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Optimal Bidding Strategy in Electricity Markets Under Uncertain Energy and Reserve Prices

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Executive Summary

This report describes the mathematics of finding optimal bidding strategies in multi-period electricity market auctions of energy and reserve markets, taking full account of generator costs and operating constraints, and exogenous price uncertainty. The operational constraints of the generator include such items such as minimum up and down times, minimum power production requirements, maximum ramp times. The report addresses five specific concerns: bidding into multiple markets, the impact of market design rules (particularly the requirement that bids be non-decreasing with increasing amounts offered), the non-convexity of cost curves (often the results of start-up and shut-down costs), the effect of inter-temporal constraints, and the effect of price uncertainty. The analysis of this problem requires the use of nested Dynamic Programming (DP) techniques; in particular, we show how to use the innermost-nested DP to find a bidding function that satisfies market design rules. This type of analysis provides new insights into anticipated market behavior in auctions.

The methods offered in the report are practical to implement and have minimal data needs. The methods are applicable to price-taking generators as well as to generators with the ability to influence price.

The bidding strategy methods are illustrated with numerous examples. One of the most revealing observations of this report is that bidding situations with only minor difference in the bidding rules can give rise to significantly different optimal bidding strategies and expected profits. This makes assessment of observed bidding behavior more challenging. For example, we show that requiring bids to have non-decreasing offer prices can produce very different bidding strategies from a market design without this requirement.

Knowledge of optimal bidding strategies would help both generators and regulators. Generators can improve profits using such strategies. A regulator can use these optimal bidding strategies to develop tests that check if a generator behaved like a price-taker or exercised market power.

The implications of this report on market design are important. Fundamentally, a market that appears on the surface to be simple can lead to the need for complex analysis on the part of the generators and the bidders in order to attain profitability as a result of the numerous concerns pertaining to costs and operational restrictions that must be faced in real life. It will also make the job of a regulator or auditor harder because more factors must be taken into consideration when assessing market power issues. Therefore, it is essential that the both bidding strategies and auditing and monitoring procedures be sophisticated enough to capture all constraints and uncertainties. This report provides the basis for such analysis.

Contents

1 Introduction

In deregulated electricity markets, generators must either submit profit-maximizing bids to a pool or optimally self-schedule in response to prices. In some market designs bids are simple bids (offers) to sell a certain amount of energy or some other service the unit is able to provide at a given price or better. In other market designs, bids may include other attributes that constitute the cost at which the unit is willing to provide a service, such as start-up or shut-down costs, energy constraints and more. In some other market designs, a generator is permitted to "self-schedule."

Regardless of market design, the generator's bidding (or self-scheduling) problem is complicated by several factors, in particular, the presence of multiple markets, market design rules, non-convexity of cost curves, inter-temporal constraints, and price uncertainty. We discuss each of these factors below.

1. **Multiple markets**: A generator typically has a choice of multiple markets into which it can sell its capacity. For example, a generator can allocate its output to the energy market, or it can make its output available to one of the operating reserves markets. In addition to deciding which market to allocate their output, energy-limited hydroelectric generators must also decide whether to allocate their output *now* or in *later* periods.

2. **Market design rules**: The design of electricity market auctions significantly affects the bidding strategy of the generator. For example, in "uniform price" auctions, every winning bidder gets the same market clearing price, while in "pay-as-bid" auctions, every winning bidder gets paid its winning bid. In uniform price auctions, price-taking generators tend to bid their incremental costs, while in pay-as-bid auctions, generators tend to bid close to their expectation of the market price. Moreover, markets where energy and reserve markets are *simultaneously* cleared require different strategies than markets where energy and reserve services are *sequentially* cleared. The effect of market design rules on generator behavior has been examined in detail by Harvey and Hogan (Harvey and Hogan 2001).

3. **Non-convexity of cost curves**: Non-convexity arises mainly from start-up and shut-down costs, increasing generator efficiencies at higher loading levels, "valve point" effects and other such effects. Non-convex costs complicate generator bidding behavior because generators are typically required to use non-decreasing bid curves (as a function of MW offered) in electricity auctions.

4. **Inter-temporal constraints**: Inter-temporal constraints (such as start-up and shut-down times, total energy limits, ramp rate limits, etc.) also complicate bidding strategies.

5. **Price uncertainty**: At the time a decision to commit or dispatch is made, a generator typically faces price uncertainty in future periods. Because of inter-temporal constraints, future price uncertainty affects the generator's bids in the present period.

There is a distinction between *optimal scheduling* strategies and *optimal bidding* strategies. Under some conditions, the two are essentially equivalent. However, under other conditions they can be quite different, as this report demonstrates. The problem of finding an optimal scheduling strategy is essentially equivalent to the traditional unit commitment problem. The problem of finding the optimal bidding strategy is related to but not identical to the unit commitment problem.

Unit commitment refers to the problem of deciding when to start and when to shut-down generators in anticipation of changing demand. In traditional utility systems, the problem of unit commitment was formulated and solved as a multi-period optimization problem. In the traditional problem formulation, the anticipated demand was an input variable. The problem was solved for multiple generators, generally owned by the same entity (a utility). The start-up, shut-down and operating costs of the generators were assumed known. There has been extensive literature on this topic over the last thirty years. See, for example, (Wood and Wollenberg 1996). A brief review of the topic is given below.

Prior to the 1980's, the unit-commitment problem was solved using Dynamic Programming (DP) (Bertsekas 1987). Bertsekas and others in the 1980's (Bertsekas, Lauer, Jr., and Posbergh 1983, Merlin and Sandrin 1983) introduced the concept of Lagrangian relaxation to attack the problem. In Lagrangian relaxation, the system-wide energy and reserve requirements constraints are "relaxed" by introducing *prices*. An economic interpretation is that, in Lagrangian relaxation, the unit commitment problem is solved using an iterative, 2-step dual formulation. In one step of the dual solution, each generator independently maximizes its profits over a number of periods taking the prices for energy and reserves in each period as *given*, while in the second step, these *prices* are varied to coax the generators to meet the required system demand for each period. At the optimal solution, each generator maximizes its profits, while the price in each period is such that the total output offered by all generators satisfies the system demand and reserve requirements in the period. The profit maximization problem for the generators in the dual formulation can be generally solved using backward DP techniques (Bertsekas 1987).

Other methods of unit commitment proposed in recent years include the method of decommitment (Li, Johnson, and Svoboda 1997), (Tseng, Oren, Svoboda, and Johnson 1997). Hobbs (B. F. Hobbs et al. 2001) and the references contained therein give the state-of-the-art on unit commitment methods.

The problem of finding optimal bids for generators has interested several researchers. See for example (Gross and Finlay 1996). Pereira and his co-workers (Kelman and Pereira 1998), (Pereira, Barroso, and Kelman 2002) addressed methods dealing with bidding strategies in hydro-dominant regions using Monte-Carlo methods, DP and heuristics to arrive at Cournot-Nash solutions. Guan et. al (Guan, Ni, Luh, and Ho 2001) use a bid selection method based on ordinal optimization to obtain "good enough" bidding strategies. Most of these papers generally try to solve the bidding strategy problem for *all* generators in the system, with each generator having to take full account of other generators' bidding strategies. Consequently, the problems that are solved are highly complex because the optimization is carried out over many different variables and over a range of generator bidding behaviors. The intensive computational burden in such problems is alleviated through approximations or heuristics. Moreover, prior work on the unit commitment problem often neglects the added complexity of locational variation of prices.

In our opinion, while recent papers have contributed significantly to the literature and have introduced ingenious computational methods, the bidding problem that they pose (and attempt to solve) is unnecessarily complex. In particular, trying to model the bidding behavior of every generator in the system is heroic. As a practical matter, even if we assumed that every generator was a *price-taker*, the problem of finding a generator bidding strategy is difficult because a market participant does not have accurate enough data on competitors' generators. Moreover, the effect of market design rules on generator behavior can be quite complex (Harvey and Hogan 2001). This is because, even under the price-taker assumption, a generator's bidding behavior could depend on its forecast of market-clearing prices, and generator price forecasts may vary (Rajaraman and Alvarado 2002a), (Rajaraman and Alvarado 2002b).

