# A Framework for Studying Demand in Hierarchical Networks (Preliminary draft*)

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**Abstract**

The aim of this preliminary draft paper is to identify a framework for studying demand in hierarchical networks. A specific choice of technical conditions are given for precisely formulating four equivalent well-known ways to specify the demand of a buyer: by a demand function, an inverse demand function, a valuation function, or a demand set. Unimodality (or quasi-concavity) of profit is a basic property insuring the ability of a seller to find a profit maximizing operating point. Conditions on the demand of a buyer are identified which insure that the profit of a seller with fixed production cost is a unimodality function, of either the price set or quantity sold. Five methods are identified for aggregating the demand of a buyer, and three methods are indentified for combining competing sellers.

## 1 Introduction

The Internet is a confederation of thousands of autonomous systems (ASs). Such systems are individually administered, and can have different routing mechanisms. Many autonomous systems are individually financed. They can be for-profit entities such as a service provider, or not-for-profit entities such as university campus networks. Hierarchy plays a critical role in the structure of the Internet, as a handful of the autonomous systems have global extent, and carry the bulk of transcontinental internet traffic. Regional networks bridge tier one networks to local service providers and campus networks. The actual connectivity among

*This is a preliminary draft released for comments at the *Workshop on Control and Pricing in Communication and Power Networks,* Institute for Mathematics and Its Applications, March 8-13, 2004. The material of this paper closely related to, and quite possibly wholly subsumed by, the classical literature on microeconomics. It is written in an attempt to collect some of the calculations and issues that seem to recur in considerations of hierarchical networks. The authors are grateful for feedback.

ASs is a complicated mesh, so that hierarchy is not strictly observed. On the other hand, due to the existence of a few tier one networks, hundreds of regional networks, and thousands of campus networks, all separately administered, the Internet topology is not a flat one either.

The goal in this paper is to provide a framework to examine some of the economic incentives that may be in force in a network with a simple hierarchical structure. Our vision is that studies of hierarchical networks and flat networks can eventually be combined to give a more realistic idea of the the economic allocation mechanisms and incentives in the Internet.

The paper is organized as follows. Section 2 gives notation for defining the demand of a single buyer, the profit of a seller faced with a single buyer, and the demand of a buyer plus selling agent. The section closes with the observation that, by induction, a chain of agents plus a buyer also defines a demand. A rational seller may be expected to set prices or quantity in order to maximize total payoff (profit). In a distributed environment with partial information, a seller may attempt to find the most profitable price or quantity point by local experimentation and hill-climbing. Hill-climbing for global maximization is feasible if the profit function is unimodal. Section 3 addresses the question of when the demand of a buyer leads to unimodal profit functions for an agent faced with the demand.

Section 4 presents five methods for aggregating the demand of multiple buyers. Two of the methods are appropriate for multitcast or broadcast communication, in which all buyers find value in a common data stream such as a video broadcast. Such communication is, in the terminology of economics, a public good. The other three methods are appropriate for unicast communication, in which each buyer wishes to receive a distinct information flow. Three of the methods are rather simple, involving the sum of demand functions, sum of inverse demand functions, or maximum of demand functions. The fourth method involves efficient allocation to the buyers, with payments according to a Vikrey-Clarke-Groves mechanism to give buyers incentive to truthfully report their valuation functions. The fifth mechanism is based on proportional allocation of capacity to buyers that settle into a Nash equilibrium.

Section 5 addresses competition among agents. Three scenarios of competition are addressed. The first two are the classical Bertrand and Cournot competition scenarios or oligopoly theory. The third scenario can also be viewed as a Cournot competition among the selling agents, in which the selling agents not only compete for the demand of a buyer, but they also compete for quantity to sell.

# 2 One buyer at a time

This section opens with a discussion of the demand of a single buyer, and the profit of an agent selling quantity to a single buyer. The agent is assumed to be able to acquire quantity to sell at a fixed price. An agent, or a chain of agents, plus a buyer again defines a demand.

## 2.1 The demand of a single buyer

The demand of a buyer of a divisible good is expressed as a set $D \subset \mathbb{R}_+^2$ of pairs $(x, p)$, such that $x$ represents quantity and $p$ represents price. It is natural to assume that $D$ is coordinate convex, meaning that if $0 \leq x' \leq x$ and $0 \leq p' \leq p$ and $(x, p) \in D$ then $(x', p') \in D$. Another assumption is imposed on $D$ for tractability. Namely, it is assumed that either $D = \{0, 0\}$, or that the interior of $D$ is not empty and $D$ is the closure of its interior. Equivalently, $D$ is closed, nonempty, and, with $x_{\max}$ and $p_{\max}$ the values in $[0, +\infty]$ defined by

$$x_{\max} = \sup\{x : (x, 0) \in D\} \quad p_{\max} = \sup\{p : (0, p) \in D\},$$

either $x_{\max} = p_{\max} = 0$, or both points $(x_{\max}, 0)$ and $(0, p_{\max})$ are limits of sequences from the interior of $D$. A set $D$ satisfying these conditions is called a *demand set*.

Associated with a demand set $D$ are two functions, $x$ and $p$, defined as follows:

$$x(p) = \sup\{x : (x, p) \in D \text{ or } x = 0\} \tag{1}$$

$$p(x) = \sup\{p : (x, p) \in D \text{ or } p = 0\} \tag{2}$$

Following standard terminology in economics, $p(x)$ is called the demand function and $x(p)$ is called the inverse demand function. Both functions are elements of $\mathcal{D}$, defined as follows.

**Definition 2.1** *Class $\mathcal{D}$ is the set of extended real valued functions $f : [0, +\infty] \to [0, +\infty]$ such that $f$ is nonincreasing, left-continuous, and continuous at 0.*

Equations (1) and (2) show how the functions $x$ and $p$ can be determined by a demand set $D$. The set $D$ can be recovered from $x$ by

$$D = \{(x, p) \in \mathbb{R}^2_+ : p \leq p_{\max} \text{ and } x \leq x(p)\} \tag{3}$$

where $p_{\max} = \sup\{p : x(p) > 0 \text{ or } p = 0\}$, or from $p$ by

$$D = \{(x, p) \in \mathbb{R}^2_+ : x \leq x_{\max} \text{ and } p \leq p(x)\} \tag{4}$$

where $x_{\max} = \sup\{x : p(x) > 0 \text{ or } x = 0\}$. Moreover, for any function $x \in \mathcal{D}$, a demand set $D$ can be defined by (3). Also, for any function $p \in \mathcal{D}$, a demand set $D$ can be defined by (4). In summary, the demand of a buyer can be described in one of three equivalent ways: by a demand set $D$, by the demand function $p$ in $\mathcal{D}$, or by the inverse demand function $x$ in $\mathcal{D}$. Equations (1)-(4) hold, and moreover they imply

$$x(p) = \sup\{x : p(x) \geq p \text{ or } x = 0\}, \tag{5}$$

$$p(x) = \sup\{p : x(p) \geq x \text{ or } p = 0\}, \tag{6}$$

$x_{\max} = x(0)$, and $p_{\max} = p(0)$.

A fourth way to describe the demand of a buyer is through the use of a valuation function in the set $\mathcal{U}$, defined as follows.

**Definition 2.2** *Class $\mathcal{U}$ is the set of extended real valued functions $U : [0, +\infty) \to [-\infty, +\infty)$ such that $U$ is nondecreasing, concave, and right-continuous.*

Given $p \in \mathcal{D}$, define $U \in \mathcal{U}$ as follows. If $p \equiv +\infty$, let $U \equiv -\infty$. If $p(x_o) < \infty$ for some finite $x_o \geq 0$ and $c$ is an arbitrary finite constant, define $U(x)$ for $x \in [0, +\infty)$ by

$$U(x) = \int_{x_o}^{x} p(y)dy + c,$$

with the usual convention that $\int_{x_o}^{x}$ is the same as $-\int_{x}^{x_o}$. This defines $U$ uniquely up to an arbitrary finite additive constant, and $U \in \mathcal{U}$. Conversely, given $U \in \mathcal{U}$, define $p \in \mathcal{D}$ by

$$p(x) = \begin{cases} \partial^- U(x) & \text{if } 0 < x < +\infty \text{ and } U(x) > -\infty \\ \partial^+ U(0) & \text{if } x = 0 \text{ and } U(0) > -\infty \\ +\infty & \text{if } 0 \leq x < +\infty \text{ and } U(x) = -\infty \\ \lim_{y \to +\infty} p(y) & \text{if } x = +\infty \end{cases}$$

where $\partial^+$ and $\partial^-$ denote the right and left derivative operators. The above mappings expressing $U$ in terms of $p$ and vice versa define a bijection between $\mathcal{D}$ and the elements of $\mathcal{U}$ modulo finite additive constants.

The inverse demand function $x$ can be expressed in terms of $U$ as follows:

$$x(p) = \max \arg \max_{x \geq 0} [U(x) - xp] \tag{7}$$

In other words, $x(p)$ is the largest value of $x$ that maximizes the payoff $U(x) - xp$ of the buyer with valuation function $U$ and fixed price $p$. In the special case that $p$ is continuous, the relation between $U$ and the demand function $p$ reduces to $p(x) = U'(x)$, so that $p$ is the marginal valuation function for $U$.

Consider two buyers, buyer 1 and buyer 2. For $i = 1$ or $i = 2$, let $D_i$ denote the demand set, $p_i$ the demand function, and $x_i$ the inverse demand function, for buyer $i$. The demand of buyer 1 is said to be no stronger than the demand of buyer 2, or equivalently the demand of buyer 2 is no weaker than the demand of buyer 1, if any of the following three equivalent conditions hold: $D_1 \subset D_2$, $x_1(p) \leq x_2(p)$ for all $p \geq 0$, or $p_1(x) \leq p_2(x)$ for all $x \geq 0$. Such condition is illustrated in Figure 1.

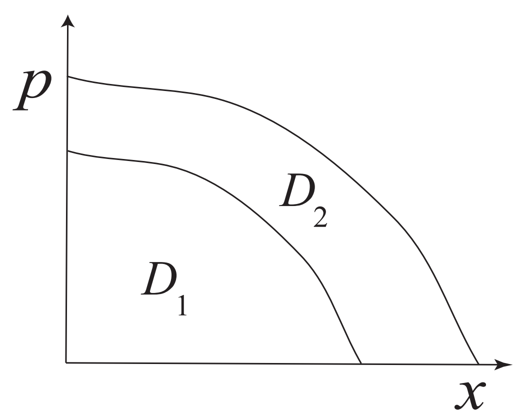


Figure 1: Demand for buyer 1 is not stronger than the demand for buyer 1

## 2.2 A single agent selling to a single buyer

Suppose a single agent (or seller) can purchase or produce quantity at price $v$, and sell it to the buyer. Suppose that the agent can determine the price $p$. Then the buyer will purchase quantity $x(p)$, and the resulting profit to the agent is $\Pi^v(p) = x(p)(p-v)$ for $0 < p < \infty$. Alternatively, it might be assumed that the agent determines the quantity $x$ to be sold, and the resulting price is $p(x)$. Then the profit to the agent can be written as $\Pi^v(x) = x(p(x)-v)$ for $0 < x < \infty$. Either way, the salient point is that the agent is acting as a leader, and the buyer is acting as a follower. Note that we did not yet define the profit functions $\Pi^v(x)$ or $\Pi^v(p)$ at the end points $x \in \{0, +\infty\}$ or $p \in \{0, +\infty\}$. To be definite we define $\Pi^v$ at these points by continuity, whenever the required limits exist.

The profit can also be formulated in terms of the demand set $D$ of the buyer. Given $v \geq 0$, define $\Pi_D^{v,*} = \sup\{x(p-v) : (x,p) \in D\}$. In view of (3) and (4), it is clear that the supremum of the profits is the same for all three formulations:

$$\Pi_D^{v,*} = \sup_{p \geq 0} \Pi^v(p) = \sup_{x \geq 0} \Pi^v(x).$$

In the remainder of this subsection, the existence and properties of profit maximizing price or quantity, and the demand of the agent plus buyer, are examined.

Suppose a buyer is given with demand set $D$ and associated functions $x$ and $p$ in $\mathcal{D}$. Consider the following two conditions:

**Condition I**: $\lim_{x\to 0} xp(x) = 0$, or equivalently, $\lim_{p\to\infty} px(p) = 0$.

**Condition II**: $\lim_{x\to+\infty} p(x) = 0$, or equivalently, $x(p) < +\infty$ for $0 < p < +\infty$.

The proof of the stated equivalences is left to the reader.

**Proposition 2.3 (Existence and bijection property of maximizers)** *Suppose Conditions I and II hold, $v > 0$ and $\Pi_D^{v,*} > 0$. Then the two sets $\arg\max_{x\geq 0} \Pi^v(x)$ and $\arg\max_{p\geq 0} \Pi^v(p)$ are nonempty compact subsets of $[0,\infty)$ and the functions $x$ and $p$ define a bijection between the sets, meaning:*
*(a) If $x^* \in \arg\max_{x\geq 0} \Pi^v(x)$ then $p(x^*) \in \arg\max_{p\geq 0} \Pi^v(p)$ and $x^* = x(p(x^*))$.*
*(b) If $p^* \in \arg\max_{p\geq 0} \Pi^v(p)$ then $x(p^*) \in \arg\max_{x\geq 0} \Pi^v(x)$ and $p^* = p(x(p^*))$.*
*Also, $\arg\max_{x\geq 0} \Pi^v(x) \subset [0, x(v)]$ and $\arg\max_{p\geq 0} \Pi^v(p) \subset [v, +\infty)$.*

**Proof.** By Condition I, there exits $p_1$ large enough so that $px(p) < \Pi_D^{v,*}/2$ whenever $p > p_1$. Hence, $x(p-v) \leq xp \leq \Pi_D^{v,*}/2$ whenever $(x,p) \in D$ with $p > p_1$. Let $x_1 = x(v)$. By Condition II, $x_1$ is finite. Note that $p(x) < v$ if $x > x_1$, so $x(p-v) < 0$ whenever $(x,p) \in D$ with $x > x_1$. Thus, letting $\widehat{D}$ denote the compact set $\widehat{D} = D \cap ([0,x_1] \times [0,p_1])$, it follows that

$$D^* \triangleq \arg\max_{(x,p)\in D} x(p-v) = \arg\max_{(x,p)\in\widehat{D}} x(p-v).$$

The set $D^*$ is pictured in Figure 2. Being the intesection of the compact set $\widehat{D}$ and the closed set $\{(x,p) : x(p-v) = \Pi_D^{v,*}\}$, the set $D^*$ is compact. The set $\arg\max_{x\geq 0} \Pi^v(x)$ is the projection of $D^*$ onto the $x$ axis, and $\arg\max_{p\geq 0} \Pi^v(p)$ is the projection of $D^*$ onto the $p$ axis. The bijective property is clear. Since $D^* \subset \widehat{\widehat{D}}$ it follows that $\arg\max_{x\geq 0} \Pi^v(x) \subset [0, x(v)]$. Since $\Pi_D^{v,*} > 0$ it follows that $\arg\max_{p\geq 0} \Pi^v(p) \subset [v, +\infty)$. □

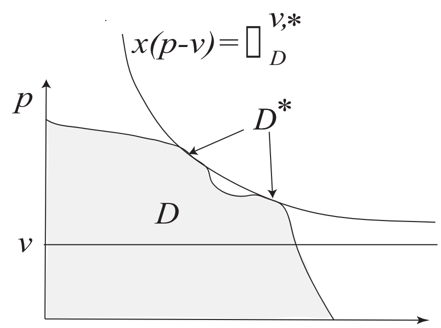


Figure 2: The sets $D$ and $D^*$.

