate allocation, so that such  $w$  is not an NEP. Similarly, in case  $(\frac{w_0}{w_0+w_2} \geq \frac{C_1}{C_2}, w_0 > 0, w_1 = 0, w_2 > 0)$ ,  $w$  is not an NEP. Consequently, by elimination of other possibilities, all NEPs are contained in  $W$ .

Next, we seek the NEP in  $W$ .

**Lemma 5.1** *If  $\bar{w} \in W$ , then  $\frac{\partial x_i}{\partial w_i}$  exists at point  $\bar{w}$ . Furthermore,*

$$\left. \frac{\partial x_i(w_i, \bar{w}_{-i})}{\partial w_i} \right|_{\bar{w}_i} = \frac{1}{\bar{x}_i \frac{\bar{w}_0}{\bar{x}_0^2} + \frac{\bar{w}_1}{\bar{x}_1} + \frac{\bar{w}_2}{\bar{x}_2}} \quad (8)$$

where  $\bar{x}$  is the rate allocation corresponding to  $\bar{w}$ . (If  $\bar{w}_i = 0$  then (8) gives the right directional derivative.)

**Proof.** From (7), for  $w$  in a neighborhood of  $\bar{w}$ , the network rate allocation is given by

$$\frac{w_0}{x_0} = \frac{w_1}{x_1} + \frac{w_2}{x_2}, \quad \text{where } x_1 = C_1 - x_0, \quad x_2 = C_2 - x_0 \quad (9)$$

Thus  $x_0$  is a function of  $w_0, w_1$  and  $w_2$ , which is determined implicitly by,

$$F(w, x_0(w)) = \frac{w_0}{x_0} - \frac{w_1}{C_1 - x_0} - \frac{w_2}{C_2 - x_0} = 0 \quad (10)$$

Obviously,  $F$  is differentiable with respect to  $w$  and  $x_0$  over the set  $W$ . In  $W$ ,

$$\begin{aligned} \frac{\partial F}{\partial x_0} &= - \left( \frac{w_0}{x_0^2} + \frac{w_1}{(C_1 - x_0)^2} + \frac{w_2}{(C_2 - x_0)^2} \right) \\ &= - \left( \frac{w_0}{x_0^2} + \frac{w_1}{x_1^2} + \frac{w_2}{x_2^2} \right) \end{aligned} \quad (11)$$

is continuous and non-zero. The Implicit Function Theorem implies that  $\frac{\partial x_0(w)}{\partial w_0}$  exists if  $w \in W$ . Futhermore,

$$\begin{aligned} \frac{\partial x_0(w)}{\partial w_0} &= - \frac{\partial F}{\partial w_0} \left( \frac{\partial F}{\partial x_0} \right)^{-1} \\ &= \frac{1}{x_0} \frac{1}{\frac{w_0}{x_0^2} + \frac{w_1}{x_1^2} + \frac{w_2}{x_2^2}} \end{aligned} \quad (12)$$

By exactly the same method,  $\frac{\partial x_1(w)}{\partial w_1}$  and  $\frac{\partial x_2(w)}{\partial w_2}$  can be derived. So (8) is proved for  $i = 0, 1, 2$ .  $\square$

The payoff function of each buyer is also differentiable at point  $\bar{w} \in W$ , and

$$\frac{\partial P_i(w_i, \bar{w}_{-i})}{w_i} \Big|_{\bar{w}_i} = U'_i(x(\bar{w})) \frac{\partial x_i(w_i, \bar{w}_{-i})}{\partial w_i} \Big|_{\bar{w}_i} - 1 \quad (13)$$

A necessary condition for a vector  $\bar{w}$  to be an NEP is that  $\bar{w}$  must be in  $W$  and satisfy the first order necessary condition (FONC): For each buyer  $i$

$$\frac{\partial P_i(w_i, \bar{w}_{-i})}{w_i} \Big|_{\bar{w}_i} \leq 0 \quad \text{with equality if } \bar{w}_i > 0. \quad (14)$$

Combining the equations (8), (13) and (14) yields that a vector  $w \in W$  satisfies the FONC if and only if

$$\frac{U'_i(x_i)}{x_i} \leq \frac{w_0}{x_0^2} + \frac{w_1}{x_1^2} + \frac{w_2}{x_2^2} \quad \text{with equality if } w_i > 0, \quad i = 0, 1, 2 \quad (15)$$

where  $x_0, x_1, x_2$  are determined by  $w_0, w_1, w_2$  in (9).