This report presents the mathematics for finding the multi-period optimal bidding strategy for a generator in electricity markets under *exogenous* uncertain energy and reserve prices; the report represents an improvement over previous work by the authors (Rajaraman, Kirsch, Alvarado, and Clark 2001). In an electricity market, one can readily observe prices by location. The historical archive of these prices (that vary by location) captures the past bidding behavior of all the generators in the system under different system conditions and load levels. The formulation can also be used to find profit-maximizing bidding strategies for generators that exercise market power (e.g., by withholding output). We do this by using "residual demand" curves (Samuelson and Nordhaus 1998); that is, we let the exogenous prices be a function of the amount of generator output. Using price (or residual demand curves) as exogenous inputs over modeling the behavior of many generators in the system is numerically more tractable (fewer variables to model) and leads to greater insights into the structure of the strategy-selection problem.

Our report also pays special attention to how market design rules, price uncertainty, and non-convexity of costs affect "allowed" bidding strategies; this is a neglected feature in most work on bidding strategies. Example 4.1 shows that even if a generator has convex costs, the effect of price uncertainty is such that the optimal bidding strategy (if there were no restrictions on the bidding function) is to bid a declining curve of MW versus price. However, since electricity market auctions typically disallow declining price bids (usually the MWs offered must be a non-decreasing function of price), the optimal bidding strategy must be changed to satisfy this constraint. The optimal bidding strategy in this case involves solving multi-level, nested backward DP problems.

Knowledge of optimal bidding strategies (under both the price-taking as well as price-influencing conditions) has at least a couple of uses. The most obvious one is that generators would improve profits using such strategies.

A less obvious use of knowing the optimal bidding strategy is that a regulator can judge if a generator behaved like a price-taker or exercised market power. Tests based on optimal bidding strategies help determine whether market power was exercised (Rajaraman and Alvarado 2002a, Rajaraman and Alvarado 2002b). The tests check whether the behavior of each generator in a market participant's portfolio is indeed the behavior one would expect to observe if the generator were a price-taker given the market design rules, multiple markets, non-convex operational constraints, non-convex cost structure, etc., in the presence of forecast uncertainty.

The basic idea behind the tests is as follows (Rajaraman and Alvarado 2002b). If a generator withholds infra-marginal output in some periods, then in order to "explain" the withheld infra-marginal output, the generator has to claim that (a) the expected future prices at the time the decision to withhold was made were lower than what actually materialized, and (b) at the price levels that were expected by the generator, the withheld output would be “out-of-the-money” (and not infra-marginal). If the generator were exercising market power by consistently withholding infra-marginal output during (what everyone expected to be) high price periods, then the generator’s claim that forecast prices were consistently and significantly lower than the actually realized prices would not be credible. On the other hand, if price spikes in some periods were due to unforeseen events, then the generator’s behavior of withholding what only in hindsight turned out to be infra-marginal generation could be explained by the lack of perfect foresight of prices at the time the decision to withhold was made. For the case of hydroelectric generators, the nature of inter-temporal constraints and the possibility of storage complicate dispatch and state transition decisions, because lack of foresight of prices in later periods could have an impact on dispatch now.

The report is organized as follows. Section 2 gives an annotated glossary of the symbols used for the convenience of the reader. Section 3 presents the mathematics of the optimal bidding strategy problem and solves it using multiple-level nested backward Dynamic Programming (DP) approaches. Section 4 illustrates our methods with a number of different examples. Section 5 briefly discusses how the methods can be applied to real electricity auctions. Section 6 shows how one may use the ideas in Section 3 to derive the optimal bidding strategy for a generator that is not a price-taker; i.e., the generator is capable of setting prices. We use the concept of residual demand curves to derive the bidding strategy. Section 7 briefly discusses computational issues. Section 8 concludes the report. The appendix gives a brief overview of DP methods.

2 Assumptions and Annotated Glossary

2.1 Assumptions

1. A single generator is does not directly constrain the output of other generators. For example, a hydroelectric system with three hydro units cascaded in series would be considered a single generator.

2. The generator is a *price-taker*; i.e., it takes prices as a given and has no influence on market prices. In Section 6, we relax this assumption by assuming that the generator has some influence on prices; the generator maximizes its profits given the exogenous, uncertain residual demand curve facing the generator.

2.2 Glossary

K There are K (sequential) uniform-price auctions, or more explicitly, auction rounds. Each auction round can correspond to a time period (e.g., 24 auctions could be for a 24-hour day-ahead period). However, an auction round can also correspond to a particular market within a time period. For example, one auction round can be used to acquire ancillary services, while another auction round can be used to secure energy, both for the same time period. In the terminology of the appendix, an auction, an auction round or a time period all are associated with the notion of "stage" in a sequential optimization process. In what follows, we will use the terms *auction*, *auction round*, *time period* and *stage* somewhat interchangeably, with a preference for using *auction round*. The concept of stage, however, is a bit more general, as the appendix illustrates, since there are other notions that may be associated with stages.

$\mathbf{p}_k$ For auction round k, the *exogenous* energy and reserve prices are represented by a price vector $\mathbf{p}_k$. If the generator can provide only one type of service (e.g., energy), then $\mathbf{p}_k$ is a scalar. The prices from any one period can be correlated to the prices for a subsequent period following some Markov random process. We assume that the probability $\mathbf{prob}(\mathbf{p}_{k+1}|\mathbf{p}_k)$ is known for all k. Figure 1 illustrates our exogenous price model.

$\mathbf{x}_k$ The generator state for auction round k. For a thermal generator, generator states could include at a minimum UP and DOWN, but there can also be many other states to indicate generator readiness or the amount of fuel remaining. For example, a generator with a minimum two-period up-time requirement can be represented by having two up-states, and only the second up-state has a permissible transition to the DOWN state. For a hydro generator, a generator state for auction round k could be the reservoir level for auction round k. The states can also indicate the fuel (or water) available for future use, the capability of a generator to deliver reserves, or any other parameter necessary to completely characterize the generator during any time period. The state $\mathbf{x}_k$ can be a set, a scalar, a vector, a set of vectors, or almost anything else.

$\mathcal{X}_k$ The set of *allowed* states of the generator during auction round k. In other words, $\mathbf{x}_k \in \mathcal{X}_k$. For example, a generator may be required to be online for some particular auction round k. As another example, the generator may need to be taken down for maintenance at some time.

$\mathbf{y}_k$ The generator dispatch quantity during auction round k. For an energy-only model, this is a scalar representing energy. In general, a generator may dispatch any quantity of energy and various kinds of reserves and other possible services during any period, subject to some validity rules. Although the dispatch vector components can in many cases be continuous variables, we often discretize the number of permissible dispatch choices. The generator dispatch $\mathbf{y}_k$ is constrained to lie in some subset that could be a function of the state $\mathbf{x}_k$. The generator dispatch is our main decision variable. It affects the state transition. Or rather, it is the means by which one can achieve a desired optimal state transition $\mathbf{x}^*_{k+1}$.

$\mathcal{Y}_k$ The set of permissible dispatch choices for $\mathbf{y}_k$. This set may in turn be a function of the state $\mathbf{x}_k$. In other words, $\mathbf{y}_k \in \mathcal{Y}_k(\mathbf{x}_k)$. In some cases, we also restrict $\mathbf{y}_k$ to be a "biddable" function, i.e., $\mathbf{y}_k$ must be bid into an auction, and therefore must satisfy certain rules of the auction. For example, it is a typical requirement of electricity market auctions that the amount of MW offered must be an increasing function of the bid price. Therefore, we assume that $\mathbf{y}_k$ also has to satisfy certain constraints based on price (in addition to the state $\mathbf{x}_k$), i.e., $\mathbf{y}_k \in \mathcal{Y}_k(\mathbf{x}_k, \mathbf{p}_k)$.

$\mathcal{T}_k$ State transition rules specify allowed transitions from $\mathbf{x}_k$ (which depend on the dispatch $\mathbf{y}_k$) to $\mathbf{x}_{k+1}$, i.e., $\mathbf{x}_{k+1} \in \mathcal{T}_k(\mathbf{x}_k, \mathbf{y}_k)$.

c_k The known transition cost associated with moving from generator state $\mathbf{x}_k$ in auction round k to $\mathbf{x}_{k+1}$ in auction round $k+1$. It is generally a function $c_k(\mathbf{x}_k, \mathbf{x}_{k+1})$.

R_k The auction round k revenues that accrue from dispatching $\mathbf{y}_k$ in state $\mathbf{x}_k$ when the price is $\mathbf{p}_k$. This includes the revenues from reserves as well as from energy sales. In many practical cases, these revenues can be expected to be independent of the state and be a function of only the price and the dispatch.