**Proposition 2.4 (Monotonicity of sets of maximizers)** *Suppose Conditions I and II are satisfied and* $0 < v_1 < v_2$.
*(a) Let* $x_i \in \arg\max_{x\geq 0} x(p(x) - v_i)$. *Then* $x_1 \geq x_2$.
*(b) Let* $p_i \in \arg\max_{p\geq 0}(p - v_i)x(p)$*, and also assume that* $p_i = v_i$ *if* $\Pi_D^{v_i,*} = 0$. *Then* $p_1 \leq p_2$.

**Proof.** By the definition of $x_1$, $x(p(x) - v_1) \leq x_1(p(x_1) - v_1)$ for all $x \geq 0$. Since $v_2 > v_1$ it follows that $x(p(x) - v_2) < x_1(p(x_1) - v_2)$ for all $x > x_1$. Hence by the definition of $x_2$, $x_2 \leq x_1$. Part (a) of the proposition is thus proved, and it will be used to help prove part (b).

If $\Pi_D^{v_2,*} = 0$ then $x(v_2 + \epsilon) = 0$ for all $\epsilon > 0$, in which case $p_1 \leq v_2 = p_2$, so that (b) holds if $\Pi_D^{v_2,*} = 0$. Thus, assume $\Pi_D^{v_2,*} > 0$. Then also $\Pi_D^{v_1,*} > 0$. By the bijection property, $x(p_i) \in \arg\max_{x\geq 0} \Pi^{v_i}(x)$ for $i = 1, 2$, so by part (a) already proved, $x(p_1) \geq x(p_2)$. Again by the bijection property, $p_i = p(x(p_i))$ for $i = 1, 2$. Thus, by the monotonicity of the demand function $p$, $p_1 \leq p_2$. □

For $v$ fixed the set $\arg\max_{x\geq 0} \Pi^v(x)$ can be viewed as the response of the buyer plus agent to price $v$. We shall focus on the case that, if there are multiple profit maximizing quantities, the buyer selects the largest. That is, for $v > 0$, let

$$\widehat{x}(v) = \max\{\arg\max_{x\geq 0} x(p(x) - v)\}$$

and define $\widehat{x}(0)$ so that $\widehat{x}$ is continuous at 0. We shall consider $\widehat{x}$ to represent the response of the buyer plus agent, considered as a unit. It defines a demand in its own right.

**Proposition 2.5** *Suppose the buyer's demand satisfies Conditions I and II. Then* $\widehat{x} \in \mathcal{D}$ *and* $\widehat{x}$ *satisfies Conditions I and II.*

**Proof.** By the monotonicity of maximizers, $\widehat{x}$ is nonincreasing. Suppose $v > 0$ and $(v_n)$ is an increasing sequence with limit $v$. Then $\widehat{x}(v_n)$ is a monotone nonincreasing sequence so it has a limit, denoted $\bar{x}$. By the definition of $\widehat{x}$, $\widehat{x}(v_n)(p(\widehat{x}(v_n)) - v_n) \geq x(p(x) - v_n)$ for all $x \geq 0$. Taking the limit as $n \to \infty$ on each side of this inequality, and using the upper semicontinuity of $p$, yields that $\bar{x}(p(\bar{x}) - v) \geq x(p(x) - v)$ for all $x \geq 0$. Hence, $\bar{x} \in \arg\max_{x\geq 0} \Pi^v(x)$. By the monotonicity of maximizers, $\arg\max_{x\geq 0} \Pi^v(x) \subset [0, x(v_n)]$ for all $n$, so $\arg\max_{x\geq 0} \Pi^v(x) \in [0,\bar{x}]$. Thus, $\bar{x} = \widehat{x}(v)$. Therefore $\widehat{x}(v)$ is left-continuous. By its definition, $\widehat{x}(v)$ is continuous at $v = 0$. Therefore $\widehat{x} \in \mathcal{D}$. Since $x$ satisfies Conditions I and II, and since, by the last part of Proposition 2.3, $\widehat{x}(v) \leq x(v)$ for all $v$, it follows that $\widehat{x}$ also satisfies Conditions I and II. □

For fixed $v > 0$, if $\Pi_D^{v,*} > 0$, then, by the bijection property, associated with the particular maximizer $\widehat{x}(v)$ there is a unique corresponding price, denoted by $\widehat{v}(v)$, charged by the agent. If the profit, $\Pi_D^{v,*}$, is zero, we take $\widehat{v}(v) = v$. Thus $\Pi_D^{v,*} = \widehat{x}(v)(\widehat{v}(v)-v)$. The next proposition gives not only a simple expression for $\widehat{v}(v)$, but also a simple expression for $\widehat{x}$.

**Proposition 2.6 (Buyer's price vs. agent's price)** *Assume the demand represented by $x$ satisfies Conditions I and II. Then*

$$\widehat{v}(v) = \min\{\arg\max_{p\geq v} \Pi^v(p)\} \tag{8}$$

$$\widehat{x}(v) = x(\widehat{v}(v)) \tag{9}$$

$$\widehat{v}(v) = \max\{v, p(\widehat{x}(v))\} \tag{10}$$

*and $\widehat{v}$ is left-continuous and nondecreasing in $v$ over $v > 0$.*

**Proof.** Consider two cases. If $\Pi_D^{v,*} = 0$, then $\widehat{v}(v) = v$, $\widehat{x}(v) = x(v)$, and $p(\widehat{x}(v)) \leq p(0) \leq v$, so (8)-(10) hold. If $\Pi_D^{v,*} > 0$ then (8)-(10) follow from the bijection property, Proposition 2.3. An argument similar to that about $\widehat{x}$ given in the proof of Proposition 2.5 can be used to show that $\widehat{v}(v)$ is left-continuous and nondecreasing over the region $v > 0$. □

The demand of the buyer plus agent has been defined by specifying the inverse demand function $\widehat{x} \in \mathcal{D}$. Let $\widehat{D}$ denote the corresponding demand set for the buyer plus agent, and let $\widehat{p}(x)$ denote the corresponding demand function for the buyer plus agent. Note that any one of $D$ or functions $x$ or $p$ determines $\widehat{D}$ and the functions $\widehat{x}$ and $\widehat{p}$. We comment briefly on expressing $\widehat{p}$ directly in terms of $p$. Suppose first that $p$ is continuously differentiable, and that $xp(x)$ is concave. Differentiating $\Pi^v(x)$ with respect to $x$ and seeking the largest zero yields that

$$\widehat{x}(v) = \sup\{x : p(x) + xp'(x) \geq v \text{ or } x = 0\} \tag{11}$$

Comparison of (11) to (5) and (6) shows that $\widehat{p}(x) = p(x) + xp'(x)$. The correspondence between $p$ and $\widehat{p}$ is more complicated if $xp(x)$ is not concave. In that case $\Pi^v(x) = x(p(x) - v)$ can have local maxima that are not global maxima. For example, if $(xp(x))'$ is not nonincreasing but instead has a single local minima that is not a global minima, $\widehat{p}$ is obtained from $p$ as shown in Figure 3. The height of the flat portion of the graph of $\widehat{p}$ is selected to equalize the areas of the two shaded regions.

## 2.3 Demand of a chain of agents plus a buyer

Consider a chain of $L$ agents, numbered 1 through $L$, and a single buyer, as pictured in Figure 4. Agent 1 sets a price $v_1$ for the buyer and agent $i$ for $2 \leq i \leq L$ sets a price $v_i$ for agent $i-1$. Agent $L$ purchases quanitity at a fixed price $v > 0$. A set of prices $(v_1, \ldots, v_L)$

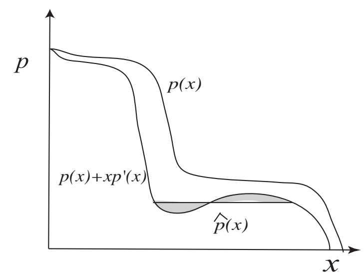


Figure 3: Demand function $\widehat{p}$ for a nonconcave $xp(x)$

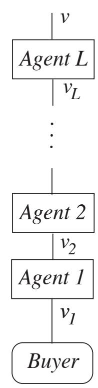


Figure 4: A buyer plus a chain of $L$ agents

is defined to be a nested leader-follower equilibrium if the following is true. Agent $i$ selects the price $v_i$ to maximize the profit of agent $i$ with the following understanding. Agents take prices from above as given, and they know the demand resulting from the buyer plus agents below. In case of ties, an agent selects the smallest price that maximizes the profit of the agent. Assuming that the demand of the buyer satisfies Conditions I and II, it follows by induction on $i$ that the demand of the buyer plus agents 1 though $i$ satisfies the same conditions. In particular, as a function of $v$, the demand of the buyer plus chain of $L$ agents is in $\mathcal{D}$ and satisfies Conditions I and II.

# 3 Unimodality of profit

## 3.1 A necessary and sufficient condition for unmodality

Consider a seller desiring to find a price $p$ to maximize its profit $\Pi^v(p)$ for a fixed $v > 0$. One method is to sequentially select prices $p_1, p_2, \ldots$ with $p_{k+1}$ close to $p_k$ such that $\Pi^v(p_0) < \Pi^v(p_1) < \cdots$, and identify the maximizer $p^*$ as the limit $p^* = \lim_{k\to\infty} p_k$. The idea for selecting $p_{k+1}$ once $p_k$ is given is to determine whether $\Pi^v(p)$ is increasing or decreasing near $p_k$, and then select $p_{k+1}$ accordingly. Such a hill-climbing approach succeeds if $\Pi$ is upper semi-continuous and unimodal. A function $F$ on an interval $I$ of the real line is defined to be unimodal (or quasi-concave) if $I = I_1 \cup I_2$ where $I_1$ and $I_2$ are disjoint intervals with $x_1 \leq x_2$ whenever $x_i \in I_i$ for $i = 1, 2$, and $F$ is nondecreasing over $I_1$ and nonincreasing over $I_2$. If $F$ is monotone increasing over $I$, or if $F$ is monotone decreasing over $I$, then $F$ is unimodal. A function $F$ defined on an interval $I$ is unimodal if and only if it does not have a local minimum that is not a global minimum.

**Proposition 3.1** *Let $v \geq 0$ be fixed. Then the following six conditions are equivalent.*
*(a)* $\Pi^v(p)$ *is unimodal over* $[0, +\infty]$,
*(b)* $\Pi^v(p)$ *is unimodal over* $(0, p_{\max})$,
*(c)* $\Pi^v(p)$ *is unimodal over* $(v \vee p_{\max}, p_{\max})$,
*(d)* $\Pi^v(x)$ *is unimodal over* $[0, +\infty]$,
*(e)* $\Pi^v(x)$ *is unimodal over* $[0, x_{\max})$,
*(f)* $\Pi^v(x)$ *is unimodal over* $(0, x(v))$

**Proof.** Trivially (a) imples (b), (b) implies (c), (d) implies (e), and (e) implies (f). It will be shown that (c) implies (a) , (f) implies (d), and (c) is equivalent to (f).

(Proof that (c) implies (a)): Note that: (i) $\Pi^v(p)$ is nondecreasing and less than or equal to zero over $(0, v]$, (ii) $\Pi^v(p) > 0$ for $v < p < p_{\max}$, and (iii) $\Pi^v(p) = 0$ for $p > p_{\max}$. So if $p_{\max} \leq v$, $\Pi^v(p)$, is monotone nondecreasing over $(0, +\infty)$. If $p_{\max} > v$, $\Pi^v(p)$ is monotone nondecreasing and less than or equal to zero over $(0, v]$, unimodal and nonnegative over $(v, p_{\max})$ (or over $(0, p_{\max}]$ if $p_{\max}$ is finite since $\Pi^v(p)$ is left-continuous), and is zero over $(p_{\max}, +\infty)$. Hence, $\Pi^v(p)$ is unimodal over $(0, +\infty)$. Since $\Pi^v(0)$ and $\Pi^v(+\infty)$ are defined by taking limits, it follows that $\Pi^v(p)$ is unimodal over $[0, +\infty]$. Thus (c) implies (a).

(Proof that (f) implies (d)). Suppose (f) is true. Over the interval $(x(v), +\infty)$, $p(x) - v$ is negative and nonincreasing, so $\Pi^v(x)$ is also negative and nonincreasing. Condition (f) and the left-continuity of $\Pi^v(x)$ impies that $\Pi^v(x)$ is greater than or equal to zero and unimodal over $(0, x(v)]$. Thus, $\Pi^v(x)$ is unimodal over $(0, +\infty)$. Since $\Pi^v(0)$ and $\Pi^v(+\infty)$ are defined by taking limits, $\Pi^v$ is unimodal over $[0, \infty]$. Thus (f) implies (d).

(Equivalence of (c) and (f).) If $v > p_{\max}$ then $x(v) = 0$ and (c) and (f) are both trivially true. If $v = p_{\max}$ then (c) is trivially true and $\Pi^v(x) = 0$ for $0 \leq x \leq x(v)$, so that (f) is also true. Thus, for the remainder of the proof it is assumed that $v < p_{\max}$ and therefore also $x(v) > 0$. Let $D_1$ denote the upper boundary of the set $D \cup ([0, x(v)] \times [v, \infty))$. An illustration of $D_1$ together with the level contours of $x(p - v)$ are illustrated in Figure 5. The function $x(p - v)$ is nonnegative over $D_1$. Consider the value of $x(p - v)$ as the point $(x, p)$ traces through $D_1$ with increasing $x$. Then $x(p - v)$ is strictly increasing along horizontal segments of $D_1$ and strictly decreasing along vertical segments of $D_1$. Therefore, conditions (c) and (f) are both seen to be equivalent to the condition that $x(p - v)$ is unimodal as the point $(x, p)$ traces through $D_1$. □

Henceforth, for fixed $v \geq 0$, it is simply said that $\Pi^v$ is unimodal if any of the equivalent conditions of Proposition 3.1 are true.

