Heavy-tailed GARCH Models: Pricing and Risk Management Applications in Power Markets

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Agenda

- Background and motivations
- Quantile-based GARCH Models
  - Parameter inference
  - Applications
    - Electricity derivatives pricing
    - Risk management measures
- Semi-parametric Estimation of Confidence Intervals of Conditional Quantiles of GARCH Models
  - Normal approximation and data-tilting
  - Applications
- Conclusion

Background and Motivations

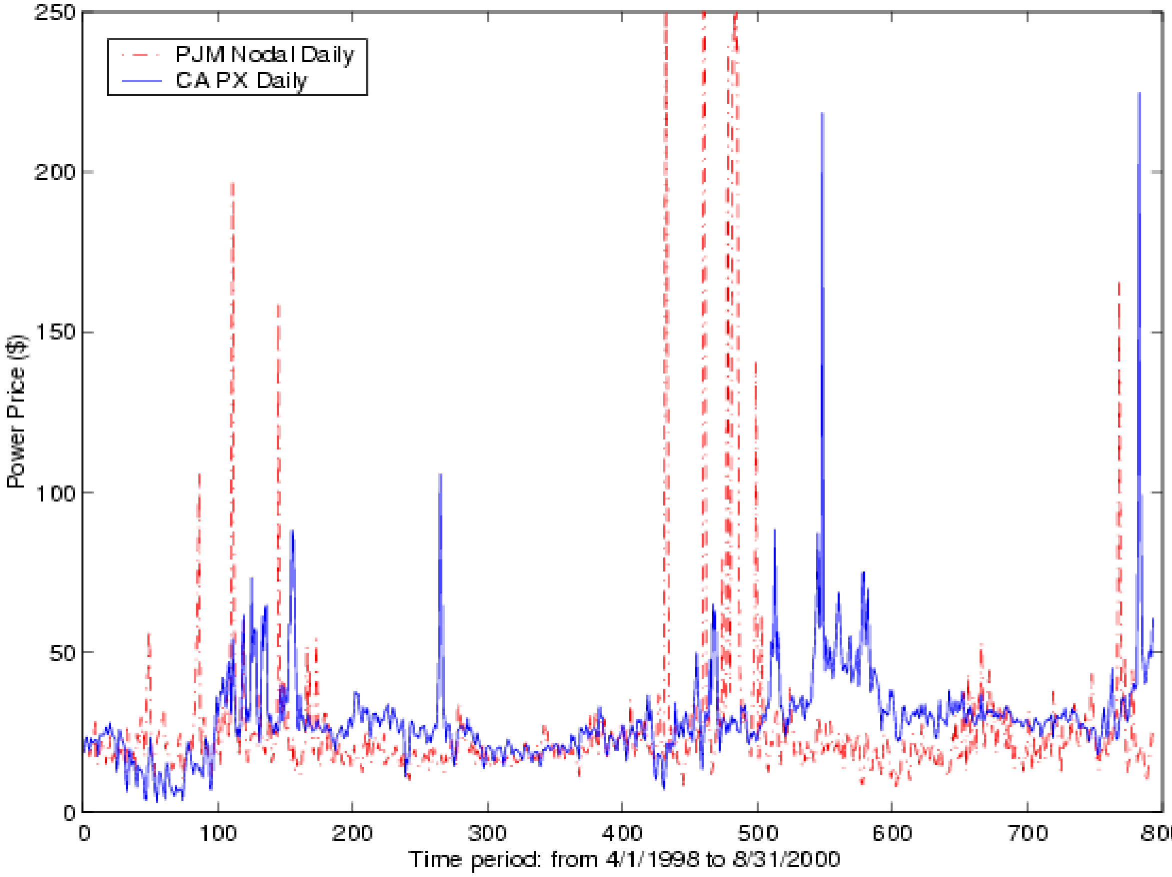
- Rapid developments of power markets starting in the mid-1990s
  - Power Exchange/ISO in California.
  - PJM, NY, NE power pools.
- Market setbacks since 2000
  - Fallen or financially distressed power merchants
  - Dropping liquidity in power exchange/OTC markets
- The nature of incompleteness in energy (power) markets
  - Almost non-storable underlying
  - Limited physical supply and inelastic demand
  - Tremendous price and quantity volatility (e.g., price spikes)
  - Limited ability in hedging quantity risks

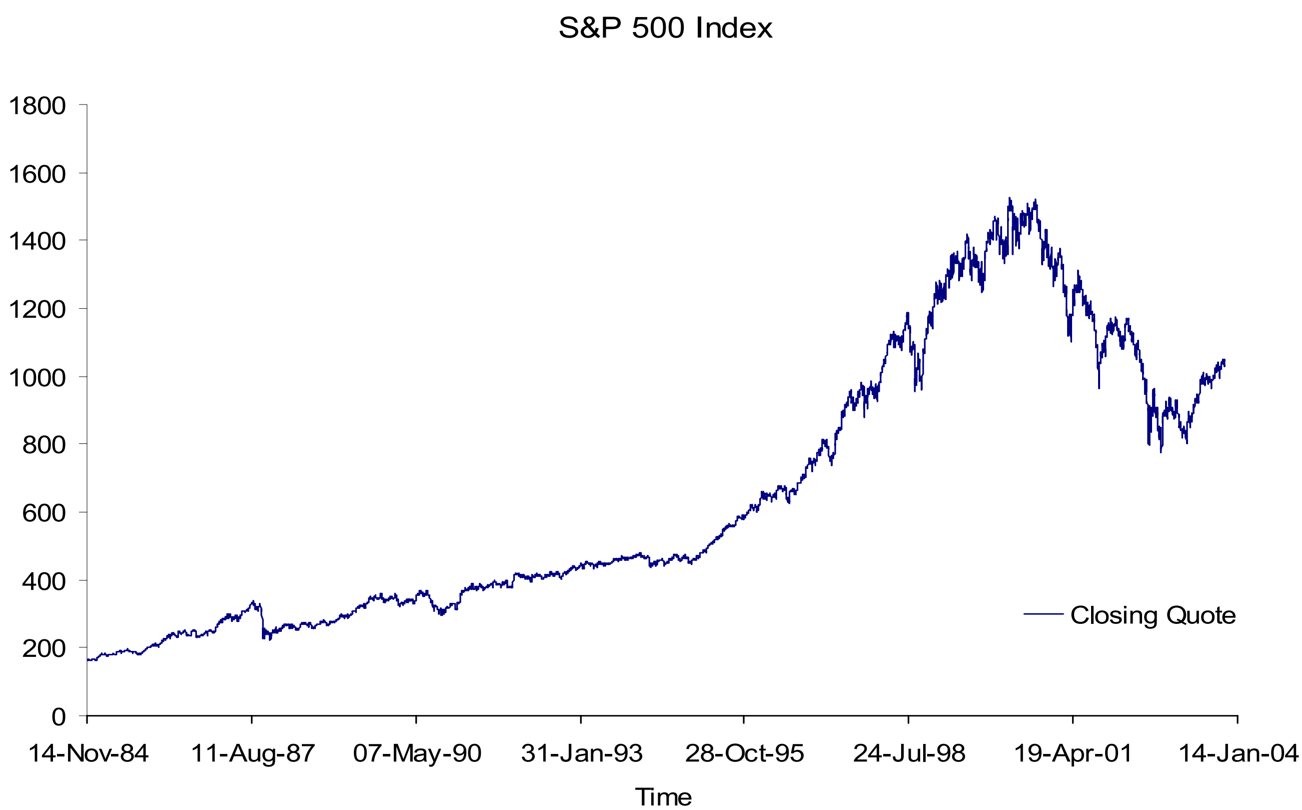
Background and Motivations (con't)

- Redesign of power markets
  - FERC Standard Market Design
  - Development of futures/contract markets