**Proposition 5.2** *There exists a bid vector  $w^*$  in  $W$  satisfying the FONC. Let  $x^* = (x_0^*, C_1 - x_0^*, C_2 - x_0^*)$  be the corresponding rate allocation. The value of  $x_0^*$  for all such  $w^*$ 's is uniquely defined by*

$$\frac{U'_0(x_0)}{x_0} = \max \left\{ \frac{U'_1(C_1 - x_0)}{C_1 - x_0}, \frac{U'_2(C_2 - x_0)}{C_2 - x_0} \right\} \quad (16)$$

*If the three curves  $\frac{U'_0(x_0)}{x_0}$ ,  $\frac{U'_1(C_1 - x_0)}{C_1 - x_0}$  and  $\frac{U'_2(C_2 - x_0)}{C_2 - x_0}$  do not all meet at a single point (i.e. there is not a "triple intersection"), then the value of  $w^*$  is also unique.*

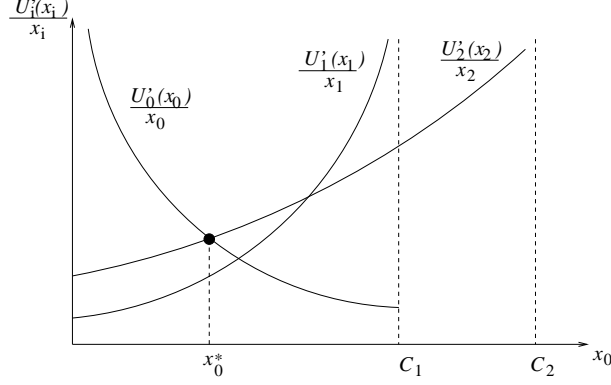


Figure 2: The uniqueness of  $x^*$

**Proof.** Let  $w^* \in W$  satisfy the FONC, and let  $x^* = x(w^*)$ . Since  $w_0^* > 0$ , (15) implies that  $\frac{U'_0(x_0^*)}{x_0^*} \geq \max \left\{ \frac{U'_1(x_1^*)}{x_1^*}, \frac{U'_2(x_2^*)}{x_2^*} \right\}$ . Since  $w_1^* + w_2^* > 0$ , (15) implies that  $\frac{U'_0(x_0^*)}{x_0^*} \leq \max \left\{ \frac{U'_1(x_1^*)}{x_1^*}, \frac{U'_2(x_2^*)}{x_2^*} \right\}$ . So  $x_0^*$  must satisfy (16).

Next we show the existence and uniqueness of  $x^*$ . From the definition of the valuation function, we know that  $U'_i(0) > 0$ . Thus  $\lim_{x_i \rightarrow 0} \frac{U'_i(x_i)}{x_i} \rightarrow +\infty$ . There must be an intersection point of the curve  $\frac{U'_0(x_0)}{x_0}$  and  $\max \left\{ \frac{U'_1(C_1 - x_0)}{C_1 - x_0}, \frac{U'_2(C_2 - x_0)}{C_2 - x_0} \right\}$ . Also because the function  $\frac{U'_i(x_i)}{x_i}$  is monotonically decreasing, the intersection point is unique. This result can be easily seen in Figure 2. Let  $x^* = (x_0^*, x_1^*, x_2^*)$  where  $x_0^*$  is the unique value of  $x_0$  satisfying (16).