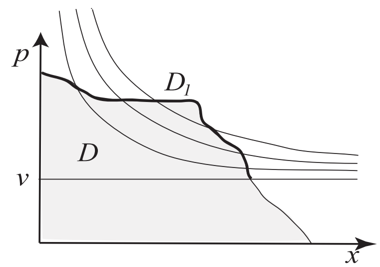


Figure 5: Set $D_1$ and the level contours of $x(p-v)$.

**Proposition 3.2 (A necessary and sufficient condition for unimodality for all $v$)** *Let a demand set $D$ be given with associated functions $p, x \in \mathcal{D}$. The profit $\Pi^v$ is unimodal for all $v \geq 0$ if and only if for some $x^* \leq \infty$: $xp(x)$ is concave over $(0, x^*)$ and is monotone nonincreasing over $(x^*, +\infty)$.*

**Proof.** The proof is obvious from a sketch of $\Pi^v(x) = xp(x) - xv$ (see Figure 6) although we write out a proof. First the "if" part of the proposition is proved. Suppose $xp(x)$ satisfies

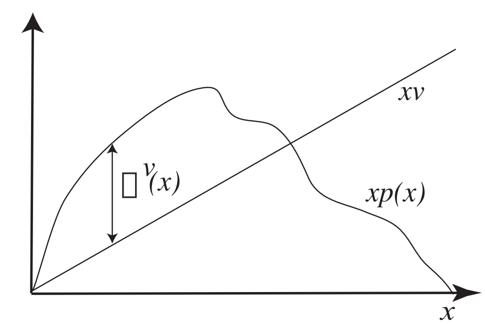


Figure 6: Obtaining $\Pi^v(x)$ from $xp(x)$.

the stated conditions for some $x^*$, and let $v \geq 0$. Then $\Pi^v(x)$ is also concave over $(0, x^*)$ (over $(0, x^*]$ if $x^* < +\infty$, by the left-continuity of $\Pi^v(x)$) and monotone nonincreasing over $(x^*, +\infty)$. Consequently, $\Pi^v$ is unimodal, and the "if" part of the proposition is proved.

Turning to the "only if" part, assume that $\Pi^v$ is unimodal for any $v \geq 0$. Then $\Pi^v(p)$ is unimodal in $p$ for $v = 0$, so that $xp(x)$ is unimodal. Therefore, with $x^* = \inf\{x > 0 : xp(x) \geq x'p(x') \text{ for all } x' > x\}$, it follows that $xp(x)$ is monotone nondecreasing over $(0, x^*)$ and monotone nonincreasing over $(x^*, +\infty)$. It remains to show $xp(x)$ is concave over $(0, x^*)$. So suppose $0 < x_0 < x_1 < x^*$ and $0 \leq \lambda \leq 1$. Let $x^\lambda = x_0(1-\lambda) + x_1\lambda$ and $v = \frac{x_1 p(x_1) - x_0 p(x_0)}{x_1 - x_0}$. Note that $v \geq 0$. This choice of $v$ yields $\Pi^v(x_0) = \Pi^v(x_1)$, so the unimodality of $\Pi^v$ implies

$$\Pi^v(x^\lambda) \geq \min\{\Pi^v(x_0), \Pi^v(x_1)\} = \Pi^v(x_0)(1-\lambda) + \Pi^v(x_1)\lambda$$

Therefore, $x^\lambda p(x^\lambda) \geq x_0 p(x_0)(1-\lambda) + \lambda x_1 p(x_1)$. Thus, $xp(x)$ is concave over $(0, x^*)$, and the proposition is proved. □

## 3.2 Sufficient conditions for unimodality of $\Pi^v$

One condition stronger than unimodality of $\Pi^v$ is that $xp(x)$, or eqivalently, $\Pi^v(x)$, be concave over $(0, x_{\max})$, instead of only being concave over $(0, x^*)$. Another condition stronger than unimodality of $\Pi^v$ is that $\Pi^v(p)$ be concave over $(v, p_{\max})$. Other conditions for unmodality involve derivatives, so two classes of functions are introduced. Recall that to each function $f \in \mathcal{D}$ there is a corresponding value $c_{\max} \in [0, +\infty]$ given by $c_{\max} = \sup\{c : f(c) > 0 \text{ or } c = 0\}$.

**Definition 3.3** *Classes of functions $\mathcal{C}^1$ and $\mathcal{C}^2$ are defined as follows.*

$$\mathcal{C}^1 = \{f \in \mathcal{D} : f \textit{ is continuously differentiable over } (0, c_{\max}), \\ f'(c) < 0 \textit{ over } (0, c_{\max}), \textit{ and } f(c_{\max}) = 0\}$$

$$\mathcal{C}^2 = \{f \in \mathcal{C}^1 : f \textit{ is twice continuously differentiable over } (0, c_{\max})\}$$

Let $i = 1$ or $i = 2$. By the inverse function theorem, a demand function $x(p)$ is in $\mathcal{C}^i$ if and only if the inverse demand function $p(x)$ is in $\mathcal{C}^i$. If $x \in \mathcal{C}^1$ the derivatives are related (for $0 < p < p_{\max}$ or equivalently $0 < x \leq x_{\max}$) by $x'(p) = \frac{1}{p'(x)}$, and if $x \in \mathcal{C}^2$ then

$$x''(p) = (-x'(p))^3 p''(x) = \frac{p''(x)}{(p'(x))^2}.$$

If $p \in \mathcal{C}^1$ then $(xp(x))' = xp'(x) + p(x)$ and if $p \in \mathcal{C}^2$ then $(xp(x))'' = (\Pi^v)''(x) = 2p'(x) + xp''(x)$. Therefore, according to Proposition 3.2, if $p \in \mathcal{C}^1$ then $\Pi$ is unimodal if and only if there is some $x^*$ so that:
(i) $p(x) + xp'(x)$ is nonincreasing for $x \leq x^*$ (if $p \in \mathcal{C}^2$ this is equivalent to $2p'(x) + xp''(x) \leq 0$ for $x \leq x^*$), and
(ii) $p(x) + xp'(x) \leq 0$ for $x > x^*$.

Recall that a condition stronger than unmodality of $\Pi^v$ is that $xp(x)$ be concave over $(0, x_{\max})$. Use of the second derivative test and inverse function theorem yields the following. For $p \in \mathcal{C}^2$, the following conditions, all implying that $\Pi^v$ is unimodal for all $v$, are equivalent:

1. $\Pi^v(x)$, or equivalently $xp(x)$, is concave over $(0, x_{\max})$
2. $p(x) + xp'(x)$ is nonincreasing over $(0, x_{\max})$
3. $2p'(x) + xp''(x) \leq 0$ over $(0, x_{\max})$
4. $-2x'(p)^2 + x(p)x''(p) \leq 0$ over $(0, p_{\max})$.

Yet another condition stronger than unimodality of $\Pi^v$ is that $\Pi^v(p)$ be concave in $p$ over $(v, p_{\max})$. If $x \in \mathcal{C}^1$ then $\Pi'(p) = x'(p)(p-v) + x(p)$, and concavity of $\Pi$ over $(v, p_{\max})$ is equivalent to $\Pi'(p)$ being monotone nonincreasing over $(v, p_{\max})$.

Since $(\Pi^v)''(p) = x''(p)(p-v) + 2x'(p)$ and $x'(p) < 0$, the condition $(\Pi^v)''(p) \leq 0$ for all $v, p$ with $0 \leq v \leq p$ holds if and only if it holds for all $p \geq 0$ and $v = 0$. Thus, concavity of $\Pi^v(p)$ over $(v, p_{\max})$ for all $v$ is equivalent to concavity of $x(p)p$ over $(0, p_{\max})$.

Examining the conditions for concavity of $\Pi^v(x)$ over $(0, x_{\max})$ and for concavity of $\Pi^v(p)$ over $(v, p_{\max})$ shows that for $p, x \in \mathcal{C}^2$, both conditions are true if $x$ is concave over $(0, p_{\max})$, or equivalently, if $p$ is concave over $(0, x_{\max})$, or equivalently, if the demand set $D$ is convex. This result remains true if the smoothness assumption are dropped (as can be shown directly or by an approximation argument). That is, if $x \in \mathcal{D}$ and $x$ is concave over $(0, p_{\max})$, or equivalently, if $p \in \mathcal{D}$ and $p$ is concave over $(0, x_{\max})$, or equivalently if the demand set $D$ is convex, then $\Pi^v(x)$ is unimodal.

## 3.3 Elasticity and one more sufficient condition for unimodality of $\Pi^v$

The elasticity of the demand at a given price $p$ is a dimensionless number. In common usage it is the fractional change in quantity per (small) fractional change in price. To typically arrive at a nonnegative number the ratio is multiplied by minus one. For example, in common usage, if a 2% increase in price causes a 3% drop in quantity demanded, then the elasticity of demand is 1.5. Therefore, assuming $x \in \mathcal{C}^1$, the elasticity of the demand at a price $p$ is defined by

$$\epsilon(p) = \lim_{\epsilon \to 0} -\left(\frac{x(p+\epsilon) - x(p)}{x(p)}\right) / \left(\frac{\epsilon}{p}\right) = \frac{-x'(p)p}{x(p)}$$

Denote by $\epsilon(x)$ the same elasticity function, but parameterized by $x$. Since $x'(p(x)) = 1/p'(x)$, it follows that

$$\epsilon(x) = \epsilon(p(x)) = \frac{p(x)}{-p'(x)x}$$

Since $x(p)$ is decreasing in $p$, it follows that $\epsilon(p)$ is nondecreasing in $p$ over $(0, p_{\max})$ if and only if $\epsilon(x)$ is nonincreasing in $x$ over $(0, x_{\max})$. The derivative of $xp(x)$ can be written as

$$(xp(x))' = p(x)\left[1 - \frac{1}{\epsilon(x)}\right].$$

Suppose $\epsilon(x)$ is nonincreasing in $x$ over $(0, x_{max})$. Let $x^* = \inf\{x : x \geq 0 \text{ and } \epsilon(x) \leq 1\}$. Then $(xp(x))'$ is nonincreasing in $x$ for $0 \leq x < x^*$, and $(xp(x))' \leq 0$ for $x^* < x < x_{\max}$. Thus, $xp(x)$ is concave in $x$ over $[0, x^*)$ and is nonincreasing in $x$ for $x > x^*$. The following proposition is proved.

**Proposition 3.4** *If $\epsilon(x)$ is nonincreasing in $x$ over $(0, x_{\max})$, or equivalently, if $\epsilon(p)$ is nondecreasing in $p$ over $(0, p_{\max})$, then the profit $\Pi^v$ is unimodal for any $v \geq 0$.*

If $p \in \mathcal{C}^2$, then

$$\epsilon'(x) = \frac{-p'(x)^2 x + p(x)p'(x) + p(x)p''(x)x}{(p'(x)x)^2}$$

Note that if $p''(x) \leq 0$, then $\epsilon''(x) \leq 0$. Therefore, if $p \in \mathcal{C}^1$, concavity of the demand (i.e. concavity of $x(p)$ over $(0, p_{\max})$, or equivalently concavity of $p(x)$ over $(0, x_{\max})$, or equivalently convexity of $D$) is a condition stronger than the condition that $\epsilon(x)$ be monotone nonincreasing over $(0, x_{\max})$, or eqivalently, that $\epsilon(p)$ be monotone nondecreasing over $(0, p_{\max})$.

The various conditions that imply unimodality of the profit function $\Pi^v(x)$ for all $v \geq 0$ are summarized in Figure 7. None of the three conditions in the middle column of the figure

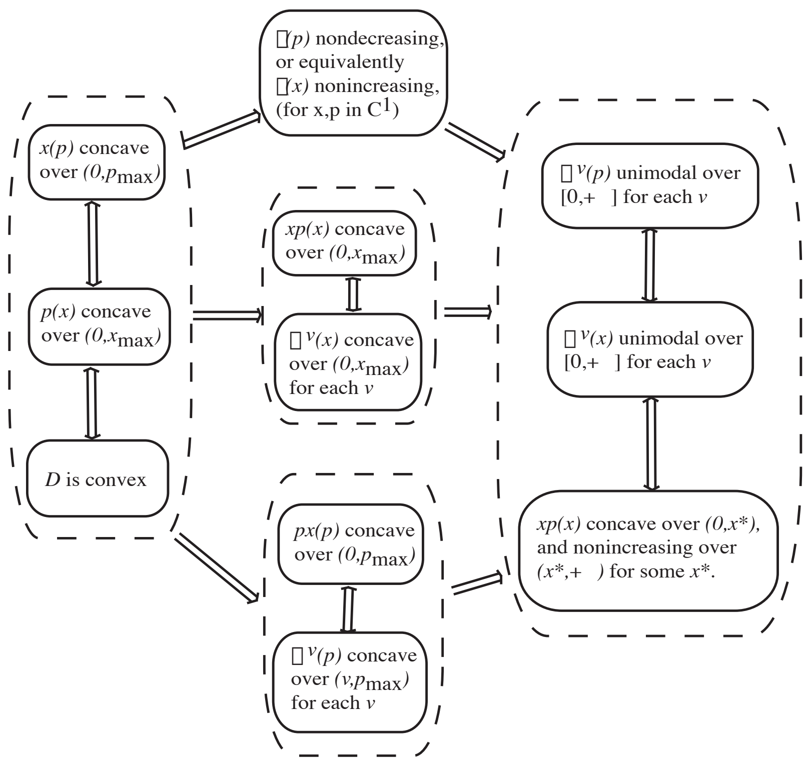


Figure 7: Constellation of conditions implying unimodality of profit.

imply either of the other two.

## 3.4 Examples

**Example 1: Power law demand** The power law demand is defined by $x(p) = p^{-\theta}$ or $p(x) = x^{-1/\theta}$ where $\theta > 0$. The corresponding valuation function is given for $\theta \neq 1$ by

$$U(x) = \frac{x^{1-\frac{1}{\theta}}}{1 - \frac{1}{\theta}} + \text{constant}$$

and by $U(x) = \log(x) + \text{constant}$ if $\theta = 1$. The elasticity is constant and equal to $\theta$: $\epsilon(x) \equiv \epsilon(p) \equiv \theta$.

If $0 < \theta < 1$ then $\Pi^v(x) = x^{1-\frac{1}{\theta}} - xv$, which is strictly decreasing, and so is maximized at $x = 0$, corresponding to $p = +\infty$. This is a rather degenerate choice of $\theta$.

If $\theta = 1$ then $\Pi^v(x) = 1 - vx$, which is also maximized at $x = 0, p = +\infty$. This is also a rather degenerate choice of $\theta$. Basically a buyer with valuation function $\log(x)$ will make the same payment no matter the price. The agent takes advantage by letting $x \to 0$.