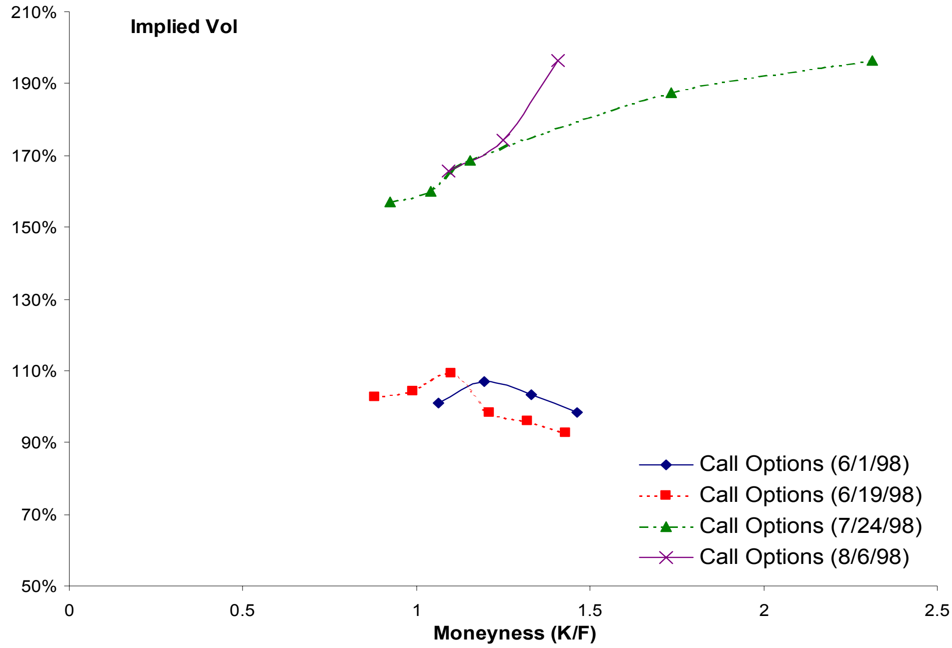
- Risk management needs
  - Independent power producers hedge their production.
  - Power marketers quantify, monitor and control trading risks in wholesale and retail markets.

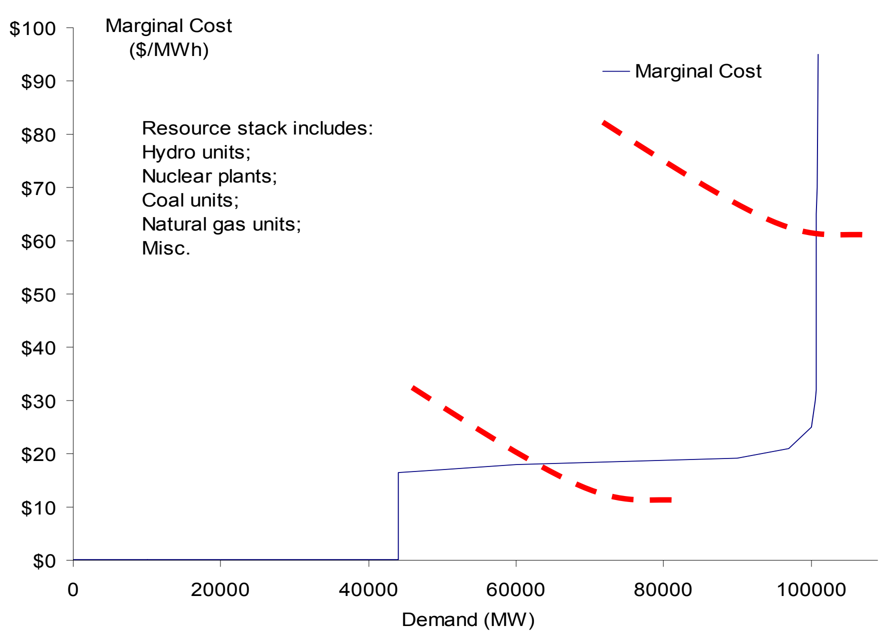
- Trading, Asset valuation, and project selection and financing
  - Pricing and risk management tools to support trading
  - Evaluation of potential investment opportunities in power generation
  - Support for project financing


PJM Nodal Daily
CA PX Daily
Power Price ($)
250
200
150
100
50
0
0
100
200
300
400
500
600
700
800
Time period: from 4/1/1998 to 8/31/2000


S&P 500 Index
1800
1600
1400
1200
1000
800
600
400
200
0
Closing Quote
14-Nov-84
11-Aug-87
07-May-90
31-Jan-93
28-Oct-95
24-Jul-98
19-Apr-01
14-Jan-04
Time

Implied Volatility of Call (Sept.) Options at Cinergy
(Inferred from Black-Scholes formula based on broker quotes for calls and forwards.)





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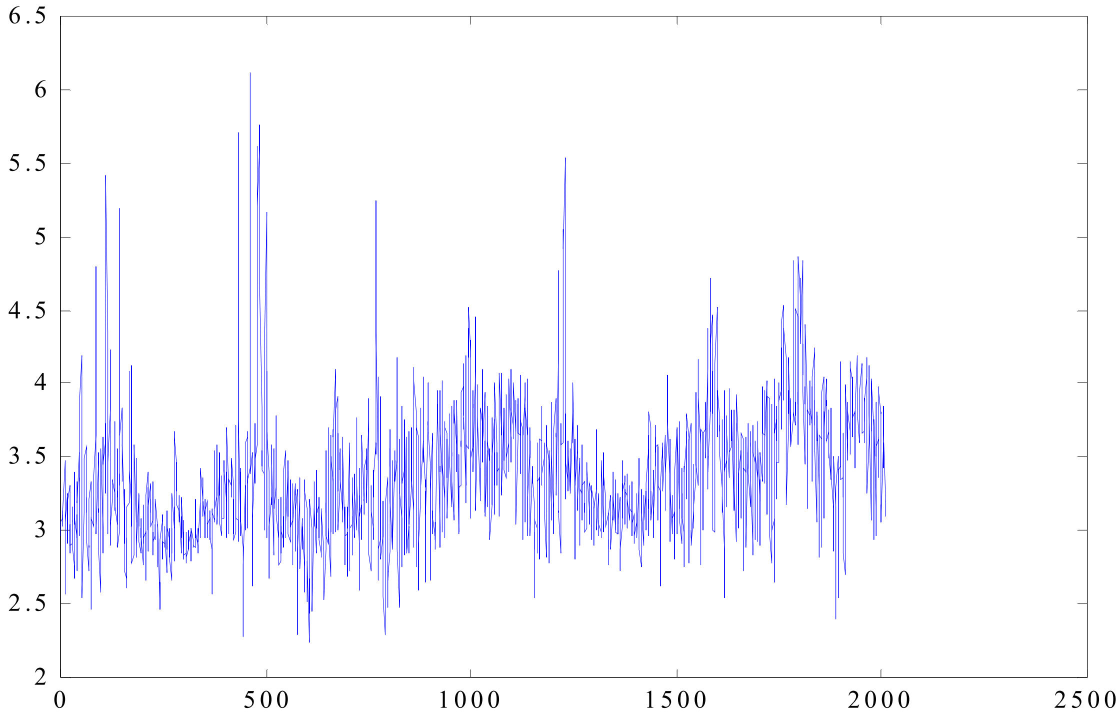
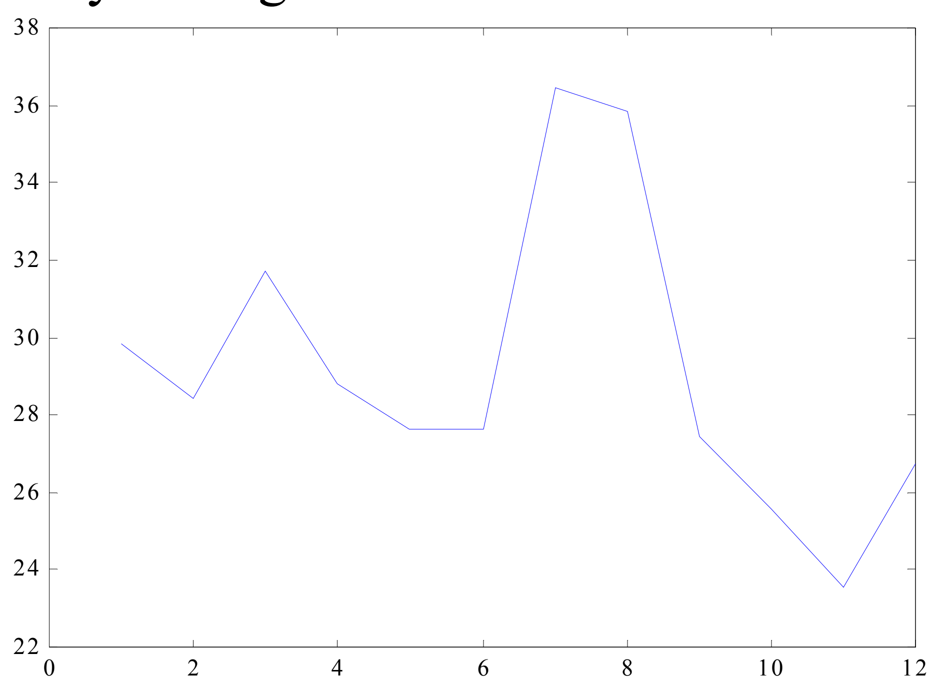
- More empirical observations: PJM log-price