C_k The costs (or profits) from dispatching $\mathbf{y}_k$ in state $\mathbf{x}_k$ when the price is $\mathbf{p}_k$. In general it is a function $C_k(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}_k)$. (As a practical matter, this cost function is often independent of the price $\mathbf{p}_k$.)

$*$ The asterisk $*$ is associated with optimal solutions.

$\mathbf{y}^*_k(\mathbf{x}_k, \mathbf{p}_k)$ The optimal dispatch strategy in auction round k as a function of state $\mathbf{x}_k$ and price $\mathbf{p}_k$.

$\mathbf{x}^*_{k+1}(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}_k)$ The optimal state transition strategy given state $\mathbf{x}_k$, price $\mathbf{p}_k$ and dispatch $\mathbf{y}_k$ in auction round k. That is, the next state in auction round $k+1$, $\mathbf{x}_{k+1} = \mathbf{x}^*_{k+1}(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}_k)$.

$\mathbf{x}^*_{k+1}(\mathbf{x}_k, \mathbf{p}_k)$ The optimal state transition strategy[1], given state $\mathbf{x}_k$ and price $\mathbf{p}_k$ corresponding to the optimal dispatch in auction round k, $\mathbf{y}_k = \mathbf{y}^*_k(\mathbf{x}_k, \mathbf{p}_k)$. That is, $\mathbf{x}^*_{k+1}(\mathbf{x}_k, \mathbf{p}_k) \equiv \mathbf{x}^*_{k+1}(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}^*_k(\mathbf{x}_k, \mathbf{p}_k))$.

[1] This represents a technical abuse of notation; however, this definition is intuitive and we use it for ease of exposition.

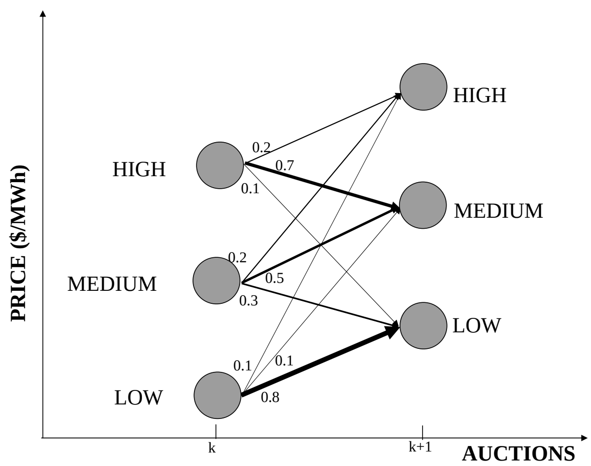

Figure 1: Price states and transition probability from one auction round to the next. Each auction round has three price states: LOW, MEDIUM and HIGH. The numbers next to the transition arrows show the probability of transition $\mathbf{prob}(\mathbf{p}_{k+1}|\mathbf{p}_k)$. The thickness of the arrows is proportional to the transition probability. For example, the probability that the price will be LOW in auction round $k+1$ given that it is HIGH in auction round k is 0.1. The actual LOW, MEDIUM and HIGH values in auction round k will in general be different from the corresponding ones in auction round $k+1$.

$\mathbf{J}_k^*(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}_k|\mathbf{x}_{k+1})$ The optimal cumulative profits from auction round k onward assuming that the state reached in auction round $k+1$ is $\mathbf{x}_{k+1}$. This is equal to $R_k - C_k$ minus the cost c_k of transitioning to the next state $\mathbf{x}_{k+1}$ in auction round $k+1$ from state $\mathbf{x}_k$ in the current auction round k, plus the expected cumulative profits from auction round $k+1$ onwards, assuming the optimal state transition and dispatch strategy is followed from auction round $k+1$ onwards.

$\mathbf{J}_k^*(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}_k)$ The optimal cumulative profits[2] from auction round k onward, assuming that the optimal state transition strategy is followed from auction round k onward; i.e, $\mathbf{J}_k^*(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}_k) \equiv \mathbf{J}_k^*(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}_k, \mathbf{x}_{k+1} = \mathbf{x}_{k+1}^*(\mathbf{x}_k, \mathbf{p}_k))$.

In the examples in Section 4, the following terms are also used:

$M(\mathbf{x}_k, \mathbf{p}_k, \mathbf{x}_k)$ When solving the DP problem associated with the biddability of a particular strategy, this matrix characterizes the profits from each bidding stage. The stages are associated with price level, not with the auction round or time period. For a discrete-valued set of price levels, this matrix is obtained as the probability of $\mathbf{p}_k$ times $\mathbf{J}_k^*(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}_k)$; i.e., $M(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}_k) = \mathbf{prob}(\mathbf{p}_k)\mathbf{J}_k^*(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}_k)$. The role this matrix plays in the biddability DP is the same as the role that $R_k - C_k$ plays in the primary optimization problem, the profits from one stage. Section 4.2 contains more details.

[2]See previous footnote.

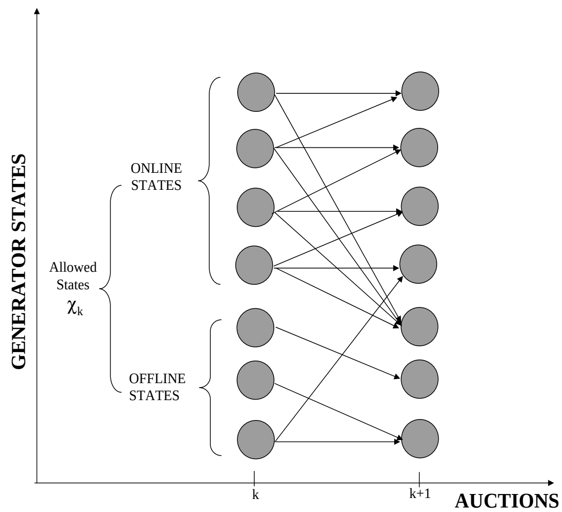

Figure 2: Generator states and state transitions from auction round k to auction round $k+1$. For thermal generators, their states can be divided into online and offline states. (For hydroelectric generators, their states can be parameterized by the amount of water remaining in the reservoir.) The state transition rules $\mathcal{T}_k$ are the arrows that show the permissible transitions from one state in auction round k to another state in auction round $k+1$.

$V(\mathbf{x}_k, \mathbf{p}_k, \mathbf{x}_k)$ When solving the DP problem associated with the biddability of a particular strategy, this matrix characterizes the optimal *cumulative* profits for each state for each bidding stage. This matrix is constructed from $M(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}_k)$ using a backward DP approach. In the examples we use, this function to determine the optimal $\mathbf{y}_k$ is a *non-decreasing* function of $\mathbf{p}_k$. In effect, this matrix serves the same role in the biddability DP as the cumulative profits $\mathbf{J}_k^*$ serves in the DP across time periods. A more precise definition of this quantity is given in equations 55 and 56.

$\tilde{c}(z_1, z_2)$ When solving the DP problem associated with the biddability of a particular strategy, this is the state transition cost matrix that forbids transitions from higher dispatch values to lower dispatch values by penalizing such transitions; that is, $\tilde{c}(z_1, z_2) = 0$ if $z_1 \leq z_2$ and $\tilde{c}(z_1, z_2) = \infty$ if $z_1 > z_2$. This plays the same role in the biddability DP as the role c_k plays in the primary optimization problem. Section 4.2 has more details.

3 Optimal State Transition and Biddable Dispatch Strategy

A generator has to solve the problem of finding the optimal state transition and "biddable" dispatch *strategies* that maximize expected total profits over the K auction rounds. More formally, given exogenous energy and reserve prices:

$$\max \sum_{k=1}^{K} \mathbf{E}\left[R_k(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}_k) - C_k(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}_k) - c_k(\mathbf{x}_k, \mathbf{x}_{k+1})\right] \quad (1)$$

$$\text{subject to } \left\{ \begin{array}{l} \mathbf{y}_k \in \mathcal{Y}_k(\mathbf{x}_k, \mathbf{p}_k) \\ \mathbf{x}_{k+1} \in \mathcal{T}(\mathbf{x}_k, \mathbf{y}_k) \\ \mathbf{x}_k \in \mathcal{X}_k \end{array} \right\} \quad k = 1, \ldots, K \quad (2)$$

where the maximization is over biddable state transition and dispatch strategies $\mathbf{x}_{k+1}(\mathbf{x}_k, \mathbf{p}_k)$ and $\mathbf{y}_k(\mathbf{x}_k, \mathbf{p}_k)$. $\mathbf{E}$ is the expectation operator over uncertain prices $\mathbf{p}_k$, with known Markov probabilities $\mathbf{prob}(\mathbf{p}_{k+1}|\mathbf{p}_k)$. We seek a solution to problem 1 of the form:

$$\left. \begin{array}{rcl} \mathbf{y}_k & = & \mathbf{y}_k^*(\mathbf{x}_k, \mathbf{p}_k) \\ \mathbf{x}_{k+1} & = & \mathbf{x}_{k+1}^*(\mathbf{x}_k, \mathbf{p}_k) \end{array} \right\} \quad k = 1, \ldots, K \quad (3)$$

In other words, we want to find the optimal dispatch and optimal state transition *strategies* for auction round k as *functions* of state $\mathbf{x}_k$ and price $\mathbf{p}_k$.