Next, we show there is a  $w^*$  so that  $x^* = x(w^*)$  and  $w^*$  satisfies FONC. Suppose  $x^*$  is not a triple intersection of the curves  $\frac{U'_i(x_i)}{x_i}$ . Then we can let  $\frac{U'_1(x_1^*)}{x_1^*} < \frac{U'_0(x_0^*)}{x_0^*}$  without loss of generality as shown in Figure 2. Thus  $w_1^* = 0$  and the FONC can be written as

$$\begin{cases} \frac{U'_0(x_0^*)}{x_0^*} = \frac{w_0}{x_0^{*2}} + \frac{w_2}{(C_2 - x_0^*)^2} \\ \frac{w_0}{x_0^*} = \frac{w_2}{x_2^*} \end{cases} \quad (17)$$

There is a unique solution  $w^*$  to (17). On the other hand, if  $x^*$  is the triple intersection. Then the FONC holds if

$$\begin{cases} \frac{U'_0(x_0^*)}{x_0^*} = \frac{w_0}{x_0^{*2}} + \frac{w_1}{(C_1 - x_0^*)^2} + \frac{w_2}{(C_2 - x_0^*)^2} \\ \frac{w_0}{x_0^*} = \frac{w_1}{x_1^*} + \frac{w_2}{x_2^*} \end{cases} \quad (18)$$

There are two linear equations in (18), while there are three variables  $(w_0, w_1, w_2)$ . Thus, the solution  $w^*$  is not unique.  $\square$

**Remark** Except in the case that  $x^*$  is a point of triple intersection, then either buyer 1 or buyer 2 has the bid 0. For example, if  $w_1^* = 0$ , then the price of the link 1 is 0. In this case, buyer 0 and buyer 2 bid for the capacity on link 2. Buyer 1 enjoys some free capacity on link 1, but is not willing to compete for more capacity.

To study the existence of the NEP further, we find the second derivative of  $x_i$  with

respect to  $w_i$  in  $W$ . We find

$$\begin{cases} \frac{\partial^2 x_0}{\partial w_0^2} = -\frac{1}{x_0^3} \frac{1}{\left(\frac{w_0}{x_0} + \frac{w_1}{x_1} + \frac{w_2}{x_2}\right)^3} \left[ \frac{w_1 C_1}{x_1^3} + \frac{w_2 C_2}{x_2^3} \right] < 0 \\ \frac{\partial^2 x_1}{\partial w_1^2} = -\frac{2}{x_1^3} \frac{1}{\left(\frac{w_0}{x_0} + \frac{w_1}{x_1} + \frac{w_2}{x_2}\right)^3} \left[ \frac{w_0 C_1}{x_0^3} + \frac{w_2}{x_2^3} (C_2 - C_1) \right] \\ \frac{\partial^2 x_2}{\partial w_2^2} = -\frac{2}{x_2^3} \frac{1}{\left(\frac{w_0}{x_0} + \frac{w_1}{x_1} + \frac{w_2}{x_2}\right)^3} \left[ \frac{w_0 C_2}{x_0^3} + \frac{w_1}{x_1^3} (C_1 - C_2) \right] \end{cases} \quad (19)$$

The relations (19) show that if  $C_1 = C_2$ , then  $x_i$  is a concave function of  $w_i$  over the region such that  $w \in W$ . Actually, if  $C_1 = C_2$ ,  $x_i$  is a concave function of  $w_i$  over the whole feasible region  $\{w : w_i > 0\}$ . The detailed proof is not shown here. Therefore, if  $C_1 = C_2$ , then  $x^*$  is the unique NEP. Unfortunately, an NEP need not always exist, as shown in the next section.

## 6 An example for which there is no NEP for the sum bid game

The following example based on the two link network considered in the previous section demonstrates that there can fail to be an NEP for the sum bid game in a two link network. Suppose that  $C_1 = 5$  and  $C_2 = 8$ , and that the valuation functions of the buyers are  $U_0(x) = U_2(x) = x$ . The valuation function  $U_1(x)$  can be any concave function with  $\left. \frac{U_1'(x)}{x} \right|_{x=4} < \frac{1}{4}$ . By (16) and (17) there is a unique bid  $w^*$  satisfying the FONC given by  $w^* = (2, 0, 2)$  with the associated rate vector  $x^* = (4, 1, 4)$ .