- Monthly average



Literature on Energy Price Modeling and GARCH Modeling

- Energy commodity spot price models.
  - Schwartz (J.of Fin 1997)
  - Miltersen and Schwartz (JFQA 1998)
  - Hilliard and Reis (JFQA 1998)
- Electricity spot price models and electricity derivatives.
  - Kaminski (Risk Book 1997)
  - Barz and Johnson (1998)
  - Deng (PSERC 1998)
  - Mount and Ethier (PSERC 1998)
  - Deng, Sun and Meliopoulos (PSERC 2003)
- ARCH/GARCH modeling and option pricing.
  - Engle (Econometrica 1982), Bollerslev (J. Econ. 1986)
  - Nelson (Econometrica 1991)
  - Hall and Yao (Econometrica, to appear)
  - Duan (Math Fin. 1995), Heston and Nandi (RFS 2002)

Quantile-based GARCH Models
(Deng and Jiang, 2003)

- Exhibit heavy, flexible and asymmetric tail behaviors
- Enable maximum likelihood estimation (MLE)/quasi-MLE combined with quantile-based estimation
- Have explicit conditional quantile functions
- Allow efficient computation in pricing and risk management applications
  - Easy and fast simulation

Quantile GARCH Models

- Quantile GARCH(1,1)

$$X_t = \sigma_t \cdot \varepsilon_t$$

$$\sigma_t^2 = c + b \cdot X_{t-1}^2 + a \cdot \sigma_{t-1}^2 \quad \text{where}$$

$$\varepsilon_t \sim \text{Class I or Class II}$$

- Extension: GARCH type AR(1) process

$$X_t = X_{t-1} + \kappa(\theta - X_{t-1}) + \sigma_t \cdot \varepsilon_t$$

$$\sigma_t^2 = c + b \cdot X_{t-1}^2 + a \cdot \sigma_{t-1}^2 \quad \text{where}$$

$$\varepsilon_t \sim \text{Class I or Class II}$$

Quantile Class-I and II Distributions

Quantile Function and Quantile Modelling

- Definition: Suppose that $F(x)$ is a probability distribution function, then the quantile function of F is the generalized inverse

$$F^{-1}(x) = \inf\{y : F(y) \geq x\}$$

- Quantile modelling: Directly specify the quantile function of the distribution in which you are interested.

- The advatages of quantile modelling:
  - Fast sampling
  - Easy Q-Q plots
  - Easy probability calculation
  - Explicit Quantile function

Two classes of new distributions

The first Class $Q_I(\alpha, \beta, \delta, \mu)$ is defined by the following quantile function:

$$q(y; \alpha, \beta, \delta, \mu) = \delta^{\frac{1}{\alpha}} \Big\{ \log \frac{y^\beta}{1 - y^\beta} \Big\}^{(\frac{1}{\alpha})} + \mu$$

where $\delta, \alpha, \beta \in \mathbf{R}_+$, $\mu \in \mathbf{R}$, and the superscript '(α)' for $\alpha > 0$ represents the operation below.

$$x^{(\alpha)} = \begin{cases} x^\alpha & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -(-x)^\alpha & \text{if } x < 0 \end{cases}$$

- μ: location
- δ: scaling parameter
- α: tail thickness
- β: tail balancing factor

Properties of Class I Distributions

- Explicit form of probability distribution function

$$q^{-1}(x;\alpha,\beta,\delta,0) = \left\{\frac{1}{1+\mathrm{e}^{-\frac{1}{\delta}x^{(\alpha)}}}\right\}^{\frac{1}{\beta}}$$

- Explicit form of probability density function

$$p(x;\alpha,\beta,\delta,0) = \frac{\frac{1}{\delta\beta}\alpha.\frac{x^{(\alpha)}}{x}\mathrm{e}^{-\frac{1}{\delta}x^{(\alpha)}}}{(1+\mathrm{e}^{-\frac{1}{\delta}x^{(\alpha)}})^{1+\frac{1}{\beta}}},$$

- Potentially different tail behaviors at two sides
  - Right side: as $x \to +\infty$ the right tail is about $Cx^{\alpha-1}\mathrm{e}^{-\frac{1}{\delta}x^{\alpha}}$.
  - Left side: as $x \to -\infty$ the left tail is about $C(-x)^{\alpha-1}\mathrm{e}^{-\frac{1}{\delta\beta}(-x)^{\alpha}}$.

The Second Class: $Q_{II}(\alpha_-, \alpha_+, \delta_-, \delta_+, \mu)$

The quantile function of the second Class is

$$q(y; \alpha_-, \alpha_+, \delta_-, \delta_+, \mu)$$
$$= -\delta_-^{\frac{1}{\alpha_-}} (\log \frac{1}{y})^{\frac{1}{\alpha_-}} + \delta_+^{\frac{1}{\alpha_+}} (\log \frac{1}{1-y})^{\frac{1}{\alpha_+}} + \mu$$

where $\alpha_-, \alpha_+, \delta_-, \delta_+ \in \mathbf{R}_+, \mu \in \mathbf{R}$. We are mostly interested in the cases $\alpha_- \leq 1, \alpha_+ \leq 1$.

- μ: location
- δ_+ / δ_-: scaling parameters at the right / left hand side.
- α_+ / α_-: tail thickness parameters at the right / left hand side.

Remark: α_- and α_+ provide the flexibility for this class of distributions to have different tail thickness at the two sides.