If $\theta > 1$ then $\Pi^v(x) = x^{1-\frac{1}{\theta}} - xv$ is strictly concave over $[0, \infty)$. The demand for a buyer with this demand function plus an agent is given by $\widehat{p}(x) = p(x) + xp'(x) = (1 - \frac{1}{\theta})x^{-\frac{1}{\theta}}$, or alternatively, by $\widehat{x}(v) = (\frac{\theta v}{\theta - 1})^{-\theta}$. The price charged by the agent to the buyer is $\widehat{v}(v) = p(\widehat{x}(v)) = \frac{\theta v}{\theta - 1}$. Thus, the agent marks the price $v$ up by the factor $\frac{\theta}{\theta-1}$, and the result is that the inverse demand function $\widehat{x}$ is smaller than $x$ by a constant factor, and the demand function $\widehat{p}$ is smaller than $p$ by a constant factor. The profit of the agent as a function of $v$ is given by $\Pi_D^{v,*} = \widehat{x}(v)(\widehat{v}(v) - v) = v^{1-\theta}\left[\frac{(\theta-1)^{(\theta-1)}}{\theta^\theta}\right]$.

**Example 2: Linear demand functions** The linear demand with parameters $a$ and $b$ is given by $x(p) = (a - p)_+/b$ or $p(x) = (a - xb)_+$ or by the quadratic valuation function

$$U(x) = \begin{cases} ax - \frac{bx^2}{2} & \text{if } x < \frac{a}{b} \\ \frac{a^2}{2b} & \text{if } x \geq \frac{a}{b}. \end{cases}$$

The elasticity functions are given by $\epsilon(x) = \frac{a}{bx} - 1$ for $0 < x < \frac{a}{b}$, and $\epsilon(p) = \frac{a}{a-p}$ for $0 < p < a$. Note that $\epsilon(x)$ is decreasing in $x$ and $\epsilon(p)$ is increasing in $p$. The agent maximizes its profit $\Pi^v(x) = x(p(x) - v)$ by selecting the price $\widehat{v}(v) = \frac{a+v}{2}$. That is, the agent charges the buyers a price half way between $v$, the price to the agent, and $a$, the largest price the buyer would ever accept. The demand function of the buyer plus agent is given by $\widehat{x}(v) = x(\widehat{v}(v)) = \frac{(a-v)_+}{2b}$ or equivalently, $\widehat{v}(x) = (a - 2bx)_+$. The demand $\widehat{x}$ of the agent plus the buyer is thus half the demand of the buyer alone.

**Example 3: Exponential inverse demand function** Consider the inverse demand function $x(p) = \exp(-\gamma p)$, for a constant $\gamma > 0$. The corresponding demand function is $p(x) = (-\log(x)/\gamma)_+$, and the valuation function is

$$U(x) = \begin{cases} \frac{x - x\log(x)}{\gamma} & \text{if } x \leq 1 \\ 1 & \text{if } x > 1. \end{cases}$$

The elasticity functions are $\epsilon(p) = \gamma p$ and $\epsilon(x) = -\log(x)$ for $0 < x \leq 1$. Since $\epsilon(p)$ is monotone increasing the payoff function $\Pi^v$ is unimodal. Indeed, $xp(x) = -\log(x)/\gamma$ over $0 \leq x \leq 1$ is concave, and $\Pi^v(x)$ is also concave. However, while being unimodal, $\Pi^v(p) = e^{-\gamma p}(p - v)$ is not concave. A profit maximizing agent with fixed production price $v$, will charge the buyer the price $\hat{v}(v) = v + \frac{1}{\gamma}$, which is a price markup by the constant $\frac{1}{\gamma}$. The resulting demand and inverse demand functions of the buyer plus agent are $\widehat{p}(x) = (-\log(ex)/\gamma)_+$ and $\widehat{x}(v) = e^{-1}e^{-\gamma p}$.

**Example 4: Exponential demand function** Consider the demand function $p(x) = \exp(-\gamma x)$, for a constant $\gamma > 0$. The corresponding inverse demand function is $x(p) = -\frac{\log(p)}{\gamma}$, and the valuation function is $U(x) = \frac{1-\exp(-\gamma x)}{\gamma}$. Although $xp(x)$ and $\Pi^v(x)$ are not concave in $x$, the elasticity is given by $\epsilon(x) = \frac{1}{\gamma x}$ is monotone decreasing in $x$, implying that $\Pi^v$ is unimodal. Indeed, $Pi^v(x) = x(\exp(-\gamma x) - v)$.

**Example 5: Bent Wing Demand** Consider the demand, called the Bent Wing Demand, given by

$$D = \{(x, p) \in \mathbb{R}^2 : 2x + p \leq 3 \text{ or } x + 4p \leq 5\}.$$

The set $D$, the function $xp(x)$, and the set $\widehat{D}$, are pictured in Figure 8. The profit function

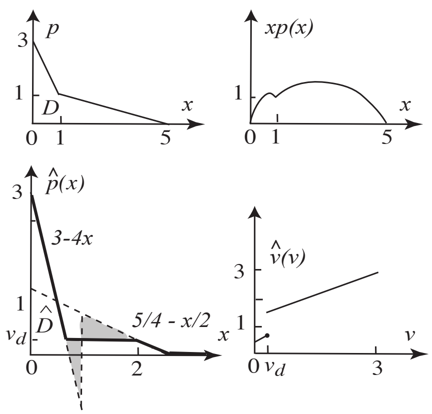


Figure 8: The Bent Wing Demand and associated functions

$\Pi^v$ is not quasiconcave if $0 \le v \le \frac{3}{4}$. The price function $\widehat{p}$ for the buyer plus an agent is also shown, and it is discontinuous at $v_d = 1 - \frac{1}{\sqrt{2}} \approx 0.29289\ldots$. The shaded triangles in the figure have equal areas.

# 4 Aggregation of Buyers

Consider $n$ buyers, and for $1 \le i \le n$, let $D_i$ denote the demand set, $p_i \in \mathcal{D}$ the demand function, $x_i \in \mathcal{D}$ the inverse demand function, and $U_i$ the valuation function, of buyer $i$. Five ways of aggregating the buyers to define a single, aggregate demand are considered in this section. The aggregate demand will be specified by a demand set $D$, or by an associated function $x$ or $p$ in $\mathcal{D}$. For each way of aggregating the buyers, if the demands of all buyers satisfy Conditions I and II, then so does the aggregate demand. The profit of a single agent facing the aggregate of buyers is defined as if there were a single buyer with demand set $D$. The demand of such an agent together with the buyers is determined by (8) and (9).

Use of both agents and aggregation of buyers can be used to define an equilibrium for a hierarchy of agents and aggregation points arranged in a tree, with buyers at the leaves, as illustrated in Figure 9. The small dots in the figure represent aggregation points, defined by one of the methods in this section. The rectangles represent agents and the rounded rectangles are buyers. Each agent takes the price from above, and each agent anticipates the responses of the demand from below. If each buyer has a demand set satisfying Conditions I and II, then aggregate demand facing each agent in the tree again satisfies Conditions I and II, and the demand of the entire tree again satisfies Conditions I and II.

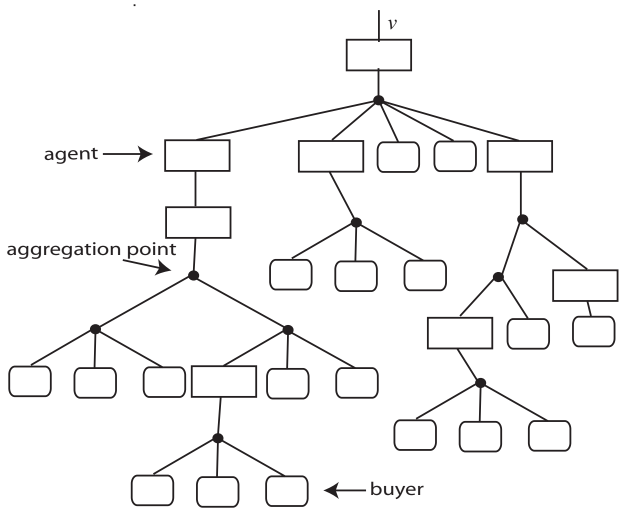


Figure 9: A tree with buyers at the leaves

After each method of aggregating the demand of multiple buyers is defined, the question of whether the profit of an agent faced with the aggregate demand is unimodal is addressed.

## 4.1 Sum of quantities demand

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The sum of quantities demand is defined by $x(p) = x_1(p) + \cdots + x_n(p)$. This corresponds to the $n$ buyers separately responding to a give price $p$. Note that if $x_i \in \mathcal{D}$ for each $i$, then $x \in \mathcal{D}$. Futhermore, if $x_i$ satisfies Conditions I and II for each $i$, then $x$ does also. Consequently, an agent with fixed production price $v$ and facing the sum of quantities aggregate demand has a profit maximizing price $\widehat{v}$ and demand function $\widehat{x}$ given by (8) and (9).

The following examples suggest that if the demands of the $n$ buyers are similar enough, and if individually they would result in a unimodal profit function, then the profit function for the sum of quantities demand is also unimodal. No general result to this effect is formulated, however.

**Example i.a1** Suppose $p_i(x) = (a - b_i x)_+$, or equivalently, $x_i(p) = \frac{(a-p)_+}{b_i}$, for $1 \leq i \leq n$. Here the maximum price $a$ is the same positive constant for all buyers (not so in the next example), and $b_i \geq 0$ for each $i$. The sum of quantities demand is given by

$$x(p) = (a-p)_+ \left( \sum_i \frac{1}{b_i} \right) \quad \text{or} \quad p(x) = (a - \frac{x}{\sum 1/b_i})_+.$$

The demand of an agent plus the aggregate of $n$ buyers is given by

$$\widehat{p}(x) = \left( a - \frac{2x}{\sum_i \frac{1}{b_i}} \right)_+ \quad \text{or} \quad \widehat{x}(p) = (a-p)_+ \left( \sum_i \frac{1}{2b_i} \right)$$

The price charged by the agent is $\frac{a+v}{2}$, which is the same the agent would charge each buyer separately if discriminating prices were used. The fact that price discrimination would not help the buyer accounts for the fact that $\widehat{x}(p) = \sum_i \widehat{x}_i(p)$ for this example.

By extension, it follows that if all buyers in the tree network of Figure 9 were to have demand of the form $x_i(p) = \frac{(a-p)_+}{b_i}$ (the same $a$ for all buyers) then the profit function of each agent in the tree is unimodal. Assuming the agent at the top of the tree is offered the fixed price $v$ with $v > a$, the price charged by the $k$th agent along any path descending from the top of the tree is $a - (v-a)_+ 2^{-k}$. Thus, the profit margin (selling price minus buying price) for such an agent is $2^{-k}(a-v)$, which is decreasing in the depth $k$.

**Example i.a2** If there are two buyers with $p_1(x) = (3-2x)_+$ and $p_2(s) = (1 - \frac{2x}{7})_+$, or equivalently, $x_1(p) = \frac{(3-p)_+}{2}$ and $x_2(p) = \frac{7(1-p)}{2}$. then the sum of quantities demand $x(p) = x_1(p) + x_2(p)$ is the same as in Example Bent Wing. The profit function of an agent faced with this demand is thus not unimodal if $0 \leq v < \frac{3}{4}$.

**Example i.b1** Suppose $x_i(p) = a_i p^{-\theta}$, or equivalently $p_i(x) = \left(\frac{x}{a_i}\right)^{1/\theta}$, where $\theta > 1$ and $a_i > 0$ for $1 \leq i \leq n$. The elasticity $\theta$ is the same for all buyers (not so in the next example). The sum of quantities demand is given by $x(p) = (\sum a_i) p^{-\theta}$, or equivalently, $p(x) = (\frac{x}{\sum a_i})^{-1/\theta}$. Therefore the demand of an agent plus the aggregate of $n$ buyers is given by

$$\widehat{p}(x) = (1 - \frac{1}{\theta})(\sum a_i)^{\frac{1}{\theta}} x^{-\frac{1}{\theta}} \quad \text{or} \quad \widehat{x}(p) = p^{-\theta}(1 - \frac{1}{\theta})^{\theta} \sum a_i$$

The agent charges price $\widehat{v}(v) = \frac{\theta v}{\theta - 1}$, which is the same the agent would charge each buyer under discriminatory pricing, explaining why $\widehat{x}(p) = \sum \widehat{x}_i(p)$ for this example.

By extension, it follows that if all buyers in the tree network of Figure 9 were to have demand of the form $x_i(p) = a_i p^{-\theta}$ ( the same $\theta$ for all buyers) then the profit function of each agent in the tree is unimodal. Assuming the agent at the top of the tree is offered the fixed price $v$ with $v > a$, the equilibrium price charged by the $k$th agent along any path descending from the top of the tree is $\left(\frac{\theta}{\theta-1}\right)^k v$. Thus, the profit margin for such an agent is $v[(\frac{\theta}{\theta-1})^k - (\frac{\theta}{\theta-1})^{k-1}] = \frac{v}{\theta-1}(\frac{\theta}{\theta-1})^{k-1}$, which is increasing in the depth $k$.

**Example i.b2** Suppose there are two buyers, with $x_i(p) = p^{-\theta_i}$ for $\theta_1 = 1 + \epsilon$ and $\theta_2 = \frac{1}{\epsilon}$, where $\epsilon$ is a small positive number. The sum of quantities demand is given by $x(p) = x_1(p) + x_2(p)$. The demand sets $D_1$, $D_2$, and the sum of quantities demand set $D$ are shown in Figure 10. The set $D$ is similar to $D_1$ in the region of small $x$ and large $p$, and is similar to $D_2$ in the region of large $x$ and small $p$. The function $xp(x)$ and the demand set $\widehat{D}$ are also shown. Observe that $xp(x)$ is increasing but not concave for $\epsilon$ sufficiently small, so that $\Pi^v$ is not unimodal for $\epsilon$ sufficiently small and for $v$ in an interval depending on $\epsilon$.

## 4.2 Broadcast with discriminatory pricing

Aggregation of the buyers for broadcast with discriminatory pricing is defined by $p(x) = p_1(x) + \cdots + p_n(x)$. The interpretation is that the seller is able to replicate and sell the quantity $x$ to all buyers, and extract from buyer $i$ the price $p_i(x)$ that buyer $i$ is willing to pay. For example, imagine that a video feed of rate $x$ is simultaneously broadcast to the buyers. In the terminology of economics, the quantity is a public good, with all buyers

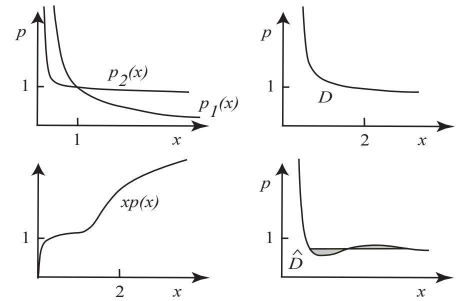


Figure 10: Sum of quantities demand for two buyers with power law demand

benefiting from it. Clearly $p \in \mathcal{D}$ if $p_i \in \mathcal{D}$ for $1 \leq i \leq n$. In addition, $p$ satisfies Conditions I and II if $p_i$ satisfies Conditions I and II for $1 \leq i \leq n$.