Fix  $w_0^*$  and  $w_1^*$ . If  $w_2 = w_2^*$ , the rate allocation to buyer 2 is  $x_2^*$ , and the payoff of buyer 2 is,

$$P_2(w_0^*, w_1^*, w_2^*) = U_2(x_2^*) - w_2^* = 2$$

While if  $w_2 = 0$ , the rate allocation to buyer 2 is  $C_2 - C_1$ , and the payoff of buyer 2 is,

$$P_2(w_0^*, w_1^*, 0) = U_2(C_2 - C_1) = 3$$

Thus,  $P_2(w_0^*, w_1^*, 0) > P_2(w_0^*, w_1^*, w_2^*)$  so that  $x^*$  is not an NEP. Also because the point satisfying the FONC is unique, we conclude that there is no NEP in this example. Intuitively, in this specific case,  $C_1$  is close to  $x_0^*$ , so that buyer 2 does not lose much capacity by decreasing her bid from 2 to 0. Figure ?? shows the payoff function  $P_2(w_0^*, w_1^*, w_2)$  as a function of  $w_2$ . Note that it is not concave, which helps explain the lack of an NEP.

## 7 Conclusions

The results of the sum bid game are mixed. On the positive side, the sum bid game accounts more fully for the interaction among strategic buyers than the itemized bid game. On the negative side, an NEP need not exist for the sum bid game. An open problem is to find an appealing variation of the sum bid game for which an NEP

exists. Or, perhaps a point satisfying the FONC exists in great generality, and is not too far from being an NEP. Of maybe an NEP exists for the sum bid game for “most” networks. These questions are left for future investigation.

## 8 Appendix: Proof of Proposition 2.1

Proposition 2.1 is proved in this appendix. First the existence of an NEP is demonstrated. The first half of the proof is the same as the proof in [1] that an NEP exists for the extended game defined therein. It begins by considering the perturbed game, called “the  $\epsilon$  game,” obtained by supposing that an outside buyer bids  $\epsilon$  on each link. The rate allocated to buyer  $r$  is thus given by  $x_r^\epsilon(\alpha) = \min_{l \in r} x_{lr}^{\alpha, \epsilon}(\alpha)$  where

$$x_{lr}^{\alpha, \epsilon}(\alpha) = \frac{\alpha_{lr} C_l}{\epsilon + \sum_{r'} \alpha_{lr'}}.$$

The payoff function for buyer  $r$  in the  $\epsilon$  game is  $P_r^\epsilon(\alpha) = U_r(x_r^\epsilon(\alpha)) - \sum_l \alpha_{lr}$ . For each fixed  $\epsilon > 0$  and each buyer  $r$ , the payoff function  $P_r^\epsilon$  is a strictly concave function of the buyer’s own bid vector  $(x_{lr} : l \in r)$  for fixed values of the other bids, and is a continuous function of the vector of all bids. Also, because of the capacity constraints, without loss of generality the vector of bids can be restricted to a compact set. Rosen’s theorem, based on Kakutani’s fixed point theorem, implies that an NEP  $\alpha^\epsilon$  exists for each  $\epsilon > 0$ . Along some subsequence of  $\epsilon \rightarrow 0$ , the set of NEP bid vectors  $\alpha^\epsilon$  and the corresponding allocations  $x^\epsilon$  converge to some respective limit vectors  $\alpha^*$  and  $x^*$ . (The proof to this point follows exactly the proof in [1]).

Let  $\mathcal{B}$  denote the set of links with at least one nonzero bid for the limit assignment  $\alpha^*$ . Since the link allocation function is continuous if the sum of link bids is bounded away from zero,  $x_{lr}^{\alpha^\epsilon} \rightarrow x_{lr}^*$  for  $l \in \mathcal{B}$ . Since  $x^\epsilon \in \mathcal{Y}^o(\alpha^\epsilon)$  for all  $\epsilon$ , it follows that  $x^* \in \mathcal{Y}^o(\alpha^*)$ . In particular,  $y_r(\alpha^*) \leq x_r^* \leq \bar{y}_r(\alpha^*)$  for all  $r$ . The next step in the proof is to show that  $x^*$  is the rate assignment for bid vector  $\alpha^*$ .