Figure 1: Density plot of the first class

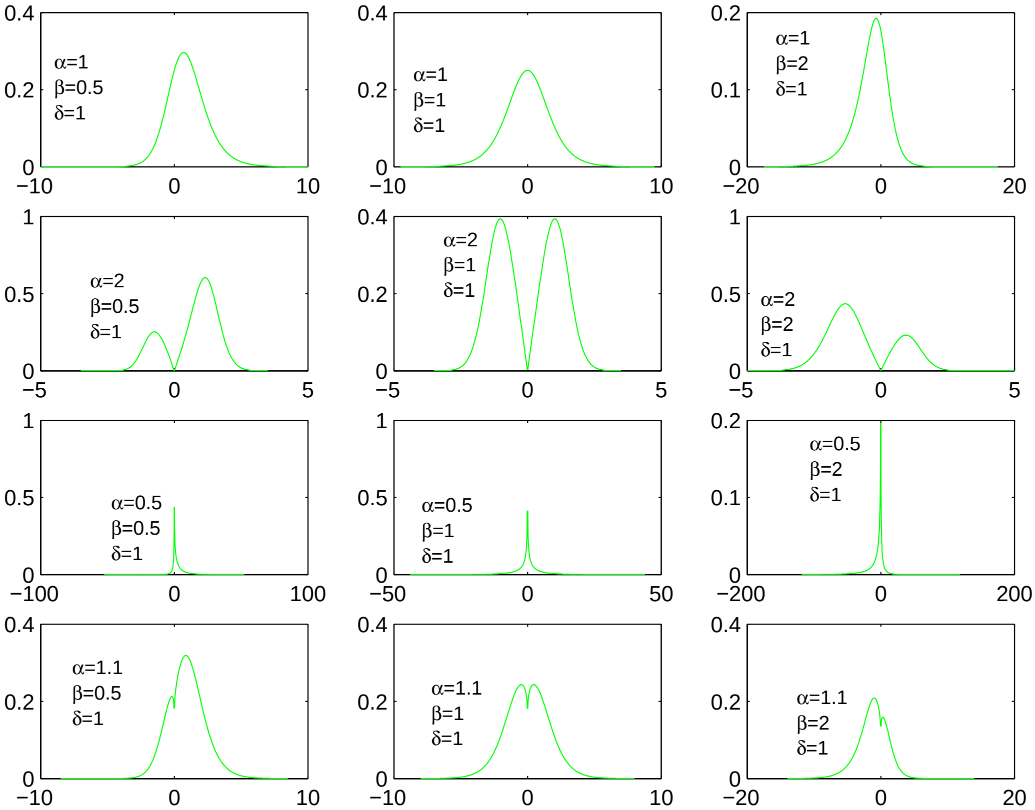
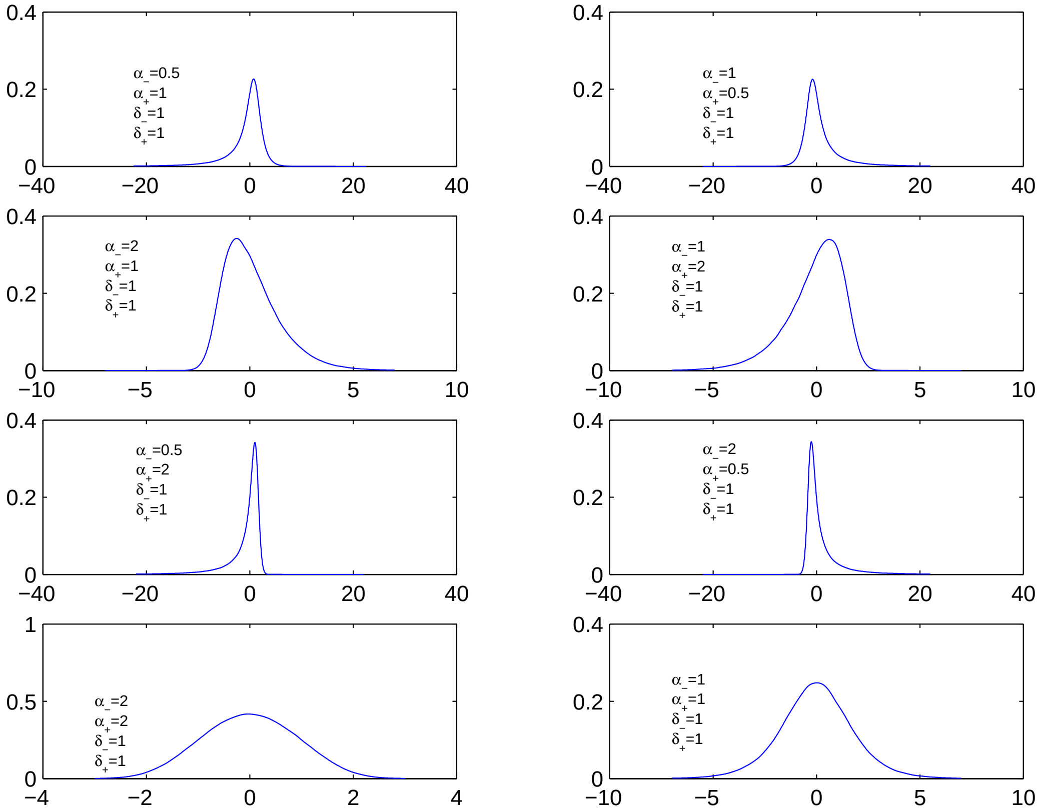
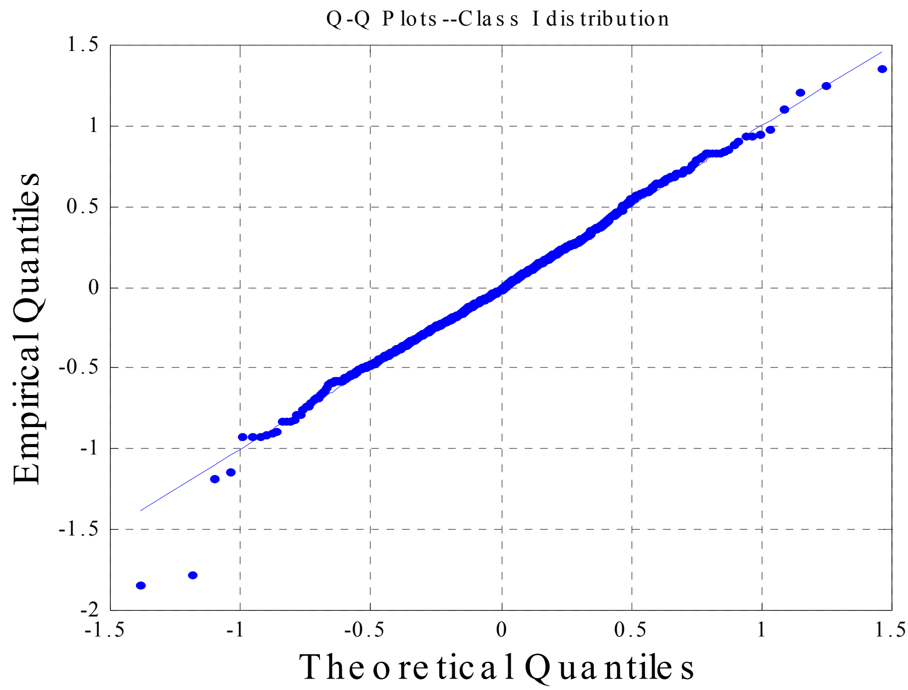