Consider an agent with constant production cost $v$ faced with the aggregate demand for broadcast with discriminatory pricing. A sufficient condition for the profit function to be unimodal is that there be an $x_{\max} \in (0, \infty]$ (not depending on $i$) such that for each $i$, $xp_i(x)$ is concave over $[0, x_{\max})$ and $p_i(x) = 0$ for $x > x_{\max}$.

**Example iia1** Suppose the $i$th buyer has the linear demand $p_i(x) = b_i(x_{\max} - x)_+$, where $x_{\max} > 0$ and $b_i > 0$ for each $i$. Then $p(x) = (\sum b_i)(x_{\max} - x)_+$, and an agent faced with this demand has a unimodal profit function. The demand of the aggregated buyers plus agent is given by $\widehat{p}(x) = (\sum b_i)(x_{\max} - 2x)_+$. Note that the support of $\widehat{p}$ is $[0, \frac{x_{\max}}{2}]$. Hence, for a possibly unbalanced tree network, if buyers with $k$ levels of agents above them have demand of the form $\pi_i(x) = b_i(x_0 2^k - x)_+$, then all agents have unimodal payoff functions.

**Example iia2** If there are two buyers with $p_1(x) = \frac{(5-x)_+}{4}$ and $p_2(x) = \frac{7(x-1)_+}{4}$, then the aggregate demand under broadcast with discriminatory pricing, $p(x) = p_1(x) + p_2(x)$, is the same considered in Example Bent Wing. The profit of an agent faced with this demand is not unimodal.

**Example iib** Suppose $x_i(p) = a_i p^{-\theta_i}$, or equivalently $p_i(x) = \left(\frac{x}{a_i}\right)^{1/\theta_i}$, where $\theta_i > 1$ and $a_i > 0$ for $1 \leq i \leq n$. The aggregate demand for broadcast with discriminatory pricing is given by $p(x) = \sum_i \left(\frac{x}{a_i}\right)^{1/\theta_i}$. Note that $xp_i(x)$ for all $i$, and $xp(x)$, are concave over $[0, \infty)$. Therefore the profit function of an agent facing this demand is unimodal. The demand of an agent plus the buyers is given by

$$\widehat{p}(x) = \sum_i \widehat{p}_i(x) = \sum_i p_i(x) + xp_i'(x) = \sum_i (1 - \frac{1}{\theta_i}) p_i(x).$$

The price charged to the $i$th buyer is $\frac{\theta_i v}{\theta_i - 1}$.

More generally, consider a tree of agents and aggregation points as shown in Figure **??**, such that the buyers have demand of the form $x_i(p) = a_i p^{-\theta_i}$, and the aggregation points are for broadcast with discriminatory pricing. Then by induction from the bottom of the tree, the demand faced by every agent in the tree has the form $\sum \left(\frac{x}{a_i}\right)^{1/\theta_i}$. It follows that the profit function of every agent is concave. Under the unique nested leader-follower equilibrium, the price charged to a buyer with demand $p(x) = \left(\frac{x}{a}\right)^{1/\theta}$ which is separated from the root of the tree by $k$ agents is $v(\frac{\theta}{\theta-1})^k$. The demand of an agent plus subtree of agents and buyers below the agent is given by $\sum(1-\frac{1}{\theta_i})^{m_i}p_i(x)$, where $p_i(x) = \left(\frac{x}{a_i}\right)^{1/\theta_i}$ is the demand of the $i$th buyer below the agent, and $m_i$ is the number of agents within the subtree above the $i$th buyer.

## 4.3 Broadcast with shared payment

Aggregation of the buyers for broadcast with shared payment is defined by $D = \cup_{i=1}^n D_i$. Equivalently $p(x) = \max_i p_i(x)$, or $x(p) = \max_i x_i(p)$. The interpretation is that the seller is able to replicate and sell the quantity $x$ to all buyers (so the quantity is a "public good"). The buyers, for their part, can pool their payments and pay for only one copy of $x$. If a particular quantity $x_o$ is purchased, the subset of buyers in $\arg\max_i p_i(x_o)$ must bear the payment $p_i(x_o)$, while other buyers pay nothing. Buyers paying little or nothing are called *free loaders.*

Clearly if $D_1, \ldots, D_n$ are demand sets, then so is $D$. Equivalently, if $x_i \in \mathcal{D}$ for all $i$ then $x \in \mathcal{D}$ and $p \in \mathcal{D}$. Since $x(p) = \max\{x_1(p), \ldots, x_n(p)\} \leq \sum x_i(p)$, it follows that if the demand for each buyer satisfies conditions I and II, then the aggregated demand satisfies Conditions I and II as well.

Consider next the profit of an agent faced with an aggregation of demand for broadcast with shared payment. If for some $i_o$, the demand for buyer $i_o$ is no weaker than the demand for any other buyer, then $D = D_{i_o}$. Trivially, if the profit function for an agent plus buyer $i_o$ alone were unimodal, then the profit function for an agent facing the aggregate demand is also unimodal.

However, very often the profit function for the agent is not unimodal. For simplicity, consider the case of $n = 2$ buyers and one agent. Consider $xp(x) = \max\{xp_1(x), xp_2(x)\}$. Suppose both $xp_1(x)$ and $xp_2(x)$ are concave, continuously differentiable, and increasing on an interval of the form $[0, x^*]$, and suppose for some $x_o \in [0, x^*]$ that $p_1(x_o) = p_2(x_o)$ and $p_1'(x_o) \neq p_2'(x_o)$. That is, the demand functions intersect transversally at $x_o$. Then at $x = x_o$, $\partial^-(xp(x)) < \partial^+(xp(x))$, so although $xp(x)$ is increasing over $[0, x^*]$ it is not concave over $[0, x^*]$. Consequently, by Proposition 3.2, the profit $\Pi^v$ of the agent is not unimodal for sufficiently small $v$. This common situation is illustrated in the following two examples.

**Example c1** If $p_1(x) = \frac{9}{2}(\frac{2}{3} - x)_+$ and $p_2(x) = \frac{3}{10}(4 - x)_+$, then the aggregate demand given by $p(x) = \max\{p_1(x), p_2(x)\}$ is the same as in Example Bent Wing.

**Example c2** Suppose $p_i(x) = x^{-\frac{1}{\theta_i}}$ for $i = 1, 2$, where $1 < \theta_1 < \theta_2$. The demands intersect at $(x, p) = (1, 1)$, and the profit function of an agent plus the aggregate demand for broadcast with shared payment is given by

$$\Pi^v(p) = \begin{cases} x_2(p)(p-v) & \text{if } v \leq p \leq 1 \\ x_1(p)(p-v) & \text{if } p \geq v \vee 1. \end{cases}$$

The profit function $\Pi_i^v(p)$ of an agent faced with the demand of buyer $i$ alone is maximized at the price $\widehat{v}_i(v) = \frac{\theta_i v}{\theta_i - 1}$, corresponding to a price markup by factor $\frac{\theta_i}{\theta_i - 1}$. The profit function of an agent faced with the aggregate demand is given by $\Pi^v(p) = \max\{\Pi_1^v(p), \Pi_2^v(p)\}$. The profit functions cross at $p = 1$, and the maximum profits cross at $v^*$ given by

$$v^* = \left( \frac{(\theta_2 - 1)^{\theta_2 - 1}\theta_1^{\theta_1}}{(\theta_1 - 1)^{\theta_1 - 1}\theta_2^{\theta_2}} \right)^{\frac{1}{\theta_2 - \theta_1}}$$

Faced with the aggregate demand, for a given $v$ the agent selects the markup $\frac{\theta_2}{\theta_2 - 1}$for $v < v^*$ and the markup $\frac{\theta_1}{\theta_1 - 1}$ for $v > v^*$. The profit $\Pi^v$ is not unimodal for $\frac{\theta_1 - 1}{\theta_1} < v < \frac{\theta_2 - 1}{\theta_2}$.

### 4.4 Aggregation by Vikrey-Clark-Groves allocation of given quantity

Aggregation by Vickrey-Clark-Groves (VCG) [3, 1, 2] allocation of given quantity is based on a game among the buyers in which an average price is determined for a given quantity. To avoid overuse of the variable $x$, for the moment the quantity is denoted by $C$. The game involves the buyers reporting their valuation functions to a distributor. The distributor is a seller that follows an allocation algorithm. The distributor computes the allocation $C_1, \ldots, C_n$ in order to maximize the sum of valuations. Specifically, if $V_i$ is the valuation function reported by buyer $i$, then the allocation $C_1, \ldots, C_n$ is selected to be a solution of the following problem:

PROBLEM $\mathcal{P}(C)$

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{n} V_i(C_i) \\ \text{subject to} \quad & \sum_{i=1}^{n} C_i \leq C \\ & C_i \geq 0 \quad 1 \leq i \leq n \end{aligned}$$

Assuming the buyers report their valuation functions truthfully, so that $V_i = U_i$ for each $i$, the resulting allocation is efficient. Problem $\mathcal{P}(C)$ determines the allocation, but it remains to specify the payments the buyers make. One possibility would be to charge all buyers the price $\lambda$, equal to the value of the Lagrange multiplier for the constraint $\sum_{i=1}^{n} C_i \leq C$, so the payment of buyer $i$ would be $\lambda C_i$. However, for such a mechanism, the buyers might not have incentive to truthfully report their valuation functions, as indicated by the following example.

**Example** Suppose $C = 1$, $n = 2$, $p_1(x) = p_2(x) = (1 - x)_+$, or equivalently $U_1(x) = U_2(x) = x(1 - \frac{x}{2})$ for $0 \leq x \leq 1$. If both buyers report truthfully, then they each buy half a unit of capacity for price $\lambda = \frac{1}{2}$, and each has payoff $U_i(C_i) - \lambda C_i = \frac{1}{8}$. If buyer 1 were to report truthfully, and buyer 2 were to lie and report the demand function $\bar{p}_2(x) \equiv \frac{1}{3}$, then the distributor would allocate quantity $\frac{1}{3}$ to buyer 2 at the price $\frac{1}{3}$. The payoff for buyer 2 would become $\frac{1}{6}$. Thus, if buyer 1 reports truthfully, buyer 2 would have incentive to lie, resulting in an inefficient allocation of capacity.

Payments for which truth telling is an optimal strategy for each buyer are defined by the well-known Vikrey-Clarke-Groves (VCG) mechanism. Under the Clarke version of the VCG mechanism, the payment made by a buyer $i$ is $\theta_i$, given as follows. Given $0 \leq y \leq C$, let $\mathcal{V}_i(y)$ denote the maximum possible sum of valuations of the other buyers, if a quantity $y$ is reserved for buyer $i$. In symbols, this is expressed as

$$\mathcal{V}_i(y) = \max_{C_{-i}:\sum_{j\neq i} C_j = C-y} \sum_{j\neq i} V_j(C_j)$$

where $C_{-i} = (C_j : j \neq i)$, and the constraints $C_j \geq 0$ are understood. The payment of buyer $i$ for the VCG mechanism is defined by $\theta_i = \mathcal{V}_i(0) - \mathcal{V}_i(C_i)$. In words, the payment $\theta_i$ is how much the maximum sum of utilities of the other buyers is diminished if buyer $i$ is awarded quantity $C_i$. The payoff function of buyer $i$, for the reported valuation functions of other buyers fixed, is $U(C_i) - \theta_i$ or

$$\text{payoff}_i = \{U_i(C_i) + \mathcal{V}_i(C_i)\} - \mathcal{V}_i(0)$$

The report of buyer $i$ does not influence the last term, $\mathcal{V}_i(0)$. If buyer $i$ reports truthfully, the distributor will seek a choice of $C_i$ that maximizes the sum of the first two terms of $\text{payoff}_i$. Thus, truthful reporting by buyer $i$ is an optimal strategy, no matter what the other buyers report. In the terminology of game theory, truthful reporting is a dominant strategy for all buyers, and the VCG mechanism is incentive compatible.

Assume the buyers truthfully report their valuation functions. The sum of payments is $\sum_i \theta_i$ so that the demand determined by VCG allocation is given by $p(C)C = \sum_i \theta_i$. The demand function $p(C)$ gives, for each $C$, the average price per unit quantity, paid by the buyers. Let $(C_1, \ldots, C_n)$ denote an efficient allocation of capacity $C$, and for each $i$ fixed, let $(C_{ij} : j \neq i)$ denote an efficient allocation of capacity $C$ if buyer $i$ is excluded. These allocations can be selected so that $C_{ij} \geq C_j$ for each $i, j$ with $i \neq j$. Then

$$\begin{aligned} \sum_i \theta_i &= \sum_i \sum_{j\neq i} [U_j(C_{ji}) - U_j(C_j)] && (12) \\ &= \left( \sum_i \sum_{j\neq i} U_j(C_{ji}) \right) - (n-1) \sum_j U_j(C_j) \\ &= \left( \sum_i (\text{max sum value with } i \text{ excluded}) \right) - (n-1)(\text{max sum value}) && (13) \end{aligned}$$

Let $p_{sum}$ denote the demand function for inverse demand $\sum_j x_j(p)$. This is the sum of quantities demand function investigated in Subsection 4.1. Similarly, let $p_{-i}$ denote the demand function for inverse demand $\sum_{j\neq i} x_j(p)$. It is the sum of quantities demand with buyer $i$ omitted.

Equation (12) yields the expression

$$p(C) = \frac{1}{C} \sum_i \sum_{j \neq i} \int_{C_j}^{C_{ji}} p_j(x)dx \tag{14}$$

The sum of lengths of the $n(n-1)$ intervals of the form $[C_j, C_{ji}]$ for $j \neq i$ is $C$, so that (14) expresses $p(C)$ as an average price. The optimality conditions for efficient allocation to all buyers imply that $p_j(x) \leq p_{sum}(C)$ for all $x > C_j$. The optimality conditions for efficient allocation to all buyers with buyer $i$ omitted imply that for $i, j$ with $i \neq j$, $p_j(x) \geq p_{-i}(C)$ for $x \leq C_{ij}$. Thus, for $i, j$ with $i \neq j$, $p_{-i}(C) \leq p_j(x) \leq p_{sum}(C)$ for $C_j < x \leq C_{ji}$. This observation and (14) imply that

$$\min_i p_{-i}(C) \leq p(C) \leq p_{sum}(C) \tag{15}$$

That is, the aggregate demand for VCG allocation lies between the sum demand for some buyer omitted, and the sum demand itself.