A key claim is that  $x^* = \bar{y}(\alpha^*)$ . The claim is now proved. Suppose for the sake of argument by contradiction that the claim is false. Then there is some route  $r$  such that  $x_r^* < \bar{y}_r(\alpha^*)$ . Let  $k$  denote the number of links  $l$  in  $r \cap \mathcal{B}^c$ , and let  $\delta$  be a small positive number such that  $U_r(\bar{y}_r(\alpha^*)) - U_r(x_r^*) - k\delta > 0$ . For each  $\epsilon$  in the subsequence being considered, let  $\tilde{x}_r^\epsilon$  denote the value of  $x_r^\epsilon$  that would result if all the bids  $\alpha_{lr}^\epsilon$  for  $l \in r \cap \mathcal{B}^c$  were increased by  $\delta$ , for a small  $\delta > 0$ , and all other bids are kept the same. Since the bids of all other buyers for the links  $l \in r \cap \mathcal{B}^c$  converge to zero as  $\epsilon \rightarrow 0$ , it follows that  $\tilde{x}_r^\epsilon \rightarrow \bar{y}_r(\alpha^*)$ . Therefore, the net gain in payoff due to adding  $\delta$  to the bids for links  $l \in r \cap \mathcal{B}^c$  converges to  $U_r(\bar{y}_r(\alpha^*)) - U_r(x_r^*) - k\delta > 0$ . This contradicts the definition of  $\alpha^\epsilon$ , so that the proof of the claim by contradiction is complete. An immediate consequence of the claim is that the limit  $x^*$  satisfies  $x^* = x(\alpha^*) = \bar{y}(\alpha^*) = y(\alpha^*)$ .

Fix a buyer  $\tilde{r}$ . Let  $v$  be a vector such that  $\alpha^* + v$  has nonnegative coordinates and  $v_{rl} = 0$  for  $r \neq \tilde{r}$ . That is,  $v$  represents a feasible deviation from  $\alpha^*$  by buyer  $r$ . It must be shown that  $P_{\tilde{r}}(\alpha^* + v) \leq P_{\tilde{r}}(\alpha^*)$ , i.e. that  $v$  is not a profitable deviation for buyer  $\tilde{r}$ . Let  $e$  denote the bid vector with  $e_{rl} = 1$  if  $r = \tilde{r}$  and  $e_{rl} = 0$  for other  $r$ , and let  $\delta > 0$ . Then  $\alpha^\epsilon + v + \delta e$  has nonnegative coordinates for all sufficiently small  $\epsilon$  in

the subsequence under consideration. Moreover,

$$\begin{aligned}
P_{\tilde{r}}(\alpha^* + v) &\stackrel{(b)}{\leq} U_{\tilde{r}}(x_{l_{\tilde{r}}}(\alpha^* + v + \delta e)) - \sum_l (\alpha_{l_{\tilde{r}}} + v_{l_{\tilde{r}}}) \\
&\stackrel{(b)}{=} \lim_{\epsilon \rightarrow 0} P_{\tilde{r}}^\epsilon(\alpha^\epsilon + v + \delta e) + L\delta \\
&\stackrel{(c)}{\leq} \lim_{\epsilon \rightarrow 0} P_{\tilde{r}}^\epsilon(\alpha^\epsilon) + L\delta \\
&\stackrel{(d)}{=} P_{\tilde{r}}(\alpha^*) + L\delta.
\end{aligned}$$

The inequality (a) comes from the fact that  $x_{\tilde{r}}$  is nondecreasing in the bids of buyer  $\tilde{r}$ . Equality (b) comes from the fact that  $x_{l_{\tilde{r}}}(\alpha^\epsilon + v + \delta e) \rightarrow x_{l_{\tilde{r}}}(\alpha^* + v + \delta)$  as  $\epsilon \rightarrow 0$  along the subsequence. Inequality (c) follows by the fact that  $\alpha^\epsilon$  is an NEP for the  $\epsilon$  game. Inequality (d) follows by the claim proved in the previous paragraph. Since  $\delta$  is an arbitrary positive constant, the required equality  $P_{\tilde{r}}(\alpha^* + v) \leq P_{\tilde{r}}(\alpha^*)$  is proved. Therefore,  $\alpha^*$  is an NEP.