The optimal state transition and dispatch strategies can be found from the following backward DP problem. We first write the DP equations for the terminal period or auction round K.

$$\mathbf{J}_K^*(\mathbf{x}_K, \mathbf{p}_K, \mathbf{y}_K, \mathbf{x}_{K+1}) = R_K(\mathbf{x}_K, \mathbf{p}_K, \mathbf{y}_K) - C_K(\mathbf{x}_K, \mathbf{p}_K, \mathbf{y}_K) - c_K(\mathbf{x}_K, \mathbf{x}_{K+1}) \quad (4)$$

$$\mathbf{J}_K^*(\mathbf{x}_K, \mathbf{p}_K, \mathbf{y}_K) = \max_{\mathbf{x}_{K+1} \in \mathcal{X}_{N+1} \cap \mathcal{T}_K(\mathbf{x}_K, \mathbf{y}_K)} \mathbf{J}_K^*(\mathbf{x}_K, \mathbf{p}_K, \mathbf{y}_K, \mathbf{x}_{K+1}) \quad (5)$$

$$\mathbf{x}_{K+1}^*(\mathbf{x}_K, \mathbf{p}_K, \mathbf{y}_K) = \operatorname*{argmax}_{\mathbf{x}_{K+1} \in \mathcal{X}_{N+1} \cap \mathcal{T}_K(\mathbf{x}_K, \mathbf{y}_K)} \mathbf{J}_K^*(\mathbf{x}_K, \mathbf{p}_K, \mathbf{y}_K, \mathbf{x}_{K+1}) \quad (6)$$

If there is no solution to the maximization problem 5, let $\mathbf{J}_K^*(\mathbf{x}_K, \mathbf{p}_K, \mathbf{y}_K) = -\infty$, and let $\mathbf{x}_{K+1}^*(\mathbf{x}_K, \mathbf{p}_K, \mathbf{y}_K)$ be the null set.

Next we carry out the following optimization to get the optimal strategies for period K:

$$\mathbf{y}_K^*(\mathbf{x}_K, \mathbf{p}_K) = \operatorname*{argmax}_{\mathbf{y}_K(\mathbf{x}_K, \mathbf{p}_K) \in \mathcal{Y}_N(\mathbf{x}_K, \mathbf{p}_K)} \mathbf{E}\left[\mathbf{J}_K^*\left(\mathbf{x}_K, \mathbf{p}_K, \mathbf{y}_K(\mathbf{x}_K, \mathbf{p}_K)\right)\right] \quad (7)$$

$$\mathbf{x}_{K+1}^*(\mathbf{x}_K, \mathbf{p}_K) = \mathbf{x}_{K+1}^*(\mathbf{x}_K, \mathbf{p}_K, \mathbf{y}_K^*(\mathbf{x}_K, \mathbf{p}_K)) \quad (8)$$

where the expectation $\mathbf{E}$ in equation 7 is over $\mathbf{p}_K$, and the optimization in equation 7 is a *functional* optimization. As we shall see in Section 4, this functional optimization can be solved using a separate backward DP optimization.

For the remaining auction rounds $k = K-1, \ldots, 1$, we can write the following recursive DP equations; the optimization is carried out in reverse order from $k = K-1$ to $k = 1$.

$$\begin{aligned}\mathbf{J}_k^*(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}_k, \mathbf{x}_{k+1}) &= R_k(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}_k) - C_k(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}_k) - c_k(\mathbf{x}_k, \mathbf{x}_{k+1}) \\ &\quad + \mathbf{E}\left[\mathbf{J}_{k+1}^*\left(\mathbf{x}_{k+1}, \mathbf{p}_{k+1}, \mathbf{y}_{k+1}^*(\mathbf{x}_{k+1}, \mathbf{p}_{k+1})\right) |\mathbf{p}_k\right] \quad (9) \\ \mathbf{J}_k^*(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}_k) &= \max_{\mathbf{x}_{k+1} \in \mathcal{X}_{k+1} \cap \mathcal{T}_k(\mathbf{x}_k, \mathbf{y}_k)} \mathbf{J}_k^*(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}_k, \mathbf{x}_{k+1}) \quad (10) \\ \mathbf{x}_{k+1}^*(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}_k) &= \operatorname*{argmax}_{\mathbf{x}_{k+1} \in \mathcal{X}_{k+1} \cap \mathcal{T}_k(\mathbf{x}_k, \mathbf{y}_k)} \mathbf{J}_k^*(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}_k, \mathbf{x}_{k+1}) \quad (11)\end{aligned}$$

If there is no solution to the maximization problem 10, then we let $\mathbf{J}_k^*(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}_k) = -\infty$, and we let $\mathbf{x}_{k+1}^*(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}_k)$ be the null set.

We next carry out the following optimization to get the optimal strategies for period k:

$$\begin{aligned}\mathbf{y}_k^*(\mathbf{x}_k, \mathbf{p}_k) &= \operatorname*{argmax}_{\mathbf{y}_k(\mathbf{x}_k, \mathbf{p}_k) \in \mathcal{Y}_k(\mathbf{x}_k, \mathbf{p}_k)} \mathbf{E}\left[\mathbf{J}_k^*\left(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}_k(\mathbf{x}_k, \mathbf{p}_k)\right)\right] \quad (12) \\ \mathbf{x}_{k+1}^*(\mathbf{x}_k, \mathbf{p}_k) &= \mathbf{x}_{k+1}^*(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}_k^*(\mathbf{x}_k, \mathbf{p}_k)) \quad (13)\end{aligned}$$

where the expectation $\mathbf{E}$ in equation 12 is over $\mathbf{p}_k$. The optimization in equation 12 is also a *functional* optimization.

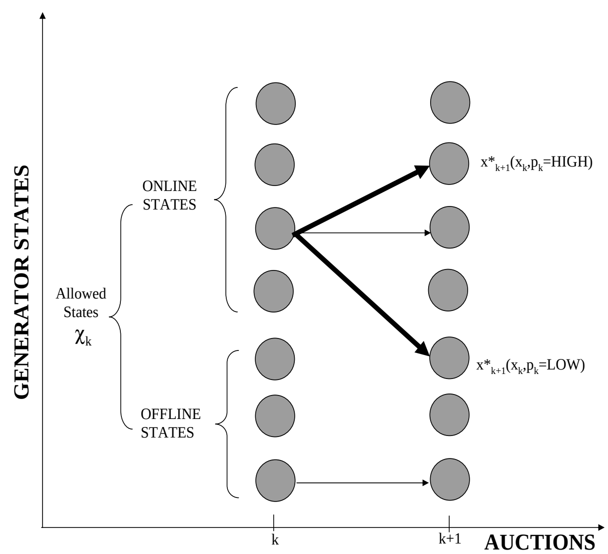

Figure 3: Optimal state transition strategy in auction round k depends upon both the generator state x_k and the price p_k. The figure illustrates that if the price p_k is LOW, it is optimal to go to one of the OFFLINE states, as shown by the lower line in bold. If the price p_k is HIGH, it is optimal to go to one of the ONLINE states, as shown by the upper line in bold. The optimal strategy is found for each state and price combination in auction round k.