The right side of (13) is absolutely continuous in $C$ and its derivative is readily identified, yielding the following expression for the derivative of the total payment with respect to $C$:

$$(p(C)C)' = \left( \sum_i p_{-i}(C) \right) - (n-1)p_{sum}(C), \tag{16}$$

Equation (16), the fact $p(C) = (p(C)C)/C$, and the product rule for differentiation yield that

$$p'(C) = \frac{1}{C} \left\{ \sum_i p_{-i}(C) - (n-1)p_{sum}(C) - p(C) \right\} \tag{17}$$

For $C$ fixed, the bound (15) implies that $p_{-i}(C) \leq p(C)$ for at least one value of $i$, and $p_{-i}(C) \leq p_{sum}(C)$ for all values of $i$. Therefore the righthand side of (17) is less than or equal to zero, proving that $p(C)$ is nonincreasing in $C$.

**Proposition 4.1** *Let $p(C)$ denote the demand function for VCG allocation of a given quantity. Then $p$ is nonincreasing, absolutely continuous, and it satisfies the bounds (15).*

**Example** Suppose $n \geq 2$ and $p_i(x) = p_1(x)$ for all $x, i$. Then an efficient allocation for all buyers is given by $C_i = \frac{C}{n}$ for $1 \leq i \leq n$, and for each $i$, an efficient allocation if buyer $i$ is excluded is given by $C_{ji} = \frac{C}{n-1}$ for $j \neq i$. Therefore, (14) yields

$$p(C) = \frac{n(n-1)}{C} \int_{C/n}^{C/(n-1)} p_1(u)du.$$

The bounds (15) reduce to $p_1(C/(n-1)) \leq p(C) \leq p_1(C/n)$.

**Example** Suppose $p_i(x) \equiv a_i$ for each $i$, and suppose the buyers are numbered so that $a_1 \geq \cdots \geq a_n$. Then all the capacity is purchased by the highest bidding buyers for the second highest price, $a_2$.

**Example** Suppose $n = 2$. The payments $\theta_1$ and $\theta_2$ are the areas under the two sections of the curve $\min\{p_1(x), p_2(C-x)\}$ shown in Figure 11. The average price $p(C)$ is the average of $\min\{p_1(x), p_2(C-x)\}$ over the interval $[0, C]$.

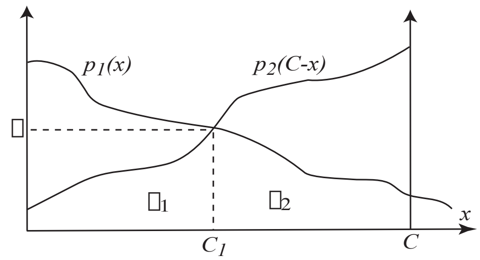


Figure 11: VCG allocation $C_1$ and payments $\theta_1$,$\theta_2$ for two buyers.

## 4.5 Aggregation by proportional allocation of given quantity

Aggregation by proportional allocation of given quantity is based on a game among the buyers in which a price is determined for a given quantity. To avoid overuse of the variable $x$, for the moment the quantity is denoted by $C$. The game involves the buyers submitting bids (i.e. payments) and the quantity $C$ is allocated to the buyers in proportion to the bids. Details are given below. Each Nash equilibrium point (NEP) has an associated price equal to the sum of the bids divided by $C$. The maximum, over all NEPs, of such equilibrium price, defines the demand, $p(C)$, for proportional allocation of capacity to the $n$ buyers.

Let the $n$ buyers with demand sets $D_i$ and associated demand functions $x_i$ and $p_i$ in $\mathcal{D}$ be given. The quantity $C > 0$ is assumed known to all buyers–hence our terminology that the quantity is given. In reality, the quantity may be set strategically by an agent, but for the purpose of this section, the buyers do not anticipate the actions of such an agent.

To set ideas, consider first the special case that for each $i$, $p_i$ is finite, continuous, strictly positive, and nonincreasing over $(0, \infty)$. Let $U_i$ be the corresponding valuation function, so that $U_i' = p_i$. Let $w_i$ denote the bid of buyer $i$ and let $W = \sum w_i$. If $W \neq 0$ the payoff of buyer $i$ is given by $U_i(\frac{w_i C}{W}) - w_i$. An vector of bids $(w_1, \ldots, w_n)$ is an NEP if for each $i$, $w_i$ is a maximizer of the payoff of buyer $i$ with $w_{-i} = (w_j : j \neq i)$ fixed. For a suitable choice of allocation if $W = 0$ (e.g. all allocations zero if $W = 0$) and assuming there are at least two buyers, all NEP's are such that $w_i > 0$ for at least two values of $i$. (This point is addressed further below.) The payoff function of each buyer $i$ is continuously differentiable and convex in $w_i$. Necessary and sufficient conditions for an NEP are thus: for each $i$

$$\frac{\partial \left[U_i(\frac{w_i C}{W}) - w_i\right]}{\partial w_i} \leq 0, \quad \text{with equality if } w_i > 0.$$

Equivalently,

$$\widetilde{p}_i(C, x_i) \leq \frac{W}{C}, \quad \text{with equality if } x_i > 0, \tag{18}$$

where $x_i = \frac{w_i C}{W}$, and $\widetilde{p}_i(C, x_i) = p_i(x_i)(1 - \frac{x_i}{C})$. As a function of $x$, $\widetilde{p}_i(C, x)$ is a new demand function. Let $\widetilde{U}_i$ denote the corresponding utility function. For brevity, the dependence of $\widetilde{U}_i$ on $C$ is suppressed in the notation. Since $p_i$ is nonincreasing and $(1 - \frac{x}{C})$ is strictly decreasing over $0 \leq x \leq C$, it follows that $\widetilde{U}_i$ is strictly concave over $[0, C]$. Moverover, (18) is the optimality condition for the following problem for given $C > 0$:

PROBLEM $\widetilde{\mathcal{P}}(C)$

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{n} \widetilde{U}_i(x_i) \\ \text{subject to} \quad & \sum_{i=1}^{n} x_i \leq C \\ & x_i \geq 0 \quad 1 \leq i \leq n \end{aligned}$$

Therefore, $w = (w_1, \ldots, w_n)$ is an NEP if and only if the corresponding allocation $x = (x_1, \ldots, x_n)$ is a solution to the optimization problem $\widetilde{\mathcal{P}}(C)$. Hence, the strategic buyers with valuation functions $U_i$ behave as socially conscious buyers with utility functions $\widetilde{U}_i$.

A simple way to find $p(C)$, then, is the following. Given $C > 0$, let $T_i(C, p)$ denote the inverse of the function $\widetilde{p}_i(C, x)$. In other words, $T_i$ for $C$ fixed is the inverse demand function for the demand function $\widetilde{p}_i$. The equilibrium price, $p(C)$, then, is the unique price $\lambda$ such that $\sum_{i=1}^{n} T_i(C, \lambda) = C$.

Next, the continuity and strict positivity assumptions about the $p_i$ are dropped. The new assumption is that for each $i$, $p_i \in \mathcal{D}$ and $p_i(x) < +\infty$ for $x > 0$. Equivalently, if $U_i$ is the valuation function associated to $p_i$, then $U_i$ is finite, concave, and nondecreasing on $(0, \infty)$. Since it is no longer assumed that $p_i(x) > 0$ for all $x > 0$, the case that all bids are zero must be addressed explicitly. For that case we adopt the extension of the buyer's game devised in Johari and Tsistsiklis [**?**]. The strategy of a buyer $i$ is a pair $\sigma_i = (w_i, \phi_i)$ such that $w_i$ is a bid and $\phi_i$ is a requested quantity. Let $\sigma = (\sigma_i : 1 \leq i \leq n)$ and $W = w_1 + \cdots + w_n$. The allocation to buyer $i$ for strategy vector $\sigma$ is defined as follows:

$$x_i(\sigma) = \begin{cases} \frac{w_i C}{W} & \text{if } W > 0 \\ \phi_i & \text{if } W = 0 \text{ and } \sum_j \phi_j \leq C \\ 0 & \text{if } W = 0 \text{ and } \sum_j \phi_j \leq C \end{cases}$$

Note that the quantity requests $\phi_i$ are used in the allocation mechanism only if $W = 0$. The payoff of buyer $i$ is $U_i(x_i(\sigma)) - w_i$. We seek an NEP $\sigma$, meaning that for each $i$, $\sigma_i$ is a maximizer of the payoff of buyer $i$ with $\sigma_{-i}$ fixed. The equivalence of NEPs and solutions to an optimization problem is investigated next. Then the aggregate demand for capacity taking buyers is considered.

As before, let $\widetilde{p}_i(C, x) = p_i(x_i)(1 - \frac{x}{C})$ for $x \geq 0$ and $C > 0$, let $\widetilde{U}_i$ be the corresponding valuation function, and let $T_i$ be the corresponding inverse demand function. Note that $\widetilde{p}_i(x, C) = \partial^- \widetilde{U}_i(x)$ for $x > 0$ and $\widetilde{p}_i(0, C) = \partial^+ \widetilde{U}_i(0)$. The function $T_i(C, a)$ is given for $C > 0$ and $a \geq 0$ by

$$T_i(C, a) = \begin{cases} \sup\{x : p_i(x)(1 - \frac{x}{C}) \geq a \text{ or } x = 0\} & \text{for } a > 0 \\ x_{i,\max} \wedge C & \text{for } a = 0 \end{cases} \tag{19}$$

where $x_{i,\max} = \sup\{x : p_i(x) > 0\} = x_i(0)$. The following simple rescaling of (19) is sometimes useful:

$$\frac{T_i(C,a)}{C} = \begin{cases} \sup\{y : p_i(yC)(1-y) \geq a \text{ or } y = 0\} & \text{for } a > 0 \\ \frac{x_{i,\max}}{C} \wedge 1 & \text{for } a = 0 \end{cases} \tag{20}$$

Some elementary properties of $T_i$ are collected in the following lemma, stated without proof.

**Lemma 4.2** *The function $T_i(C,a)$, defined over $\{C > 0, a \geq 0\}$, has the following properties:*
*(a) $T_i(C,a) = 0$ at $a = p_i(0) \leq +\infty$,*
*(b) $T_i(C,a) \leq C \wedge x_i(a)$,*
*(c) $T_i(C,a) < C$ for $C > 0, a > 0$,*
*(d) $T_i(C,a)$ is nonincreasing in $a$ for fixed $C > 0$,*
*(e) $T_i(C,a)$ is nondecreaing in $C$ for fixed $a \geq 0$,*
*(f) $T_i$ is continuous over its domain $\{C > 0, a \geq 0\}$,*
*(g) $\frac{T_i(C,a)}{C}$ is nonincreasing in $C$ (Follows from (20)).*

We return to consideration of Problem $\widetilde{\mathcal{P}}(C)$ under the more general assumptions about the functions $p_i$. Let $C_{\max} = \sum_i x_{i,\max}$.

**Lemma 4.3** *Fix $C > 0$. The vector $x = (x_1, \ldots, x_n)$ is a solution to Problem $\widetilde{\mathcal{P}}(C)$ and $\lambda$ is a corresponding Lagrange multiplier value, if and only if one of the two following conditions hold:*
*(i) $C > C_{\max}$, $x_i \geq x_{i,\max}$ for all $i$, and $\lambda = 0$*
*(ii) $x_i = T_i(C,\lambda)$ for all $i$ and $\sum_{i=1}^n x_i = C$.*

**Proof.** The lemma just gives the first order necessary and sufficient optimality condition for Problem $\widetilde{\mathcal{P}}(C)$, which has a concave objective function and convex constraint set. □

**Lemma 4.4** *Fix $i$ and $\sigma_{-i}$. Suppose $w_j > 0$ for some $j \neq i$. The payoff of buyer $i$ is maximized at $w_i$ if and only if $x_i(\sigma) = T_i(C,\lambda)$ for $\lambda = \frac{W}{C}$.*

**Proof.** If the bid of buyer $i$ were $\breve{w}_i$, the allocation to buyer $i$ would be $\breve{x}_i = \frac{C\breve{w}_i}{\breve{W}}$, where $\breve{W} = \breve{w}_i + \sum_{j \neq i} w_j$, and the payoff would be $U_i(\breve{x}_i) - \breve{w}_i$. The left derivative of this payoff with respect to $\breve{w}_i$ is $\frac{C}{\breve{W}} p_i(\breve{w}_i)(1 - \frac{\breve{x}_i}{C}) - 1$. Consequently, $w_i$ maximizes the payoff if and only if

$$w_i = \max\{\breve{w}_i : p_i(\breve{x}_i)(1 - \frac{\breve{x}_i}{C}) \geq \frac{\breve{W}}{C} \text{ or } \breve{w}_i = 0\}. \tag{21}$$

Since $p_i(\breve{w}_i)(1 - \frac{\breve{x}_i}{C})$ is strictly decreasing in $\breve{w}_i$, (21) is equivalent to

$$w_i = \max\{\breve{w}_i : p_i(\breve{x}_i)(1 - \frac{\breve{x}_i}{C}) \geq \lambda \text{ or } \breve{w}_i = 0\}. \tag{22}$$

Since $\breve{x}_i = 0$ if $\breve{w}_i = 0$, (22) is equivalent to $x_i = T_i(C,\lambda)$. □

**Proposition 4.5** *Fix $C > 0$ and $\lambda \geq 0$. If $x = (x_1, \ldots, x_n)$ is a solution to Problem $\widetilde{\mathcal{P}}(C)$ and $\lambda$ is the associated Lagrange multiplier value, then $\sigma = ((w_i, \phi_i) : 1 \leq i \leq n)$ defined by $w_i = \lambda x_i$ and $\phi_i = x_i$ is an NEP, and $x = x(\sigma)$ and $\lambda = \frac{W}{C}$. Conversely, if $\sigma$ is an NEP, then $x(\sigma)$ is a solution to Problem $\widetilde{\mathcal{P}}(C)$, and $\lambda = \frac{W}{C}$ is a corresponding Lagrange multiplier value.*

**Proof.** Suppose $x$ solves Problem $\widetilde{\mathcal{P}}(C)$ with Lagrange multiplier value $\lambda$. Let $\sigma$ be defined by $w_i = \lambda x_i$ and $\phi_i = x_i$. If $C > C_{\max}$ then $\lambda = 0$ and $x_i = x_i(\sigma) \geq x_{i,\max}$ for all $i$. Therefore $\sigma$ is an NEP. If $0 < C \leq C_{\max}$, Lemma 4.3 implies that $x_i = T_i(C, \lambda)$ for all $i$ and $\sum_i x_i = C$. If $\lambda = 0$ then either $x_i = x_{i,\max}$ for all $i$, or $x_{i_o,\max} = C_{\max}$ for some $i_o$ and $x_{j,\max} = 0$ for $j \neq i_o$. Either way, $\sigma$ is an NEP. Finally, if $0 < C \leq C_{\max}$ and $\lambda > 0$, then $x_i < C$ for each $i$, so $w_i > 0$ for at least two values of $i$. Therefore by Lemma 4.4, $\sigma$ is an NEP. The first half ot the proposition is proved.