To complete the proof it remains to show that (1)-(3) characterize the NEPs. Suppose  $\alpha$  is an NEP. Then if  $x_r \neq x_{l_r}^o$  for some buyer  $r$  and some  $l \in r \cap \mathcal{B}$ , then  $x_r < x_{l_r}^o$  and the buyer could reduce her bid on link  $l$  without reducing  $x_r$ . This would increase the payoff to buyer  $r$ , contradicting the assumption that  $\alpha$  is an NEP. Thus, the NEP  $\alpha$  satisfies (1).

If only one buyer  $r$  had a positive bid for a particular link, the bid could be decreased to a smaller positive amount. It would not decrease  $x_r$ , so that the payoff to buyer  $r$  would increase. Thus, the NEP  $\alpha$  satisfies (2).

If  $x_r > 0$  and the bids of buyer  $r$  are not all zero, a possible deviation of  $\alpha_{l_r} : l \in r$  is to decrease each nonzero element of  $\alpha_{l_r} : l \in r$  by  $\frac{\delta \sum_{r'} \alpha_{l_{r'}}}{C_l - x_r}$ . Such causes  $x_r$  to decrease by  $\delta + o(\delta)$ . In order that such deviation not be profitable for small  $\delta$ , it is required that the left hand side of (3) be greater than or equal to zero if  $x_r > 0$ . Such inequality is also true if the bids of buyer  $r$  are all zero.

If  $x_r < C_l$  for all  $l \in r$ , then a possible deviation of  $(\alpha_{l_r} : l \in r)$  is to increase each nonzero element of  $(\alpha_{l_r} : l \in r)$  by  $\frac{\delta \sum_{r'} \alpha_{l_{r'}}}{C_l - x_r}$ , and increase each zero element of  $\alpha_{l_r} : l \in r$  to  $\delta^2$ . Such causes  $x$  to increase by  $\delta + o(\delta)$ . In order that such deviation not be profitable for small  $\delta$ , it is required that the left hand side of (3) be less than or equal to zero if  $x_r < C_l$  for all  $l \in r$ .

Combining the previous four paragraphs shows that all NEP's satisfy (1)-(3). Conversely, if  $\alpha$  satisfies (1)-(3), then (2) implies that the payoff of each buyer  $r$  is a continuous, convex function of her bid vector, and (1) and (3) together imply the bid vector of any buyer  $r$  is a local maximum of the buyer's payoff function, for the bids of other buyers fixed. Hence, the point  $\alpha$  is an NEP and the proposition is proved. Proof of the proposition. First the existence of an NEP is demonstrated. The first half of the proof is the same as the proof in [1] that an NEP exists for the extended game defined therein. It begins by considering the perturbed game, called "the  $\epsilon$  game," obtained by supposing that an outside buyer bids  $\epsilon$  on each link. The rate allocated to buyer  $r$  is thus given by  $x_r^{\alpha, \epsilon}(\alpha) = \min_{l \in r} x_{l_r}^{\alpha, \epsilon}(\alpha)$  where

$$x_{l_r}^{\alpha, \epsilon}(\alpha) = \frac{\alpha_{l_r} C_l}{\epsilon + \sum_{r'} \alpha_{l_{r'}}}.$$