If the optimal dispatch and state transition strategies are followed, the expected profit

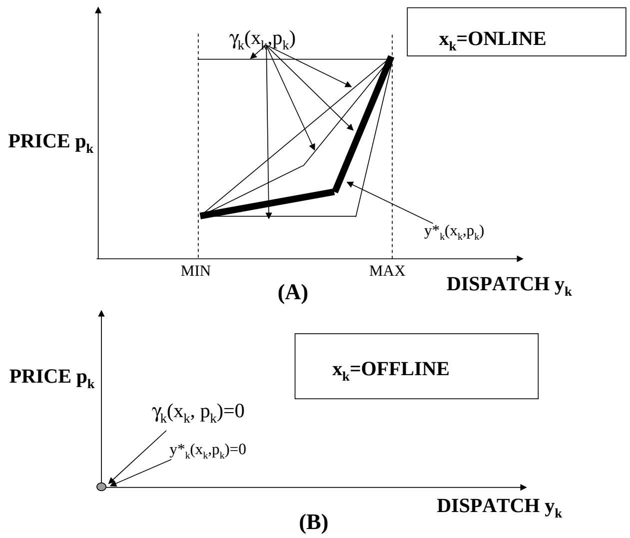

Figure 4: The optimal dispatch strategy is a function of both state and price in auction round k. Figure 4(A) illustrates the optimal bidding strategy when the generator is ONLINE in auction round k, assuming that the optimal \$/MW bid must be a non-decreasing function of MW, and that when the generator is ONLINE, the generator must produce output between MIN and MAX. Therefore, the allowed dispatch strategies $\mathcal{Y}_k$ are all non-decreasing functions of price $\mathbf{p}_k$; the optimal dispatch strategy for the case when the generator is ONLINE is shown in bold. When the generator is OFFLINE, it cannot produce any output, no matter what the price. Therefore, when the generator is OFFLINE, there is only one allowed dispatch strategy $\mathcal{Y}_k$, which is to produce 0 MW; this is also the optimal dispatch strategy $\mathbf{y}_k^*$, as shown in Figure 4(B).

from following this strategy as a function of the period 1 state $\mathbf{x}_1$ and price $\mathbf{p}_1$ is:

$$\text{Optimal expected total profits} = \mathbf{E}\left[\mathbf{J}_1^*(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1^*(\mathbf{x}_1, \mathbf{p}_1))\right] \tag{14}$$

where the expectation $\mathbf{E}$ in equation 14 is over $\mathbf{p}_1$.

Figure 3 illustrates the optimal state transition strategy as a function of state and price. Figure 4 illustrates the optimal dispatch strategy as a function of state and price. Here are a few observations:

1. The maximization in equation 12 (and equation 7) is a constrained optimization over a function space; therefore, depending on the nature of this optimization, one may need DP-like methods to solve it. In such a case, the optimization 1 is a two-level nested backward-DP problem. Such problems arise in practice because the bidding strategy is typically restricted in electricity market auctions to be an non-decreasing function of the MW offered. Equations 7 and 12 differentiate the problem from conventional self-commitment problems. This is illustrated in examples 4.2 and 4.3.

2. In the examples in the next section, we use discrete rather than a continuous mathematics to describe the DP formulation, even though the mathematics presented here supports both continuous and discrete frameworks. In other words, we discretize $\mathbf{p}_k$ and $\mathbf{y}_k$; our treatment is chosen to provide insights and to make it easier to develop computer algorithms to solve the optimal bidding strategy problem.

3. If $\mathcal{Y}_k(\mathbf{x}_k, \mathbf{p}_k)$ does not depend on $\mathbf{p}_k$ (i.e., $\mathcal{Y}_k(\mathbf{x}_k, \mathbf{p}_k) \equiv \mathcal{Y}_k(\mathbf{x}_k)$), then the maximization in equation 12 (and equation 7) can be simplified. For each $\mathbf{p}_k$, we simply pick a $\mathbf{y}_k = \mathbf{y}_k^*(\mathbf{x}_k, \mathbf{p}_k) \in \mathcal{Y}_k(\mathbf{x}_k)$ that maximizes $\mathbf{J}_k^*(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}_k)$, and we set $\mathbf{y}_k^*(\mathbf{x}_k, \mathbf{p}_k) = \mathbf{y}_k^*(\mathbf{x}_k, \mathbf{p}_k)$. We illustrate this with Example 4.1. We cannot do this when $\mathcal{Y}_k(\mathbf{x}_k, \mathbf{p}_k)$ depends on $\mathbf{p}_k$ because, even though the function $\mathbf{J}_k^*(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}_k)$ is being maximized for each $\mathbf{p}_k$, it is possible that the constraint $\mathbf{y}_k^*(\mathbf{x}_k, \mathbf{p}_k) \in \mathcal{Y}_k(\mathbf{x}_k, \mathbf{p}_k)$ may be violated for some values of $\mathbf{p}_k$. Example 4.2 illustrates this.

4 Illustrative Examples

This section shows how to use the methods from Section 3 to derive optimal dispatch and optimal bidding strategies for power market auctions. We have several objectives in mind for this analysis:

1. To demonstrate that the optimal bidding (and state transition) strategies can be derived in a methodical way. The examples are such that, with some effort, they can be done by hand. The point of these examples is to demonstrate the method for deriving optimal bidding strategies in a simple, logical, systematic way.

2. To demonstrate that the optimal dispatch strategy may require a bidding strategy with a declining MW versus price curve. However, declining bid curves are often not allowed in electricity auctions and the optimal *allowed* bidding strategy may be second-best in terms of maximizing profits.

3. To demonstrate that a generator's optimal bidding strategy can depend in a very subtle way on how price forecasts are modeled (e.g., price correlation among periods can matter).

4. To demonstrate that a generator's optimal bidding strategy could vary depending on the auction design even assuming that the auction is a single-price auction. This may seem obvious, but we show via examples that there are subtleties associated with the bidding strategy.

The examples also demonstrate that nested multiple-level backward DP problems must be solved to find the optimal bidding strategy. The nested multiple-level DP problems are:

1. *Across time periods*, starting from the final period. This is the conventional backward DP.

2. *Within each time period* (see equation 12) to find the optimal bidding strategy to satisfy bidding rules in the auction. Examples 4.2 to 4.5 illustrate this point.

3. *Across multiple markets*, starting from the final market that clears. Examples 4.4 and 4.5 illustrate this point.

Figure 5 illustrates the nested nature of these DP problems.

Therefore, to solve the optimal bidding strategy problem, one may need to solve three (or more) nested backward DP problems. This makes the problem different from other conventional DP problems. The appendix gives an overview of typical DP problems.

4.1 Energy Limited Hydro

We consider a two-period problem with a single energy market and a hydroelectric generator that has 100 MWh of energy in its reservoir. No energy (water) is added to the reservoir

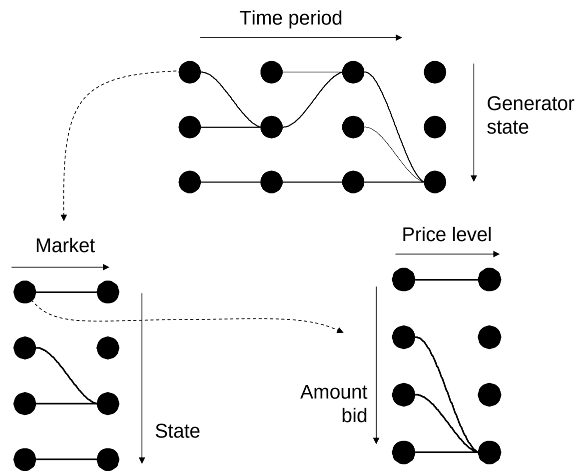

Figure 5: Illustration of the nested nature of the DP problems for the specific case of a problem involving multiple time periods with price uncertainty, multiple markets, and biddability restrictions.

over the next two periods. The direct cost of producing energy is zero. At the end of two periods, the hydroelectric generator must have no energy left in its reservoir. The hydroelectric generator's minimum and maximum hourly power limits are 0 MW and 200 MW respectively, although, as we shall see, this maximum limit will not play a role in this example because the initial energy in the reservoir is below this value.

The auction rules are that:

1. The auction is a single-price auction; i.e., all winners get paid a uniform clearing price.
2. MW bids are restricted to be in 50 MW blocks.
3. The period 1 auction is held first; after this auction clears, the period 2 bids are accepted and cleared.
4. There is no restriction on the bids; i.e., the MW versus price ($/MWh) bids can take *any* functional form, including non-increasing functions.

The hydroelectric generator is a risk-neutral profit-maximizing (in an expected value sense) price-taker. The hydroelectric generator has forecasted that the market clearing prices are uncertain and satisfy a Markov distribution. The market clearing prices have two levels, LOW and HIGH, in each period (see Table 1). Moreover, period 2 prices are perfectly correlated with period 1 prices; that is, if prices start HIGH in period 1, they will be HIGH in period 2, and vice-versa. If the are LOW in period 1, they remain LOW in period 2 (although the value of HIGH and LOW price is different for each period).