To prove the converse part, begin by assuming $\sigma$ is an NEP. If $W = 0$ then $x_i(\sigma) \geq x_{i,\max}$ for all $i$, so that $x$ solves Problem $\widetilde{\mathcal{P}}(C)$ with Lagrange multiplier value $\lambda = 0$. If $W > 0$ then $w_i > 0$ for at least two values of $i$. Therefore $x_i = T_i(C, \lambda)$ for all $i$ by Lemma 4.4. Thus $x$ solves Problem $\widetilde{\mathcal{P}}(C)$ with Lagrange multiplier value $\lambda = \frac{W}{C}$ by Lemma 4.3. □

Recall that the aggregate demand for proportional allocation of given quantity is given by the demand function $p(C) = \frac{W}{C}$ for $C > 0$, where $W = W(C)$ is the maximum equilibrium price for the NEPs of the buyers' game for allocation of quantity $C$ using proportional allocation.

**Proposition 4.6** *Let $p_i \in \mathcal{D}$ for $1 \leq i \leq n$ such that $p_i(x) < \infty$ for $x > 0$, and let $C_{\max} = \sum_i x_{i,\max}$. The aggregate demand for proportional allocation of given quantity is given by*

$$p(C) = \begin{cases} \max\{a : \sum_i T_i(C, a) = C\} & \text{if } 0 < C \leq C_{\max} \\ 0 & \text{if } C > C_{\max} \end{cases} \tag{23}$$

*and $p(0)$ is defined to make $p$ continuous at $0$. Moreover, $p(C) < \infty$ for $C > 0$ and $p \in \mathcal{D}$. The quantity purchased by buyer $i$, $T_i(C, p(C))$, is nondecreasing and left continuous in $C$ for $C > 0$. The demand $p$ is no stronger than the sum of quantities demand $x_1(p) + \ldots + x_n(p)$. Also, $p(x) \leq p^{(2)}(0)$, where $p^{(1)}(x) \geq p^{(2)}(x) \geq \cdots \geq p^{(n)}(x)$ denotes for each $x$ the numbers $p_1(x), \ldots, p_n(x)$ written in nonincreasing order. If the $p_i$ satisfy Conditions I and II for all $i$, then so does $p$. If $x_{i,\max} > 0$ for at most one value of $i$, then $p(C) \equiv 0$. Otherwise, $p(C) > 0$ for $0 \leq C < C_{\max}$.*

**Proof.** By Lemma 4.3, if $0 < C \leq C_{\max}$ then $\{a : \sum_i T_i(C, a) = C\}$ is the set of Lagrange multiplier values for Problem $\widetilde{\mathcal{P}}(C)$, and if $C > C_{\max}$ then $\lambda = 0$ is the unique Lagrange multiplier value for Problem $\widetilde{\mathcal{P}}(C)$. Proposition 4.5 implies that the set of possible Lagrange multiplier values for Problem $\widetilde{\mathcal{P}}(C)$ is equal to the set of prices for NEPs of the allocation game. Thus (23) holds.

For $C > 0$ fixed, $\sum_i T_i(C, a)$ is continuous in $a$ with value $\sum_i (x_{i,\max} \wedge C) \geq (\sum_i x_{i,\max}) \wedge C = C_{\max} \wedge C$ at $a = 0$ and limit zero as $a \to +\infty$. Thus (23) yields that $p(C) < \infty$ for $C > 0$.

It is proved that $p$ is nonincreasing over the region $C > 0$ as follows. Suppose that $0 < C < \overline{C}$. It is to be shown that $p(C) \geq p(\overline{C})$. If $p(\overline{C}) = 0$ the result is trivially true, so suppose that $p(\overline{C}) > 0$. Let $a = p(\overline{C})$. Part (g) of Lemma 4.2, and (23) applied for quantity $\overline{C}$, yield that $\sum_i T_i(C, a) \geq \frac{C}{\overline{C}} \sum_i T_i(\overline{C}, a) = C$. Therefore (23) yields that $p(C) \geq a$. Therefore $p$ is nonincreasing over the region $C > 0$. Therefore $p(0)$ is well defined and $p(C)$ is nonincreasing over the region $C \geq 0$ and continuous at $C = 0$.

Next it is proved that $p$ is left continuous. Suppose that $0 < C_1 < C_2 < \cdots$ and $C$ are given with $\lim_{k\to\infty} C_k = C$. Since $p(C_k)$ is nonincreasing the limit $a^* = \lim_{k\to\infty} p(C_k)$ is well defined. By (23) applied for the quantity $C_k$, $\sum_i T_i(C_k, p(C_k)) \geq C_k$. Taking the limit as $k \to \infty$ and using the continuity of $T_i$ yields that $\sum_i T(C, a^*) \geq C$. Therefore, by (23), $p(C) \geq a^*$. Also $p(C) \leq a^*$ since $p$ is nonincreasing. Therefore $p(C) = a^*$, establishing that $p$ is left-continuous. This completes the proof that $p \in \mathcal{D}$.

The quantity purchased by buyer $i$, $T_i(C, p(C))$, is nondecreasing and left continuous in $C$ for $C > 0$, because $p(C)$ is nonincreasing and left continuous in $C$, and $T_i(C, a)$ is nondecreasing in $C$, nonincreasing in $a$, and continuous in $(C, a)$.

It follows from the inequality $T_i(C, a) \leq x_i(a)$ that if $C \leq C_{\max}$ then

$$p(C) = \max\{a : \sum_i T_i(C, a) = C\} \leq \sup\{a : \sum_i x_i(C, a) = C\}.$$

That is, $p(C)$ is less than or equal to the price function for the aggregation of demand by sum of quantities, as was to be proved. A consequence is that if $p_1, \ldots, p_n$ satisfy Conditions I and II, then so does $p$.

The proof that $p(x) \leq p^{(2)}(0)$ is given next. The inequality is trivially true if $C_{\max} = 0$ so assume $C_{\max} > 0$. Then (23) implies that

$$p(0) = \lim_{c\to 0}\{a : \sum \frac{T_i(C, a)}{C} \geq 1\},$$

and (20) implies that for any $i$ with $p_i(0) > 0$ and $a \geq 0$, $\lim_{C\to 0} \frac{T_i(C,a)}{C} = (1 - \frac{a}{p_i(0)})_+$. Thus, if $p_i(0) > 0$ for at least two values of $i$,

$$p(0) = \max\{a : \sum \left(1 - \frac{a}{p_i(0)}\right)_+ \geq 1\} \tag{24}$$

Otherwise, $p(0) = 0$. The monotonicity of $p$ and (24) show that for all $x \geq 0$, $p(x) \leq p(0) \leq p^{(2)}(0)$, as desired.

If $x_{i,\max} > 0$ for at most one value of $i$, then $p(C) \equiv 0$. Otherwise, $p(C) > 0$ for $0 \leq C < C_{\max}$.

If $x_{i,\max} = 0$ then $T_i(C, a) = 0$ for all $C > 0$ and all $x \geq 0$. Thus if $x_{i,\max} = 0$ for all $i$, then $C_{\max} = 0$ and $p(C) = 0$ for all $C \geq 0$ by (23). If there is exactly one particular value $i$ such that $x_{i,\max} = 0$, then part (c) of Lemma 4.2 and (23) shows that again $p(C) = 0$ for all $C \geq 0$. Finally, suppose $0 < C < C_{\max}$ and $x_{i,\max} > 0$ for at least two values of $i$. Then $\sum_i T_i(C, 0) = \sum_i x_{i,\max} \wedge C > C$ and $\lim_{a\to\infty} \sum_i T_i(C, a) = 0$. Thus, by (23) and the continuity of $T_i$, it follows that $p(C) > 0$. The proof of Proposition 4.6 is complete. □

**Example** Suppose $n \geq 2$ and $p_i(x) = p_1(x)$ for $1 \leq i \leq n$. The demand for aggregation of capacity taking buyers is given by $p(C) = p_1(\frac{C}{n})(\frac{n-1}{n})$. Compare to $p(C) = p_1(\frac{C}{n})$ for the sum of quantities demand.

**Example** Suppose $p_i(x) \equiv a_i$ for $1 \leq i \leq n$, where $n \geq 2$ and $+\infty > a_1 \geq a_2 \geq \cdots \geq a_n > 0$. Then $p(C) \equiv \lambda$, where $\lambda$ is the unique positive constant such that $\sum_i (1 - \frac{\lambda}{a_i})_+ = 1$. Note that $\lambda < a_2$. That is, the equilibrium price is less than the second largest price.

**Example** Suppose $n = 4$ with

$$p_1(x) = p_2(x) = 200(1 - x)_+ \quad \text{and} \quad p_3(x) = p_4(x) = 2(1 - \frac{x}{200})_+$$

Buyers 1 and 2 are willing to pay a high price, if necessary, for a small amount of quantity, but they have no need for more than one unit of quantity each. On the other hand, buyers

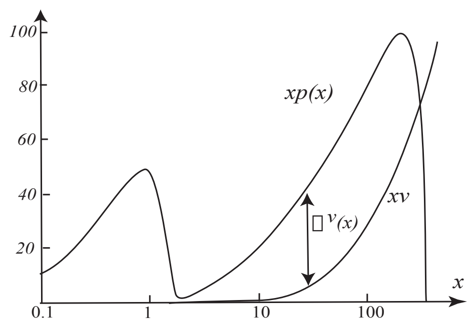


Figure 12: Payoff and $xp(x)$ for an example with four buyers

3 and 4 are willing to by relatively large quantities for a moderate price. A graph of $xp(x)$ and the corresponding payoff function $\Pi^v(x) = x(p(x) - v)$ for $v = 0.295$ is pictured in Figure 12. For this value of $v$ the payoff function $\Pi^v$ has two maximizers. The first peak in the function $xp(x$ is primarily due to competition among buyers 1 and 2 in case the offered quantity is small, and the second peak is primarily due to competition among buyers 3 and 4 for moderate amounts of quantity.

A lower bound on the aggregate demand of capacity takers (ADCT) with proportional allocation is given next. Suppose for each fixed $p$, the $n$ numbers $x_1(p), \ldots, x_n(p)$ are written as a nonincreasing list: $x^{(1)}(p) \geq \cdots \geq x^{(n)}(p)$. Thus, $x^{(i)}(p)$ is the $i$th highest quantity demanded for price $p$. Given $k \geq 1$, define $\delta_k(p)$ by $\delta_k(p) = \sum_{j=1}^{k-1} x^{(j)}(p) - x^{(k)}(p)$.

**Proposition 4.7** *Let $k$ be an integer with $1 \leq k \leq n$. The aggregate demand for capacity taking buyers with demands $x_1, \ldots x_n$ is no weaker than the demand defined by $x_{s,k} = \left(\sum_{i=1}^n x_i(\frac{kp}{k-1})\right) - \delta_k(\frac{kp}{k-1})$.*

**Proof.** Let $\breve{x}(p) = \min\{x^{(k)}(p), x_i(p)\}$ for $1 \leq i \leq n$, and let $\breve{x}^{(1)}(p) \geq \cdots \geq \breve{x}^{(n)}(p)$ denote the nonincreasing reordering of $\breve{x}_1(p), \ldots, \breve{x}_n(p)$ for each $p \geq 0$. Note that

$$\begin{aligned} \breve{x}_i(p) &\leq x_i(p) && \text{for all } i && (25)\\ \breve{x}^{(i)}(p) &= x^{(k)}(p) && \text{for } 1 \leq i \leq k && (26)\\ \breve{x}^{(i)}(p) &= x^{(i)}(p) && \text{for } k \leq i \leq n \\ \sum_i \breve{x}_i(p) &= (\sum_i x_i(p)) - \delta_k(p) \end{aligned}$$

By (25), the ADCT for $x_1, \ldots, x_n$ is no weaker than the ADCT for $\breve{x}_1, \ldots, \breve{x}_n$. If $\breve{a} = \breve{p}(C)$ denotes the equilibrium price defining the ADCT for $\breve{x}_1, \ldots, \breve{x}_n$, (26) implies that $\breve{x}_i(a) \leq \frac{C}{K}$ for all $i$. Thus, the inverse demand functions $\breve{T}_i$ associated to $\breve{p}$ satisfy $\breve{T}_i(C, \breve{a}) \geq \breve{x}_i(\frac{(k-1)\breve{a}}{k})$, so that $\sum \breve{x}_i(\frac{(k-1)\breve{a}}{k}) \geq C$. The ADCT for $\breve{x}_1, \ldots, \breve{x}_n$ is therefore no weaker than the sum of quantities demand $\sum \breve{x}_i(\frac{(k-1)\breve{a}}{k}) \ = x_{s,k}(p)$. □

An implication of Proposition 4.7 is the following. If $n >> k >> 1$, and if the demands of the users are not significanly different, then the effect of the term $\delta_k$ and the scaling by $\frac{k-1}{k}$ are not very significant, so that the aggregate demand for capacity taking buyers is nearly as large as the sum of quantities aggregate demand.

# 5 Competing Agents

Three scenarios in which multiple agents (sellers) compete for the demand of a single buyer are considered in this section. Let $m$ denote the number of agents. The demand of the buyer is represented by a demand set $D$, or equivalently a demand function $p \in \mathcal{D}$, or inverse demand function $x \in \mathcal{D}$. The buyer is a price or quantity taker, and doesn't anticipate decisions of the agents. The buyer itself can represent an aggregate of many buyers and agents, which possibly anticipate actions among themselves, but in this section, only the single buyer is given. Attention is given to the new demand represented by the competing agents plus buyer, assuming all agents have access to quantity for the same price. This demand is denoted by $\widehat{D}$, $\widehat{p}$, or $\widehat{x}$, which was used earlier in Section 1 in the special case $m = 1$.

The three scenarios considered are the classical Bertrand competition, classical Cournot competition, and a variaton of Cournot competition. In both the Bertrand and Cournot models, the agents are price takers. The difference between the two models is that in the Bertrand model each agent offers a price to the buyer, and in the Cournot model each agent offers a quantity to the buyer. For the third model, the agents take a total capacity as a given, with each agent bidding for a share of the capacity which the agent then offers to the buyer. A fourth possible scenario is that the agents take a total capacity as given, and offer agents prices. However, this is equivalent to the third scenario and is not treated separately.