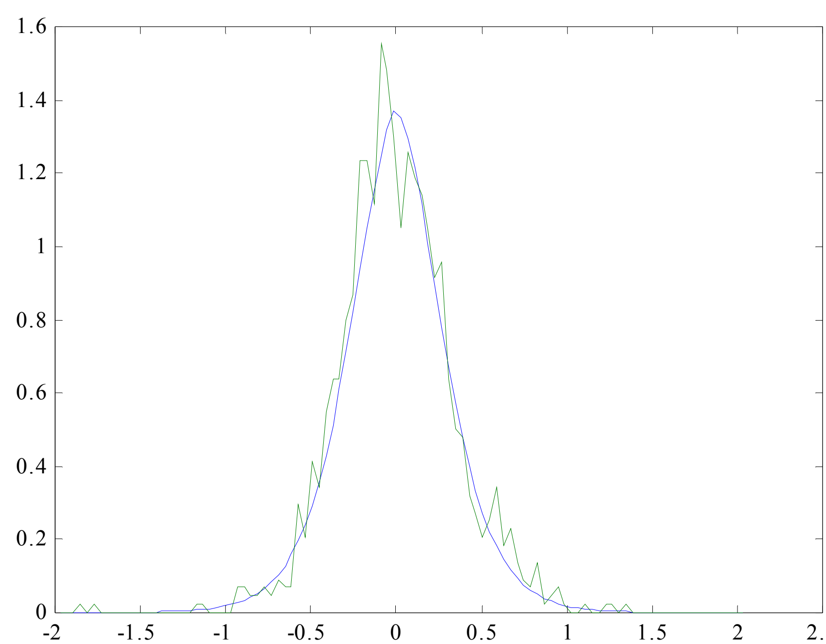
Figure 2: Density plot of the second class



Class-I Fit of PJM Daily Price Return

α=0.99245
β=0.92348
δ=0.19064
μ=-0.027395





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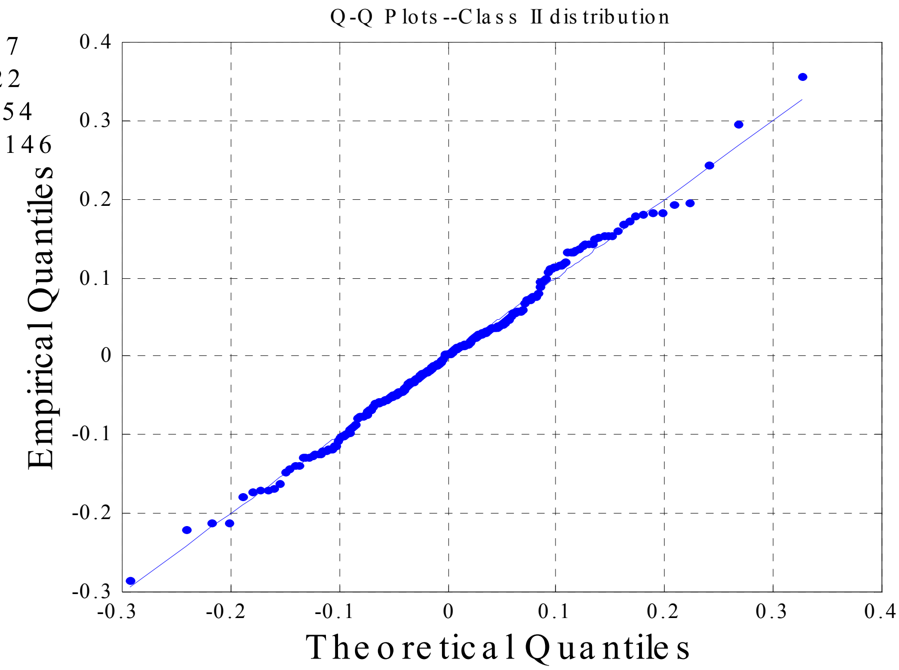
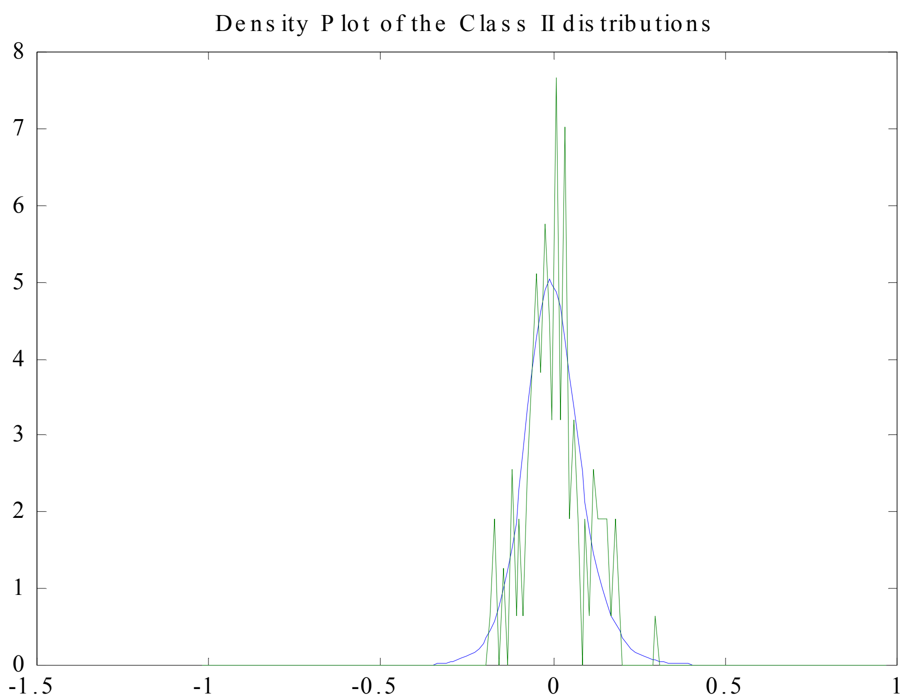
Class-II Fit of PJM Daily Load

α-left=0.91657
α-right=1.0422
δ-left=0.048554
δ-right=0.053146
μ=-0.020806



α-left=0.91657
α-right=1.0422
δ-left=0.048554
δ-right=0.053146
μ=-0.020806



A Two-step Estimation Scheme

- Step 1: Quasi-MLE for estimating GARCH coefficients
  - Hall and Yao (Econometrica, to appear)

$$L_\nu(a,b,c) = \sum_{t=\nu}^{n} \{ \frac{X_t^2}{\tilde{\sigma}_t^2(a,b,c)} + \log \tilde{\sigma}_t^2(a,b,c) \},$$

- Step 2: Quantile-based estimation for obtaining parameters for the innovation term.