Given this information, what is the hydroelectric generator's optimal dispatch and bidding strategy? To answer this question, we observe the following:

Table 1: HIGH and LOW values of $\mathbf{p}_k$ in \$/MWh for each period.

	Time Period	
$\mathbf{p}_k$	1	2
HIGH	30	40
LOW	20	15

1. The number of auction rounds is $K = 2$.
2. The state $\mathbf{x}_k$ in period k (corresponding to auction round k) is the amount of energy left (in MWh) in the reservoir. Because of the requirement that the generator bid in 50 MW blocks, the allowed states $\mathbf{x}_k$ are $\mathcal{X}_k = \{0, 50, 100\}$ MWh. Let EMPTY $= 0$, HALF $= 50$ MWh, and FULL $= 100$ MWh. Moreover, $\mathbf{x}_{K+1} =$ EMPTY $= 0$ MWh.
3. The maximum dispatch must not exceed the maximum of 200 MW or the total energy in the reservoir ($\mathbf{x}_k$). By assumption, the dispatch $\mathbf{y}_k$ must be a multiple of 50 MW. Thus, $\mathbf{y}_k$ must lie in the set $\mathcal{Y}_k = \{0, 50, \ldots, \min(\mathbf{x}_k, 200)\}$ MW. For example, if $\mathbf{x}_k = 0$, then $\mathcal{Y}_k = \{0\}$.
4. The state transition rule is given by $\mathbf{x}_{k+1} \in \mathcal{T}_k(\mathbf{x}_k, \mathbf{y}_k) = \mathbf{x}_k - \mathbf{y}_k$.
5. The revenues are $R_k(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}_k) = \mathbf{p}_k\mathbf{y}_k$. The dispatch cost $C_k(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}_k) = 0$, and so are the transition costs $c_k(\mathbf{x}_k, \mathbf{x}_{k+1}) = 0$. Thus, profits equal revenues in this example.

In order to obtain the optimal dispatch and bidding strategy, we again solve optimization problem 1. In doing so, we fill Tables 2 to 5 which show the optimal dispatch strategy, the optimal commitment strategy and the optimal cumulative profits (from period k to the terminal period) respectively, starting with the last column in each table and working backwards.

We start with the terminal period $K = 2$, and we solve equations 4, 5, 6. We note that $\mathbf{x}_3 = 0$ because all the energy must be used up at the end of three periods. It can be verified that

$$\mathbf{J}_2^*(\mathbf{x}_2, \mathbf{p}_2, \mathbf{y}_2, \mathbf{x}_3) = \mathbf{p}_2\mathbf{y}_2 \quad (15)$$
$$\mathbf{J}_2^*(\mathbf{x}_2, \mathbf{p}_2, \mathbf{y}_2) = \mathbf{p}_2\mathbf{y}_2 \quad (16)$$
$$\mathbf{x}_3^*(\mathbf{x}_2, \mathbf{p}_2, \mathbf{y}_2) = 0 \quad (17)$$

Next we solve equations 7 and 8. $\mathbf{x}_3 =$ EMPTY $= 0 = \mathbf{x}_2 - \mathbf{y}_2$ implies that $\mathbf{x}_2 = \mathbf{y}_2$. Thus, the optimal dispatch strategy in the terminal period is straightforward: to meet the boundary condition that there be no energy left at the end of the three periods, any remaining energy must be spent regardless of price in the final period. Therefore:

$$\mathbf{y}_2^*(\mathbf{x}_2, \mathbf{p}_2) = \mathbf{x}_2 \quad (18)$$
$$\mathbf{x}_3^*(\mathbf{x}_2, \mathbf{p}_2) = 0 = \text{EMPTY} \quad (19)$$

The optimal next state for time period 3 for each of the states during time period 2 is EMPTY. The values of the profits for period 2 depend on both the dispatch strategy and the uncertain price state during period 2. For each of the two possible price levels during period 2 we can calculate the profits as $\mathbf{p}_2\mathbf{y}_2$. For example, for a dispatch of 50 MW and a LOW price in period 2, the profits are $50 \cdot 15 = 750$. All other entries of this column are obtained the same way. This fills the last columns (corresponding to period 2) of Tables 2, 3 and 4.

We next turn our attention to filling the first columns (corresponding to period 1) of Tables 2, 3 and 4. To do this, we must solve equations 7 to 11 for $k = 1$.

$$\begin{aligned}
\mathbf{J}_1^*(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1, \mathbf{x}_2) &= \mathbf{p}_1\mathbf{y}_1 + \mathbf{E}\left[\mathbf{J}_2^*\left(\mathbf{x}_2, \mathbf{p}_2, \mathbf{y}_2^*(\mathbf{x}_2, \mathbf{p}_2)\right)|\mathbf{p}_1\right] && (20)\\
\mathbf{J}_1^*(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1) &= \max_{\mathbf{x}_2 \in \mathcal{X}_2 \cap \mathcal{T}_1(\mathbf{x}_1, \mathbf{y}_1)} \mathbf{J}_1^*(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1, \mathbf{x}_2) && (21)\\
\mathbf{x}_2^*(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1) &= \operatorname*{argmax}_{\mathbf{x}_2 \in \mathcal{X}_2 \cap \mathcal{T}_1(\mathbf{x}_1, \mathbf{y}_1)} \mathbf{J}_2^*(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1, \mathbf{x}_2) && (22)
\end{aligned}$$

We derive equation 23 using equations 16, 18 and 20; the fact that price transitions from period 1 to period 2 are perfectly correlated; and the prices from Table 1:

$$\mathbf{J}_1^*(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1, \mathbf{x}_2) = \mathbf{p}_1\mathbf{y}_1 + \mathbf{E}\left[\mathbf{p}_2\mathbf{x}_2|\mathbf{p}_1\right] = \begin{cases} 30\mathbf{y}_1 + 40\mathbf{x}_2 & \text{if } \mathbf{p}_1 = \text{HIGH} = \$30/\text{MWh} \\ 20\mathbf{y}_1 + 15\mathbf{x}_2 & \text{if } \mathbf{p}_1 = \text{LOW} = \$20/\text{MWh} \end{cases} \quad (23)$$

We now illustrate calculations for $\mathbf{x}_1 = \text{FULL}$. The entries corresponding to $\mathbf{x}_1 = \text{HALF}$ and $\mathbf{x}_1 = \text{EMPTY}$ can be similarly filled in the first column (corresponding to $k = 1$) of Tables 2, 3 and 4 (though, strictly speaking, they are not needed because the hydroelectric generator is in the FULL state in period 1). The three possible state transitions from $\mathbf{x}_1 = \text{FULL} = 100$ MWh are $\mathbf{x}_2 = \text{FULL} = 100$ MWh or $\mathbf{x}_2 = \text{HALF} = 50$ or $\mathbf{x}_2 = \text{EMPTY} = 0$ MWh; these correspond to $\mathbf{y}_1 = 0$, $\mathbf{y}_1 = 50$ MW, and $\mathbf{y}_1 = 100$ MW respectively. Therefore, for each value of $\mathbf{y}_1$ there is only one possible state transition. This implies that:

$$\begin{aligned}
\mathbf{J}_1^*(\mathbf{x}_1 = \text{FULL}, \mathbf{p}_1 = \text{HIGH}, 0) &= 30 \cdot 0 + 40 \cdot 100 = 4000 && (24)\\
\mathbf{J}_1^*(\mathbf{x}_1 = \text{FULL}, \mathbf{p}_1 = \text{LOW}, 0) &= 20 \cdot 0 + 15 \cdot 100 = 1500 && (25)\\
\mathbf{J}_1^*(\mathbf{x}_1 = \text{FULL}, \mathbf{p}_1 = \text{HIGH}, 50) &= 30 \cdot 50 + 40 \cdot 50 = 3500 && (26)\\
\mathbf{J}_1^*(\mathbf{x}_1 = \text{FULL}, \mathbf{p}_1 = \text{LOW}, 50) &= 20 \cdot 50 + 15 \cdot 50 = 1750 && (27)\\
\mathbf{J}_1^*(\mathbf{x}_1 = \text{FULL}, \mathbf{p}_1 = \text{HIGH}, 100) &= 30 \cdot 100 + 40 \cdot 0 = 3000 && (28)\\
\mathbf{J}_1^*(\mathbf{x}_1 = \text{FULL}, \mathbf{p}_1 = \text{LOW}, 100) &= 20 \cdot 100 + 15 \cdot 0 = 2000 && (29)\\
\mathbf{x}_2^*(\mathbf{x}_1 = \text{FULL}, \mathbf{p}_1, 0) &= \text{FULL} && (30)\\
\mathbf{x}_2^*(\mathbf{x}_1 = \text{FULL}, \mathbf{p}_1, 50) &= \text{HALF} && (31)\\
\mathbf{x}_2^*(\mathbf{x}_1 = \text{FULL}, \mathbf{p}_1, 100) &= \text{EMPTY} && (32)
\end{aligned}$$

We now fill the first column of Tables 2, 3 and 4. As a preamble to this calculation, we use the expected optimal cumulative profits $\mathbf{J}_1^*(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1)$ from period 1 to the terminal period for each of the two possible price levels and three possible dispatch choices during period 1. These values are entered into Table 5. For reasons that will become clear later on, we arrange this table as shown: rows represent dispatch levels, columns correspond to price levels (increasing from left to right).