The payoff function for buyer  $r$  in the  $\epsilon$  game is  $P^\epsilon(\alpha) = U_r(x_r^\epsilon(\alpha)) - \sum_l \alpha_{lr}$ . For each fixed  $\epsilon > 0$  and each buyer  $r$ , the payoff function  $P_r^\epsilon$  is a strictly concave function of the buyer's own bid vector  $(x_{lr} : l \in r)$  for fixed values of the other bids, and is a continuous function of the vector of all bids. Also, because of the capacity constraints, without loss of generality the vector of bids can be restricted to a compact set. Rosen's theorem, based on Kakutani's fixed point theorem, implies that an NEP Nash  $w^\epsilon$  exists for each  $\epsilon > 0$ . Along some subsequence of  $\epsilon \rightarrow 0$ , the set of NEP bid vectors  $\alpha^\epsilon$  and the corresponding allocations  $x^\epsilon$  converge to the respective limit vectors  $\alpha^*$  and  $x^*$ . (The proof to this point follows exactly the proof in [1]).

Since the link allocation function is continuous if the sum of link bids is bounded away from zero,  $x_{lr}^{o,\epsilon} \rightarrow x_{lr}^*$  for  $l \in B^*$ . Let  $\mathcal{B}^*$  denote the set of links with at least one nonzero bid for the limit assignment  $\alpha^*$ . Since  $x^\epsilon \in \mathcal{Y}^o$  for all  $\epsilon$ , it follows that  $x^* \in \mathcal{Y}^o$ . In particular,  $\underline{y}_r(\alpha^*)x_r^* \leq \overline{y}_r(\alpha^*)$  for all  $r$ . The next step in the proof is to show that  $x^*$  is the rate assignment for bid vector  $\alpha^\epsilon$ .

A key claim is that  $x^* = \overline{y}(\alpha^*)$ . The claim is now proved. Suppose for the sake of argument by contradiction that the claim is false. Then there is some route  $r$  such that  $x_r^* < \overline{y}_r(\alpha^*)$ . Let  $k$  denote the number of links  $l$  in  $r \cap \mathcal{B}^c$ , and let  $\delta$  be a small positive number such that  $U_r(\overline{y}_r(\alpha^*)) - U_r(x_r^*) - k\delta > 0$ . For each  $\epsilon$  in the subsequence being considered, let  $\tilde{x}_r^\epsilon$  denote the value of  $x_r^\epsilon$  that would result if all the bids  $\alpha_{lr}^\epsilon$  for  $l \in r \cap \mathcal{B}^c$  were increased by  $\delta$ , for a small  $\delta > 0$ , and all other bids are kept the same. Since the bids of all other buyers for the links  $l \in r \cap \mathcal{B}^c$  converge to zero as  $\epsilon \rightarrow 0$ , it follows that  $\tilde{x}_r^\epsilon \rightarrow \overline{y}_r(\alpha^*)$ . Therefore, the net gain in payoff due to adding  $\delta$  to the bids for links  $l \in r \cap \mathcal{B}^c$  converges to  $U_r(x_r^*) - U_r(\overline{y}_r^*) - k\delta > 0$ . This contradicts the definition of  $\alpha^\epsilon$ , so that the proof of the claim by contradiction is complete. An immediate consequence of the claim is that the limit  $x^*$  satisfies  $x^* = x(\alpha^*) = \overline{y}(\alpha^*) = \underline{y}(\alpha^*)$ .

Fix a buyer  $\tilde{r}$ . Let  $v$  be a vector such that  $\alpha^* + v$  has nonnegative coordinates and  $v_{rl} = 0$  for  $r \neq \tilde{r}$ . That is,  $v$  represents a feasible deviation from  $\alpha^*$  by buyer  $r$ . It must be shown that  $P_{\tilde{r}}(\alpha^* + v) \leq P_{\tilde{r}}(\alpha^*)$ , i.e. that  $v$  is not profitable for buyer  $\tilde{r}$ . Let  $e$  denote the bid vector with  $e_{rl} = 1$  if  $r = \tilde{r}$  and  $e_{rl} = 0$  for other  $r$ , and let  $\delta > 0$ . Then  $\alpha^\epsilon + v + \delta e$  has nonnegative coordinates for all sufficiently small  $\epsilon$  in the subsequence under consideration. Moreover,