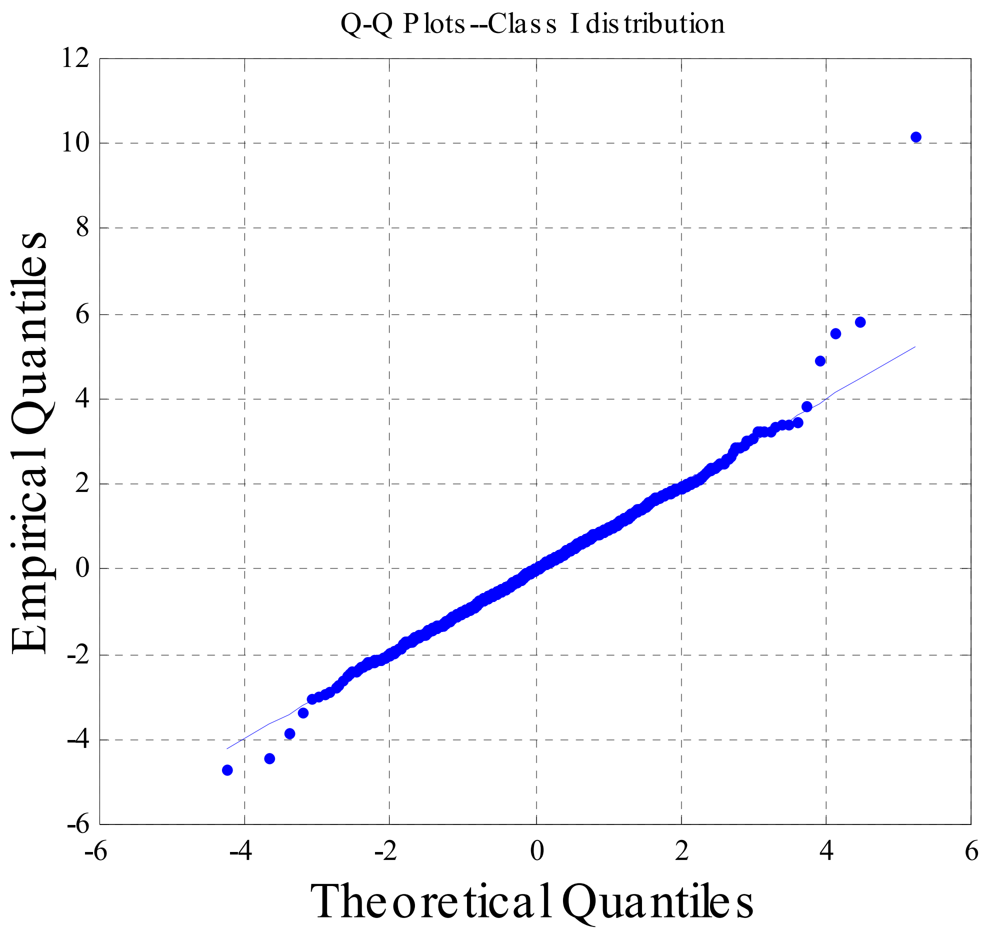
Empirical Estimation

- MLE estimation of GARCH(1,1) coefficients:

  $\alpha0= 0.0023$; $\alpha1= 0.0587$; $\beta= 0.9252$.

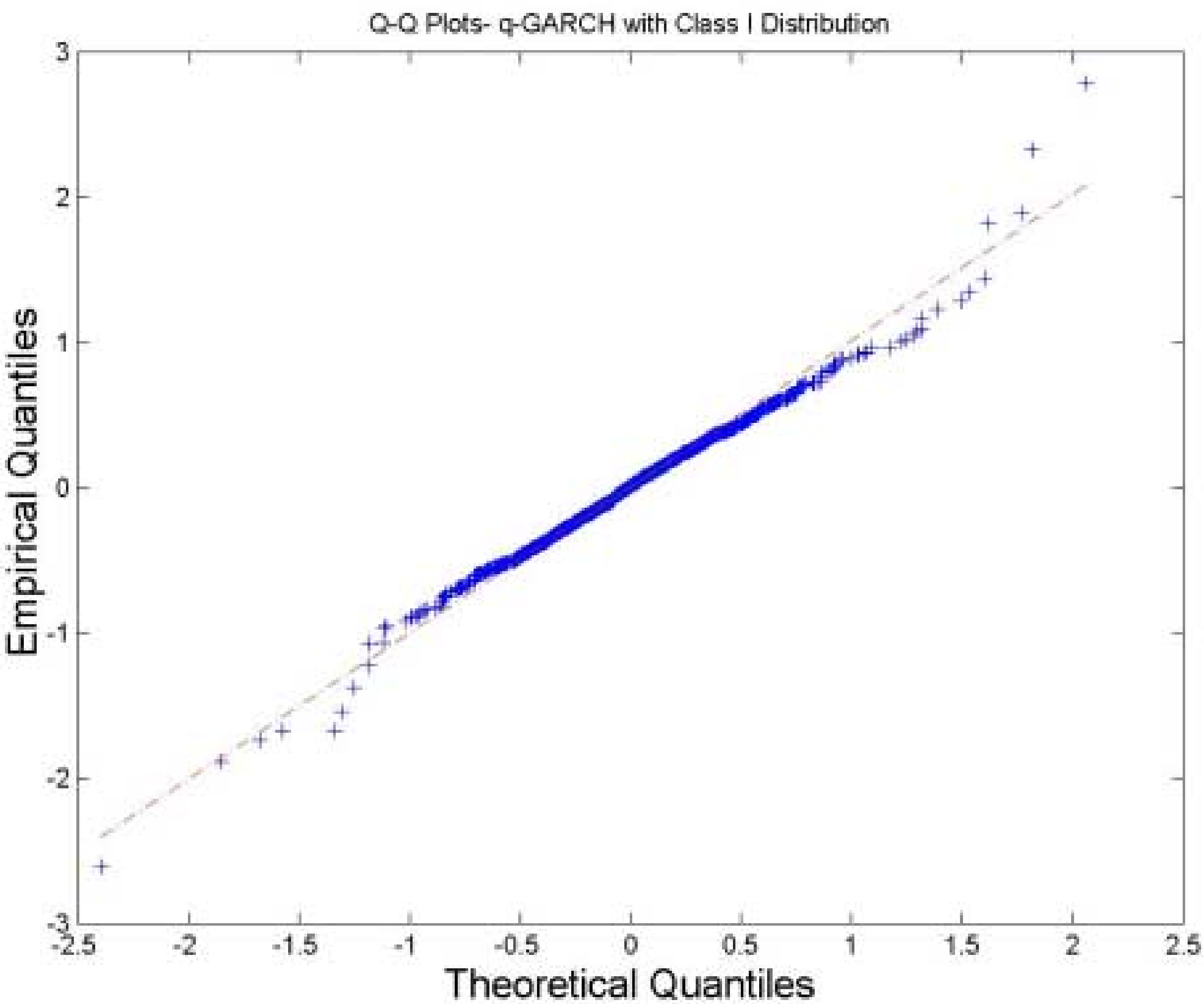
- Q-Q Plot of the innovation term.
  - Daily electricity price: PJM Western Hub

α=0.92191
β=0.77793
δ=0.56062
μ=-0.21724



Empirical Estimation (con't)

- Q-Q Plot of unconditional marginal distrib.



Application: Value Energy Contracts

- Energy (electricity) derivatives are complex financial instrument
  - Physical characteristics of underlying
  - Path-dependent and American-style (exercisable at any time)
- Examples:
  - Tolling agreements.
    - Independent power producers hedge operational risks.
    - Power merchants implement asset-light operations.
    - Fossil-fueled power producers hedge output risks.
  - Swing contracts
  - Gas storage contracts

Application: Pricing Methodology

- European-style financial contracts
  - Closed-form: Heston and Nandi (2002)
  - Simulation
- American-style path-dependent financial contracts
  - Dynamic programming least-squares approximation:

    Longstaff and Schwartz (2001), Tsitsiklis and Van Roy (2001)
  - Simulation

Interval Estimation for the Conditional Quantile of Fat-tailed GARCH Models
(Chan, Deng, Peng, and Xia, 2003)

- One-step conditional quantile estimation of heavy-tailed GARCH models
- Characterization of confidence intervals for conditional quantiles

Model Specification

- Heavy-tailed GARCH Model

$$X_t = \sigma_t \epsilon_t, \sigma_t^2 = c + \sum_{i=1}^{p} b_i X_{t-i}^2 + \sum_{j=1}^{q} a_j \sigma_{t-j}^2,$$

- Heavy-tail in innovation

$$\varepsilon_t \sim G(x) \quad (\text{cdf of } \varepsilon_t)$$

$$1 - G(x) \sim c_1 x^{-\gamma}, \quad G(-x) \sim c_2 x^{-\gamma}, \quad \text{for} x \text{ large} \& \quad \gamma > 2$$

- 100α% one step ahead conditional VaR

$$x_{\alpha,n} = \inf\{x : P(X_{n+1} \le x | X_{n+1-k}, k \ge 1) \ge \alpha\}.$$

Estimation

- Likelihood function (quasi-MLE)

$$L_\nu(a,b,c) = \sum_{t=\nu}^{n} \{ \frac{X_t^2}{\tilde{\sigma}_t^2(a,b,c)} + \log \tilde{\sigma}_t^2(a,b,c) \},$$

- Tail index

$$\hat{\gamma} = \{ \frac{1}{k} \sum_{i=1}^{k} \log \frac{\hat{\epsilon}_{m,m-i+1}}{\hat{\epsilon}_{m,m-k}} \}^{-1},$$