## 5.1 Bertrand Competition

Suppose agent $i$ can produce or purchase quantity at a constant price $v_i$, where $(v_1, \ldots, v_M)$ is given. Assume without loss of generality that $v_1 \leq \cdots \leq v_m$. In the Bertrand competition, agent $i$ offers price $p_i$ to the buyer. The buyer purchases an amount $x_i$ from agent $i$. The buyer takes the lowest price $p_{\min} = \min\{p_1, \ldots, p_m\}$, and purchases an amount $x(p_{\min})$ from the agents offering the minimum price. It multiple agents offer the same minimum price, the buyer splits the purchase in an arbitrary way among those agents. Thus, a vector $(x_1, \ldots, x_n)$ is a valid allocation vector for $(p_1, \ldots, p_m)$ if

$$\begin{aligned} \sum_i x_i &= x(p_{\min}) \\ x_i &\geq 0, \quad \text{with equality if } p_i > p_{\min}. \end{aligned}$$

The payoff of agent $i$ is $(p - p_{\min})x_i$. A point $(p_1, \ldots, p_m, x_1, \ldots, x_m)$ is defined to be a Bertrand equilibrium if no agent can be guaranteed a strictly larger payoff by deviating from price $v_i$. If $m = 1$, the Bertrand equilibrium prices are the profit maximizing prices for a single agent facing a given demand, as discribed in Section **??**. Recall that, for $m = 1$, an equilibrium exists if $v_i > 0$, $p \in \mathcal{D}$, and Conditions I and II are satisfied. The proof of the following standard proposition is straight-forward and is left to the reader.

**Proposition 5.1** *Suppose $m \geq 2$, $x \in \mathcal{D}$, $x$ satisfies Condition II, and $0 < v_1 \leq \cdots \leq v_n$. Then a Bertrand equilibrium exists. Suppose $(x_1, \ldots, x_m)$ is a valid allocation vector for $(p_1, \ldots, p_m)$. Then $(p_1, \ldots, p_m, x_1, \ldots, x_m)$ is a Bertrand equilibrium if and only if the following condition is true (which case applies depends on the given demand functions p and on the given prices $(v_1, \ldots, v_n)$):*
*(Case 1: $v_1 < p(0)$ and $v_1 < v_2$): $p_i \in \arg\max_{a \in [r_1, r_2]} (a - v_1)x(a)$, $p_{\min} - p_1$, and $x_1 = x(p_{\min})$.*
*(Case 2: $v_1 < p(0)$ and $v_1 = v_2$): $p_{\min} = v_1$, at least two agents bid $v_1$, and all $m$ profits are zero.*
*(Case 3: $v_1 \geq p(0)$ and $v_1 = v_2$): All $m$ profits are zero.*

Intuitively, agent 1 has an opportunity for a positive profit if $v_1 < v_2$. Otherwise profits are zero. In particular, in the case of a common production cost $v$ (so $v_i = v$ for all $i$) the profit for all agents is zero and the equilibrium price is $p_{\min} = v$.

The demand function of the buyer plus $m$ agents in Bertrand competition is given, for $m \geq 2$ and a fixed common production cost $v$, by the total quantity purchased, which is simply $x(v)$. That is, the demand represented by two or more agents in Bertrand competition with common production costs, is equal to the original demand of the buyers.

## 5.2 Cournot Competition

Suppose agent $i$ can produce or purchase quantity at a constant price $v_i \geq 0$, where $(v_1, \ldots, v_m)$ is given. Assume without loss of generality that $v_1 \geq \cdots \geq v_m$. (This ordering is opposite of the ordering used in the previous section). For Cournot competition, agent $i$ offers the quantity $C_i$ to the buyer. The buyer purchases the total quantity $C_1 + \cdots + C_m$ at the price per unit quantity $p(C_1 + \cdots + C_m)$. The payoff (or profit) of agent $i$ is defined by

$$\Pi_i(C_i, C_{-i}) = C_i(p(C_1 + \cdots + C_m) - v_i). \tag{27}$$

where $C_{-i} = (C_j : j \neq i)$. The vector $(C_1, \ldots, C_m)$ is said to be a Cournot equilibrium if it is an NEP for the game with payoff functions $\Pi_i$. The following result is related to, but is not a special case of, a similar result in [5].

**Proposition 5.2** *Suppose $p \in \mathcal{C}^1$ and that either $v_m > 0$ or $0 < x_{\max} < +\infty$. Suppose also that either (i) $v_m = v_{m-1}$ and $xp(x)$ is concave over $(0, x_{\max})$, or (ii) $xp(x)$ is strictly concave over $(0, x_{\max})$. Then the Cournot equilibrium exists and is unique.*

The proposition is proved after the following lemma is established.

**Lemma 5.3** *Suppose $p \in \mathcal{C}$ with $xp(x)$ concave over $(0, x_{\max})$, and suppose $y$ is fixed with $0 \leq y < x_{\max}$. (a) As a function of $x$, $xp(x+y)$ is concave over $(0, x_{\max} - y)$, (b) If $y > 0$ or if $xp(x)$ is strictly concave over $(0, x_{\max})$, then $xp(x+y)$ is strictly concave over $(0, x_{\max} - y)$.*

**Proof.** Let $0 < x_1 < x_2 < x_3 < x_{\max} - y$ and let $0 < \lambda < 1$, such that $x_2 = \lambda x_1 + (1-\lambda)x_3$. It is to be shown that

$$\lambda x_1 p(x_1 + y) + (1-\lambda)x_3 p(x_3 + y) \leq x_2 p(x_2 + y), \tag{28}$$

with strict inequality if $y > 0$ or if $xp(x)$ is strictly concave over $(0, x_{\max})$. Two cases are considered:
(Case 1: $\lambda p(x_1 + y) + (1-\lambda)p(x_3 + y) \leq p(x_2 + y)$.) In this case

$$\lambda x_2 p(x_1 + y) + (1-\lambda)x_2 p(x_3 + y) \leq x_2 p(x_2 + y), \tag{29}$$

and in general, the facts $\lambda(x_1 - x_2) < 0 < (1-\lambda)(x_3 - x_2)$, $\lambda(x_1 - x_2) = -(1-\lambda)(x_3 - x_2)$, and $p(x_1 + y) > p(x_3 + y)$, imply that

$$\lambda(x_1 - x_2)p(x_1 + y) + (1-\lambda)(x_3 - x_2)p(x_3 + y) < 0. \tag{30}$$

Combining (29) and (30) implies (28) with strict inequality.

(Case 2: $\lambda p(x_1 + y) + (1-\lambda)p(x_3 + y) > p(x_2 + y)$.) In this case

$$-\lambda y p(x_1 + y) - (1-\lambda)y p(x_3 + y) \leq -y p(x_2 + y), \tag{31}$$

with strict inequality if case $y > 0$. By the concavity of $xp(x)$,

$$\lambda(x_1 + y)p(x_1 + y) + (1-\lambda)(x_3 + y)p(x_3 + y) \leq (x_2 + y)p(x_2 + y), \tag{32}$$

with strict inequality in case $xp(x)$ is strictly concave. Combining (31) and (32) implies (28), with strict inequality in case either $xp(x)$ is concave, or $y > 0$. Lemma 5.3 is proved. □

**Proof of Proposition 5.2** Define $\overline{C}_0 = 0$ and for $1 \leq i \leq m$ define $\overline{C}_i$ by $p(\overline{C}_i) = v_i$ if $v_i \leq p(0)$ and $\overline{C}_i = 0$ if $v_i > p(0)$. Then $0 = \overline{C}_0 \leq \overline{C}_1 \leq \cdots \leq \overline{C}_m < +\infty$. By Lemma 5.3, the payoff function $\Pi_i(C_i, C_{-i})$ is concave in $C_i$, and maximization can be limited to the compact, convex set $\{C_i : C_1 + \cdots + C_m \leq \overline{C}_m\}$, over which the payoff functions are concave. By Rosen's theorem there exists an NEP $(C_1, \ldots, C_m)$. Let $C = C_1 + \ldots + C_m$. The first order optimality condition for each buyer $i$ holds, and is given by

$$p(C) + C_i p'(C) \leq v_i \quad \text{with equality if } C_i > 0. \tag{33}$$

Thus $p(C) \leq v_i$ if $C_i = 0$, and $p(C) > v_i$ if $C_i > 0$. Hence $(p(C) - v_i)_+ + C_i p'(C) = 0$. Summing over $i$ yields $\Gamma(C) = 0$, where the function $\Gamma$ is defined by

$$\Gamma(C) = \sum_{i=1}^{m} (p(C) - v_i)_+ + Cp'(C) \tag{34}$$

If $p(0) = 0$, or equivalently $x_{\max} = 0$, then by assumption $v_m > 0$, so for any $i$, $\Pi_i(C_i) < 0$, with equality if and only if $C_i = 0$. If $p(0) > 0$ and $v_m \geq p(0)$, then again $\Pi_i(C_i) \leq 0$ with equality if and only if $C_i = 0$. In either of these two degenerate cases the unique equilibrium is $C_i = 0$ for all $i$. Thus, for the remainder of the proof, suppose $0 \leq v_m < p(0)$, which implies $\Gamma(0) > 0 > \Gamma(\overline{C}_m)$.

To complete the proof it suffices to show that $\Gamma$ is strictly decreasing over $(0, \overline{C}_m)$, for then $C$ is uniquely determined by the equation $\Gamma(C) = 0$, with $0 < C < \overline{C}_m$, and the $C_i$ are uniquely determined by the first order optimality conditions (33). For $1 \leq i \leq m-1$,

$$\Gamma(C) = (m-i)p(C) + Cp'(C) - \sum_{j=i}^{m} v_j \quad \text{for } \overline{C}_i \leq C \leq \overline{C}_{i+1}.$$

For $0 \leq i \leq m-2$, on the interval $(\overline{C}_i, \overline{C}_{i+1})$, $p(C) + Cp'(C)$ is nonincreasing and $(m-i-1)p(C)$ is strictly decreasing. Hence $\Gamma(C)$ is strictly decreasing over $(\overline{C}_i, \overline{C}_{i+1})$ if $0 \leq i \leq m-2$. If condition (i) of the proposition holds, the interval $(\overline{C}_{m-1}, \overline{C}_m)$ is empty, and if condition (ii) of the proposition holds, $\Gamma(C) = (Cp(C))' + v_m$ over $(\overline{C}_{m-1}, \overline{C}_m)$, which is strictly decreasing. Therefore $\Gamma$ is strictly decreasing over $(\overline{C}_i, \overline{C}_{i+1})$ for $1 \leq i \leq m-1$, and continuous over $(0, \overline{C}_m)$. So $\Gamma$ is strictly decreasing over $(0, \overline{C}_m)$, and Proposition 5.2 is proved. □

Next consider the demand of the buyer plus $m$ agents in Cournot competition for a fixed common production cost $v > 0$. Suppose $p \in \mathcal{C}^1$ and that $xp(x)$ is concave over $(0, x_{\max})$. The demand of agents plus buyer is given by $\widehat{x}(p) = C_1 + \cdots + C_m$, where $(C_1, \ldots, C_m)$ is the unique Cournot equilibrium for production costs $v_i = v$ for all $i$. Letting $C = C_1 + \cdots + C_m$, it follows by symmetry that $C_i = C/m$ for all $i$. The optimality condition (33) becomes that $C$ satisfies

$$p(C) + \frac{C}{m}p'(C) \leq v \quad \text{with equality if } C > 0.$$

The assumption that $Cp(C)$ is concave implies that $p(C) + Cp'(C)$ is nonincreasing in $C$, and

$$p(C) + \frac{C}{m}p'(C) = (\frac{m-1}{m})p(C) + (\frac{1}{m})(p(C) + Cp'(C))$$

so that $p(C) + \frac{C}{m}p'(C)$ is also nondecreasing in $C$, so

$$\widehat{x}(v) = \sup\{C : p(C) + \frac{C}{m}p'(C) \geq v \text{ or } C = 0\}.$$

Therefore, the aggregate demand for the buyer plus $m$ agents in Cournot competition is given by $\widehat{p}(C) = p(C) + \frac{C}{m}p'(C)$.

**Examples** (a) If $\theta > 1$ and $p(C) = C^{-\frac{1}{\theta}}$, then $\widehat{p}(C) = (1 - \frac{1}{n\theta})C^{-\frac{1}{\theta}}$.
(b) If $p(C) = (a - bC)_+$ then $\widehat{p}(C) = (a - b(\frac{n+1}{n})C)_+$.
(c) If $p(C) \equiv \alpha$, then $\widehat{p}(C) \equiv \alpha$.

## 5.3 Cournot competition for capacity taking agents

Suppose the $m$ agents bid for shares of a given quantity $C$. That is, agent $i$ submits a bid $w_i$ and receives a quantity $C_i = \frac{Cw_i}{W}$, where $W = \sum_{i=1}^m w_i$. In turn, all agents sell their capacity to the buyer for the price $p(C_1 + \ldots + C_m)$. The payoff of agent $i$ is given by

$$\Pi_i(w_i, w_{-i}) = C_i \left( p(C_1 + \cdots + C_n) - \frac{W}{C} \right). \tag{35}$$

The Cournot equilibrium for capacity taking agents is the NEP for the payoff functions $\Pi_i$ in (35). The difference between (35) and the Cournot equilibrium of the previous section is that the price paid by agents in (35), namely $\frac{W}{C}$, is variable.

Equation (35) disguises the fact that $C_1 + \cdots C_n$ is specified to be $C$. An alternative expression for $\Pi_i$ is given by

$$\Pi_i(w_i, w_{-i}) = C_i \left( p(C_1 + \cdots + C_n - \frac{W}{C} \right). \tag{36}$$

Equation (36) has the following interpretation. The price taking buyer will pay $Cp(C)$ units of money, no matter what the bids are, as long as $W > 0$. The agents bit for a share of the payment. We call this the "cash giveaway" game.

The profit functions in (36) have the same form as the functions in (27), with the function $p$ in (27) taken to have the form $\widetilde{p} = \frac{A}{C}$, where $A$ is the amount of money to be given away, and the prices $v_i$ in (27) are all taken to be one. That is, the Cournot competition for capacity taking agents is a cash giveaway game, which itself is a Cournot game with the bids $w_i$ being the strategy variables. Since the production prices are equal (all one) and $x\widetilde{p}(x)$ is concave (in fact, constant), Proposition 5.2 applies to yield that there is a unique equilibrium point. By symmetry, all $m$ bids are equal, so that $C_i = \frac{C}{m}$. The optimality condition for Cournot equilibrium, (33) yields that $\widetilde{p}(W) + \frac{W}{m}\widetilde{p}'(W) = 1$, or equivalently $\frac{A}{W} - \frac{W}{m}(\frac{A}{W^2}) = 1$, or equivalently $W = \frac{m-1}{m}p(C)C$. Thus, the price per unit quantity paid by the agents is $\frac{W}{C} = \frac{m-1}{m}p(C)$. The conclusion is that the demand function for the buyer plus $m$ capacity taking agents is $\widehat{p}(C) = \frac{m-1}{m}p(C)$.

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