The bidding function is not restricted to being a non-decreasing function of price. Without a restriction on the shape of the bidding function, the optimal dispatch strategy $\mathcal{Y}_k(\mathbf{x}_k, \mathbf{p}_k)$ does not depend on $\mathbf{p}_k$. Therefore, $\mathcal{Y}_k(\mathbf{x}_k, \mathbf{p}_k) \equiv \mathcal{Y}_k(\mathbf{x}_k)$. For each price level $\mathbf{p}_k$, we pick a

$\mathbf{y}_k^*(\mathbf{x}_k, \mathbf{p}_k) \in \mathcal{Y}_k(\mathbf{x}_k)$ that maximizes $\mathbf{J}_k^*(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}_k)$, and $\mathbf{y}_k^*(\mathbf{x}_k, \mathbf{p}_k)$ is the optimal dispatch strategy.

To illustrate the solution process, we first inspect Table 5 and determine, for each possible price level (column), the corresponding dispatch. Doing this in Table 5, we find that $\mathbf{J}_1^*(\text{FULL}, \mathbf{p}_1, \mathbf{y}_1)$ is maximized for $\mathbf{p}_1 = \text{LOW}$ in period 1 when $\mathbf{y}_1 = 100$ MW; the maximum value is shown in **boldface** in the first column. Similarly, $\mathbf{J}_1^*(\text{FULL}, \mathbf{p}_1, \mathbf{y}_1)$ is maximized for $\mathbf{p}_1 = \text{HIGH}$ in period 1 when $\mathbf{y}_1 = 0$ MW; the maximum value is shown in **boldface** in the second column.

These two values, corresponding to the optimal profits when the reservoir is initially FULL, are entered into the two corresponding entries in Table 4 in positions (FULL,LOW) and (FULL,HIGH). These values are also in **boldface**. The dispatch implied by these optimal profits ($\mathbf{y}_1 = 100$ when $\mathbf{p}_1 = \text{LOW}$, for example) is entered into the corresponding entry in Table 2, and the consequent next period transition state (in this case HALF) is entered into its corresponding place in Table 3.

The remaining portion of column 1 can be similarly obtained, although it is technically not necessary because of the given condition that the initial state is FULL.

Table 2: Optimal bidding or dispatch strategy $\mathbf{y}_k^*(\mathbf{x}_k, \mathbf{p}_k)$ in MW implemented *during* period k

	Time Period	
$(\mathbf{x}_k, \mathbf{p}_k)$	1	2
(EMPTY,LOW)	0	0
(EMPTY,HIGH)	0	0
(HALF,LOW)	50	50
(HALF,HIGH)	0	50
(FULL,LOW)	100	100
(FULL,HIGH)	0	100

Table 3: Optimal state transition strategy $\mathbf{x}_{k+1}^*(\mathbf{x}_k, \mathbf{p}_k)$, implemented at the *end* of period k

	Time Period	
$(\mathbf{x}_k, \mathbf{p}_k)$	1	2
(EMPTY,LOW)	EMPTY	EMPTY
(EMPTY,HIGH)	EMPTY	EMPTY
(HALF,LOW)	EMPTY	EMPTY
(HALF,HIGH)	HALF	EMPTY
(FULL,LOW)	EMPTY	EMPTY
(FULL,HIGH)	FULL	EMPTY

From these tables, in period 1, the optimal dispatch strategy is:

$$\mathbf{y}_1^*(\text{FULL}, \text{LOW}) = 100 \text{ MW} \quad (33)$$

$$\mathbf{y}_1^*(\text{FULL}, \text{HIGH}) = 0 \text{ MW} \quad (34)$$

Table 4: Values of $\mathbf{J}_k^*(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}_k^*(\mathbf{x}_k, \mathbf{p}_k)$ in \$

	Time Period	
$(\mathbf{x}_k, \mathbf{p}_k)$	1	2
(EMPTY,LOW)	0	0
(EMPTY,HIGH)	0	0
(HALF,LOW)	1000	750
(HALF,HIGH)	2000	2000
(FULL,LOW)	**2000**	1500
(FULL,HIGH)	**4000**	4000

Table 5: Values of $\mathbf{J}_1^*(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1)$ in \$ for $\mathbf{x}_1$ = FULL

	$\mathbf{p}_1$	
$\mathbf{y}_1$ in MW	LOW (\$20/MWh)	HIGH (\$30/MWh)
0	1500	**4000**
50	1750	3500
100	**2000**	3000

Using the above equations in addition to equations 30 and 32, the optimal state transition strategy in period 1 is therefore:

$$\mathbf{x}_2^*(\text{FULL}, \text{LOW}) = \text{EMPTY} \quad (35)$$

$$\mathbf{x}_2^*(\text{FULL}, \text{HIGH}) = \text{FULL} \quad (36)$$

It is useful to ask about the expected optimal profits from following the optimal strategy. From equation 14 and Table 4 (column entries corresponding to period 1), the expected profits from following the optimal state transition and dispatch strategies are \$2000 when the period 1 state is FULL and period 1 price is LOW; the expected profits are \$4000 when the period 1 state is FULL and period 1 price is HIGH. Taking expectations over price in period 1, expected profits are $0.5 \cdot 2000 + 0.5 \cdot 4000 = \$3000$ when the period 1 state is FULL.

Assuming, as we have, that bidding a declining cost curve is permissible, the optimal bidding strategy for this example is:

- In period 1, bid a decreasing function of price; offer 100 MW at LOW price and 0 MW at HIGH price in period 1.
- In period 2 offer all remaining output (depending on what output got selected in period 1) at a price lower than LOW, i.e., at a price lower than \$15/MWh.

For this simple example, the reason for the declining bid curve is intuitive: if the price is HIGH (\$30/MWh) in period 1, it is going to be even *higher* (\$40/MWh) in period 2, so we are better off waiting to sell the entire output in period 2; if the price is LOW (\$20/MWh) in period 1, it is going to be even *lower* (\$15/MWh) in period 2, so we are better off selling the entire output in period 1.

4.2 Energy Limited Hydro, Non-Declining Bids Required

Now we consider the same problem in the previous example, except that we are required to bid non-declining prices with increasing quantity. As above, we proceed to fill-in several tables containing the optimal dispatch strategy and its implied bidding strategy (Table 6); the optimal state transition strategy(Table 7), and, of course, the optimal cumulative profits (Table 8).

As in the previous example, we can derive the following:

$$\mathbf{J}_2^*(\mathbf{x}_2, \mathbf{p}_2, \mathbf{y}_2, \mathbf{x}_3) = \mathbf{p}_2\mathbf{y}_2 \quad (37)$$

$$\mathbf{J}_2^*(\mathbf{x}_2, \mathbf{p}_2, \mathbf{y}_2) = \mathbf{p}_2\mathbf{y}_2 \quad (38)$$

$$\mathbf{x}_3^*(\mathbf{x}_2, \mathbf{p}_2, \mathbf{y}_2) = 0 \quad (39)$$

$$\mathbf{y}_2^*(\mathbf{x}_2, \mathbf{p}_2) = \mathbf{x}_2 \quad (40)$$

$$\mathbf{x}_3^*(\mathbf{x}_2, \mathbf{p}_2) = 0 = \text{EMPTY} \quad (41)$$

This helps fill the second columns of Tables 6, 7 and 8.