  - where $\hat{\epsilon}_t = X_t / \tilde{\sigma}_t(\hat{a}, \hat{b}, \hat{c})$ and $\hat{\epsilon}_{m,1} \leq \cdots \leq \hat{\epsilon}_{m,m}$ denote the order statistics of $\hat{\epsilon}_\nu, \cdots, \hat{\epsilon}_n$

- Estimator by method I:

$$\hat{x}_\alpha^0 = (1-\alpha)^{-1/\hat{\gamma}} (\frac{k}{m})^{1/\hat{\gamma}} \hat{\epsilon}_{m,m-k},$$

$$\hat{x}_{\alpha,n} = \tilde{\sigma}_{n+1}(\hat{a}, \hat{b}, \hat{c}) \hat{x}_\alpha^0$$

Theorem 1

- Suppose regularity conditions hold and

$$k = k(m) \to \infty, k/m \to 0, \sqrt{k}A(m/k) \to 0,$$

$$n^{-1}\lambda_n/A(m/k) \to 0, \log(\frac{k}{m(1-\alpha)})/\sqrt{k} \to 0$$

as n→∞. Then

$$\frac{\hat{\gamma}\sqrt{k}}{|\log(\frac{k}{m(1-\alpha)})|}\{\frac{\hat{x}_{\alpha,n}}{x_{\alpha,n}} - 1\} \xrightarrow{d} N(0,1).$$

Confidence Intervals

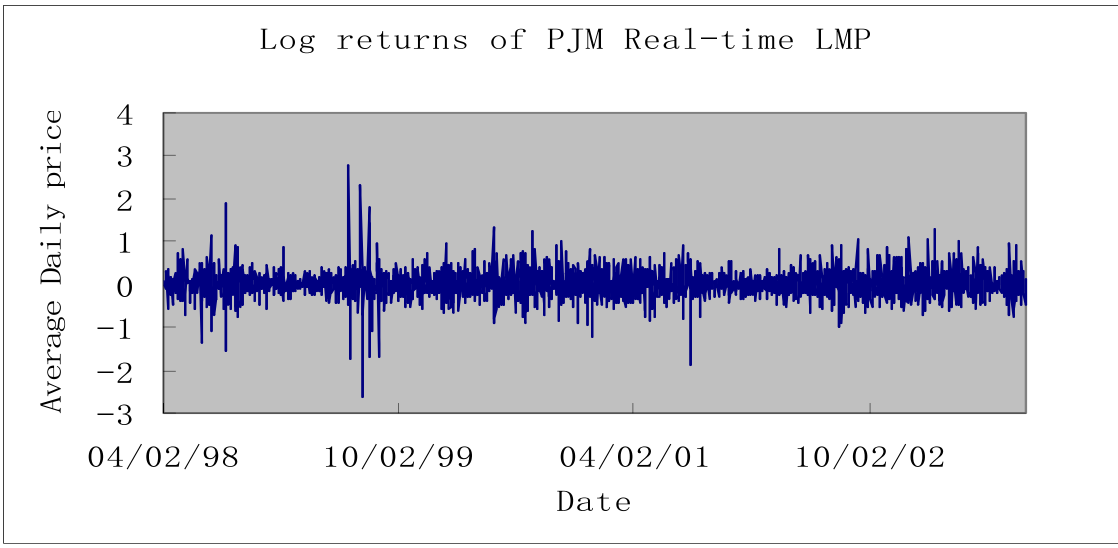
- Method I: Normal approximation method.
  - Based on Theorem 1, a confidence interval with level β for $x_{\alpha,n}$ is

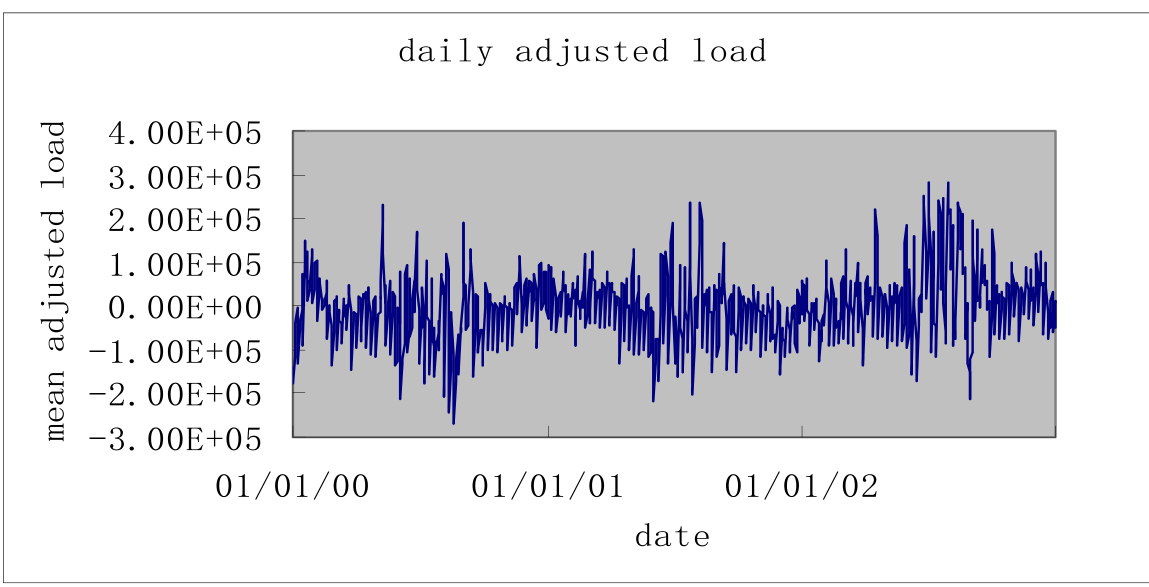
$$I_\beta^n = \left(\hat{x}_{\alpha,n}\left(1 + \frac{z_\beta}{\hat{\gamma}\sqrt{k}}\left|\log\frac{k}{m(1-\alpha)}\right|\right)^{-1}, \quad \hat{x}_{\alpha,n}\left(1 - \frac{z_\beta}{\hat{\gamma}\sqrt{k}}\left|\log\frac{k}{m(1-\alpha)}\right|\right)^{-1}\right),$$

with z_β satisfies $P(|N(0,1)| \le z_\beta) = \beta.$

Data Sets and Their Autocorrelation

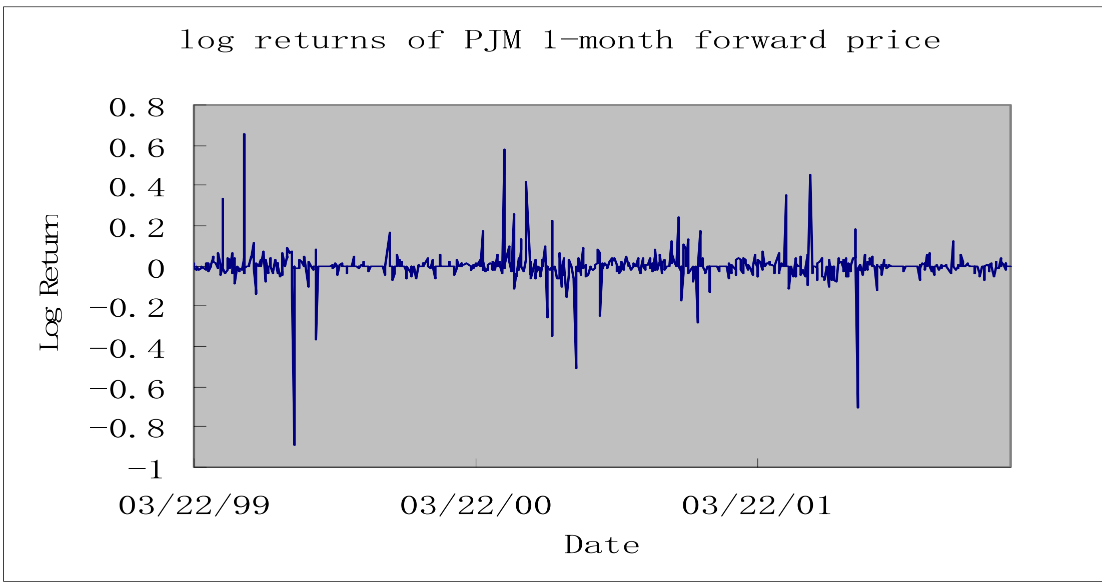
- Daily electricity price return and load change in PJM