$$\begin{aligned}
P_{\tilde{r}}(\alpha^* + v) &\stackrel{(b)}{\leq} U_{\tilde{r}}(x_{l\tilde{r}}(\alpha^* + v + \delta e)) - \sum_l (\alpha_{l\tilde{r}} + v_{l\tilde{r}}) \\
&\stackrel{(b)}{=} \lim_{\epsilon \rightarrow 0} P_{\tilde{r}}^\epsilon(\alpha^\epsilon + v + \delta e) + L\delta \\
&\stackrel{(c)}{\leq} \lim_{\epsilon \rightarrow 0} P_{\tilde{r}}^\epsilon(\alpha^\epsilon) + L\delta \\
&\stackrel{(d)}{=} P_{\tilde{r}}(\alpha^*) + L\delta.
\end{aligned}$$

The inequality (a) comes from the fact that  $x_{\tilde{r}}$  is nondecreasing in the bids of buyer  $\tilde{r}$ . Equality (b) comes from the fact that  $x_{l\tilde{r}}(\alpha^\epsilon + v + \delta e) \rightarrow x_{l\tilde{r}}(\alpha^* + v + \delta)$  as  $\epsilon \rightarrow 0$  along the subsequence. Inequality (c) follows by the fact that  $\alpha^\epsilon$  is an NEP for the  $\epsilon$  game. Inequality (d) follows by the claim proved in the previous paragraph. Since  $\delta$  is an arbitrary positive constant, the required equality  $P_{\tilde{r}}(\alpha^* + v) \leq P_{\tilde{r}}(\alpha^*)$  is proved. Therefore,  $\alpha^*$  is an NEP.

To complete the proof it remains to show that (1) and (2) characterize the NEPs.

Suppose  $\alpha$  is an NEP. Then if  $x_r \neq x_{l_r}^o$  for some buyer  $r$  and some  $l \in r \cap \mathcal{B}$ , then  $x_r < x_{l_r}^o$  and the buyer could reduce her bid on link  $l$  without reducing  $x_r$ . This would increase the payoff to buyer  $r$ , contradicting the assumption that  $\alpha$  is an NEP. Thus, the NEP  $\alpha$  satisfies (1).

If only one buyer  $r$  had a positive bid for a particular link, the bid could be decreased to a smaller positive amount. It would not decrease  $x_r$ , so that the payoff to buyer  $r$  would increase. Thus, the NEP  $\alpha$  satisfies (2).

If  $x_r > 0$  and the bids of buyer  $r$  are not all zero, a possible deviation of  $\alpha_{l_r} : l \in r$  is to decrease each nonzero element of  $\alpha_{l_r} : l \in r$  by  $\frac{\delta \sum_{r'} \alpha_{l_r'}}{C_l - x_r}$ . Such causes  $x$  to decrease by  $\delta + o(\delta)$ . In order that such deviation not be profitable for small  $\delta$ , it is required that the right hand side of (3) be greater than or equal to zero if  $x_r > 0$ . Such inequality is also true if the bids of buyer  $r$  are all zero.

If  $x_r < C_l$  for all  $l \in r$ , then a possible deviation of  $(\alpha_{l_r} : l \in r)$  is to increase each nonzero element of  $(\alpha_{l_r} : l \in r)$  by  $\frac{\delta \sum_{r'} \alpha_{l_r'}}{C_l - x_r}$ , and increase each zero element of  $\alpha_{l_r} : l \in r$  to  $\delta^2$ . Such causes  $x$  to increase by  $\delta + o(\delta)$ . In order that such deviation not be profitable for small  $\delta$ , it is required that the right hand side of (3) be less than or equal to zero if  $x_r < C_l$  for all  $l \in r$ .

Combining the previous four paragraphs shows that all NEP's satisfy (1)-(3). Conversely, if  $\alpha$  satisfies (1)-(3), then (2) implies that the payoff of each buyer  $r$  is a continuous, convex function of her bid vector, and (1) and (3) together imply the bid vector of any buyer  $r$  is a local maximum of the buyer's payoff function, for the bids of other buyers fixed. Hence, the point  $\alpha$  is an NEP and the proposition is proved.

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