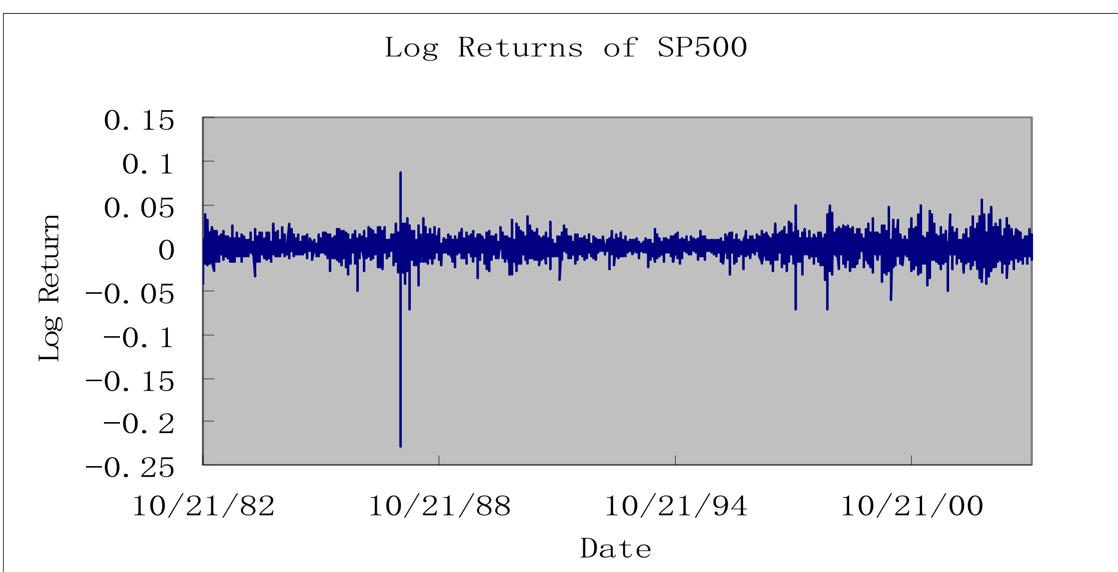




Data Sets and Their Autocorrelation

- 1-Month PJM forward price vs. SP 500





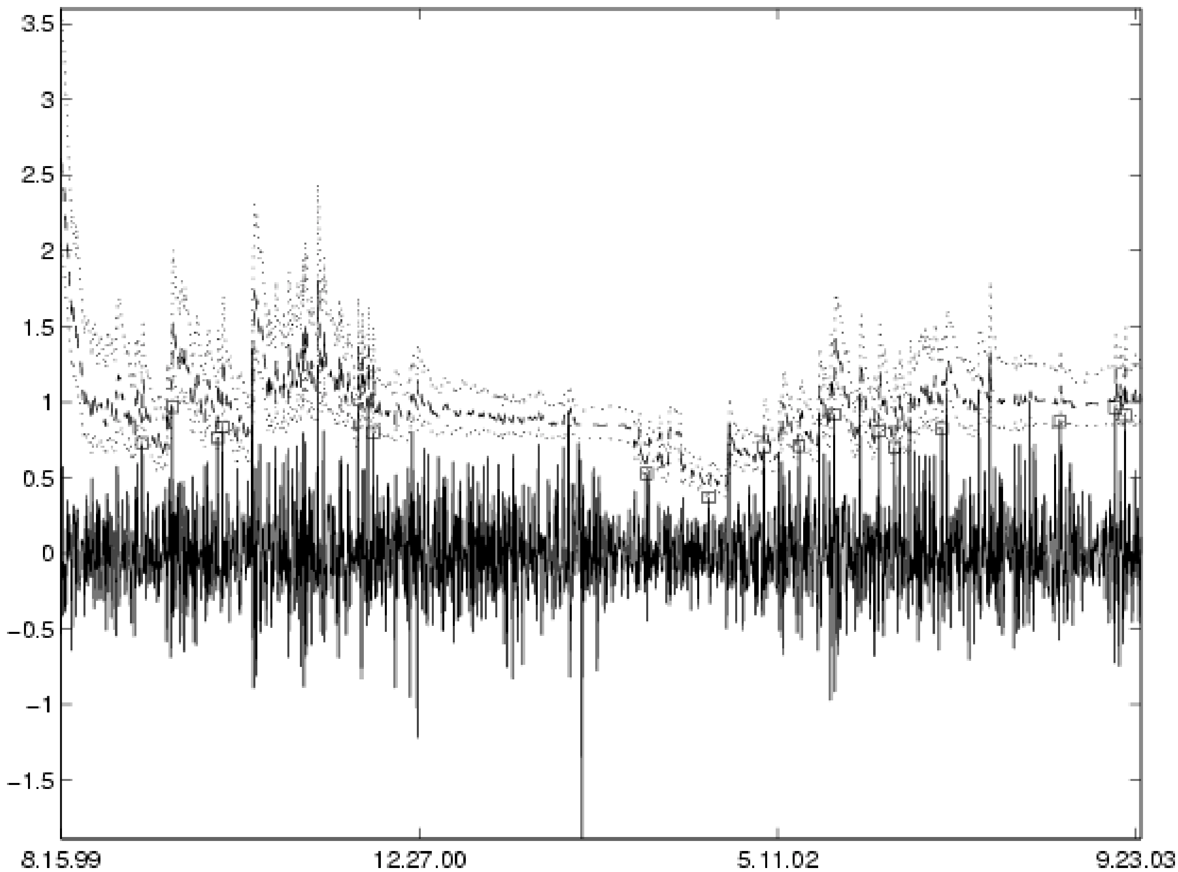
Comparison with Gaussian GARCH

k=30, alpha=0.99	Method I	Conditonal Normal
daily load	0.030	0.024
1-m PJM forward	0.009	0.014
3-m PJM forward	0.005	0.005
PJM real time LMP	0.018	0.030
SP500	0.014	0.017

k=60, alpha=0.99	Method I	Conditonal Normal
daily load	0.024	0.024
1-m PJM forward	0.009	0.014
3-m PJM forward	0.005	0.005
PJM real time LMP	0.016	0.030
SP500	0.014	0.017

k=100, alpha=0.99	Method I	Conditonal Normal
daily load	0.017	0.024
1-m PJM forward	0.005	0.014
3-m PJM forward	0.000	0.005
PJM real time LMP	0.016	0.030
SP500	0.015	0.017

Confidence Band of CVaR



Conclusion

- Methodology
  - Non-Gaussian fat-tailed GARCH type models based on quantile distributions
  - A two-step procedure for parameter inference: quasi-MLE and quantile-based estimation
  - A one-step semi-nonparametric estimation scheme of high/low quantiles and their confidence intervals
- Applications
  - Modeling financial time series data as well as energy (e.g., electricity price and load) data
  - Financial derivatives/contracts pricing
  - Risk management measures (e.g., CVaR)

Future Work

- More on Quantile-based GARCH models.
  - Parameter inference
  - Multivariate extensions
- Risk-neutralized processes corresponding to the quantile-based GARCH models.
- Efficient simulation and dynamic programming algorithms for asset pricing problems.