As before, we also derive:

$$\mathbf{J}_1^*(\mathbf{x}_1 = \text{FULL}, \mathbf{p}_1 = \text{HIGH}, 0) = 30 \cdot 0 + 40 \cdot 100 = 4000 \quad (42)$$

$$\mathbf{J}_1^*(\mathbf{x}_1 = \text{FULL}, \mathbf{p}_1 = \text{LOW}, 0) = 20 \cdot 0 + 15 \cdot 100 = 1500 \quad (43)$$

$$\mathbf{J}_1^*(\mathbf{x}_1 = \text{FULL}, \mathbf{p}_1 = \text{HIGH}, 50) = 30 \cdot 50 + 40 \cdot 50 = 3500 \quad (44)$$

$$\mathbf{J}_1^*(\mathbf{x}_1 = \text{FULL}, \mathbf{p}_1 = \text{LOW}, 50) = 20 \cdot 50 + 15 \cdot 50 = 1750 \quad (45)$$

$$\mathbf{J}_1^*(\mathbf{x}_1 = \text{FULL}, \mathbf{p}_1 = \text{HIGH}, 100) = 30 \cdot 100 + 40 \cdot 0 = 3000 \quad (46)$$

$$\mathbf{J}_1^*(\mathbf{x}_1 = \text{FULL}, \mathbf{p}_1 = \text{LOW}, 100) = 20 \cdot 100 + 15 \cdot 0 = 2000 \quad (47)$$

$$\mathbf{x}_2^*(\mathbf{x}_1 = \text{FULL}, \mathbf{p}_1, 0) = \text{FULL} \quad (48)$$

$$\mathbf{x}_2^*(\mathbf{x}_1 = \text{FULL}, \mathbf{p}_1, 50) = \text{HALF} \quad (49)$$

$$\mathbf{x}_2^*(\mathbf{x}_1 = \text{FULL}, \mathbf{p}_1, 100) = \text{EMPTY} \quad (50)$$

Table 6: Optimal dispatch strategy $\mathbf{y}_k^*(\mathbf{x}_k, \mathbf{p}_k)$ in MW implemented *during* period k

	Time Period	
$(\mathbf{x}_k, \mathbf{p}_k)$	1	2
(EMPTY,LOW)	0	0
(EMPTY,HIGH)	0	0
(HALF,LOW)	0	50
(HALF,HIGH)	0	50
(FULL,LOW)	0	100
(FULL,HIGH)	0	100

If there were no restrictions on bidding, the optimal dispatch would be obtained exactly as in the previous example. This means that we would select the largest value from each column

Table 7: Optimal state transition strategy $\mathbf{x}^*_{k+1}(\mathbf{x}_k, \mathbf{p}_k)$, implemented at the *end* of period k

	Time Period	
$(\mathbf{x}_k, \mathbf{p}_k)$	1	2
(EMPTY,LOW)	EMPTY	EMPTY
(EMPTY,HIGH)	EMPTY	EMPTY
(HALF,LOW)	HALF	EMPTY
(HALF,HIGH)	HALF	EMPTY
(FULL,LOW)	FULL	EMPTY
(FULL,HIGH)	FULL	EMPTY

Table 8: Values of $\mathbf{J}^*_k(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}^*_k(\mathbf{x}_k, \mathbf{p}_k))$ in \$

	Time Period	
$(\mathbf{x}_k, \mathbf{p}_k)$	1	2
(EMPTY,LOW)	0	0
(EMPTY,HIGH)	0	0
(HALF,LOW)	750	750
(HALF,HIGH)	2000	2000
(FULL,LOW)	1500	1500
(FULL,HIGH)	4000	4000

of Table 9. However, doing so would require that during period 1 we dispatch 100 MW when the price level is LOW and 0 MW when the price is HIGH. Under the bidding rules for this example, this cannot be done.

Because of the "biddability" constraint, filling column 1 of these tables requires a new approach. We illustrate the method only for $\mathbf{x}_1$ = FULL. The entries corresponding to $\mathbf{x}_1$ = HALF and $\mathbf{x}_1$ = EMPTY can be filled similarly.

To solve equation 12, use the following nested backward DP method. Let $\mathbf{p}_1$ be discretized into the following prices arranged in ascending order; i.e., $\mathbf{p}_1 = p_1^1, p_1^2, \ldots, p_1^\ell$. In our example, there are two possibilities: $\mathbf{p}_1$ is either HIGH or LOW, so $\ell = 2$, and our notation implies that p_1^1 = LOW and p_1^2 = HIGH. We next fill the values of $\mathbf{J}^*_1(\text{FULL}, \mathbf{p}_1, \mathbf{y}_1)$ in Table 9. From this table, we evaluate the probability weighted values $M(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1) = \mathbf{prob}(\mathbf{p}_1)\mathbf{J}^*_1(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1)$ for $\mathbf{x}_1$ = FULL; this is displayed in Table 10.

The main constraint in equation 12 is that $\mathbf{y}_1$ be a non-decreasing function of price $\mathbf{p}_1$ so that it is "biddable". Therefore, we seek a non-decreasing function of price $\mathbf{p}_1$ that solves the following problem:

$$\max \mathbf{E}\left[\mathbf{J}^*_1\left(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1(\mathbf{x}_1, \mathbf{p}_1)\right)\right] = \max \sum_{i=1}^{\ell} M(\mathbf{x}_1, p_1^i, \mathbf{y}_1(\mathbf{x}_1, p_1^i)) \quad (51)$$

$$\mathbf{y}_1(\mathbf{x}_1, p_1^j) \geq \mathbf{y}_1(\mathbf{x}_1, p_1^{j-1}) \quad 1 < j \leq \ell = 2 \quad (52)$$

If we introduce a function $\tilde{c}(z_1, z_2)$ such that[3]:

$$\tilde{c}(z_1, z_2) = \begin{cases} 0 & \text{if } z_1 \leq z_2 \\ -\infty & \text{if } z_1 > z_2 \end{cases} \tag{53}$$

then we can write equations 51 and 52 as:

$$\max \mathbf{E}\left[\mathbf{J}_1^* \left(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1(\mathbf{x}_1, \mathbf{p}_1)\right)\right] = \max \left(\sum_{i=1}^{\ell} M(\mathbf{x}_1, p_1^i, \mathbf{y}_1(\mathbf{x}_1, p_1^i)) + \sum_{i=1}^{\ell-1} \tilde{c}(\mathbf{y}_1(\mathbf{x}_1, p_1^i), \mathbf{y}_1(\mathbf{x}_1, p_1^{i+1})) \right) \tag{54}$$

The problem in equation 54 can be solved by DP methods. In DP terminology (as in the Appendix), to solve this problem we associate "price level" with "stage" (columns) and dispatch levels with states (rows), with $\tilde{c}$ being the "transition costs".

To find a non-decreasing optimal dispatch strategy, we can solve equation 54 by using the following backward DP approach:

$$V(\mathbf{x}_1, p_1^\ell, \mathbf{y}_1) = M(\mathbf{x}_1, p_1^\ell, \mathbf{y}_1) \tag{55}$$

$$V(\mathbf{x}_1, p_1^i, \mathbf{y}_1) = M(\mathbf{x}_1, p_1^i, \mathbf{y}_1) + \max_{\widetilde{y}} \left(\tilde{c}(\mathbf{y}_1, \widetilde{y}) + V(\mathbf{x}_1, p_1^{i+1}, \widetilde{y}) \right) \quad \forall \quad i = \ell - 1, \ldots, 1 \tag{56}$$

$$\mathbf{y}_1^*(\mathbf{x}_1, p_1^1) = \underset{\widetilde{y}}{\operatorname{argmax}} V(\mathbf{x}_1, p_1^1, \widetilde{y}) \tag{57}$$

$$\mathbf{y}_1^*(\mathbf{x}_1, p_1^i) = \underset{\widetilde{y} \geq \mathbf{y}_1^*(\mathbf{x}_1, p_1^{i-1})}{\operatorname{argmax}} V(\mathbf{x}_1, p_1^i, \widetilde{y}) \quad i = 2, \ldots, \ell \tag{58}$$

Therefore, the entries of V are built from the maximum value of $\mathbf{p}_1 = p_1^\ell =$HIGH, and then we work backwards until we reach the smallest value of $\mathbf{p}_1 = p_1^1 =$LOW. We have associated the notion of "stage" with "price level".

Table 11 shows the entries of V for $\mathbf{x}_1$ = FULL. The last column of Table 11 is simply the last column of Table 10 per equation 55. The first column of Table 11 is calculated using equation 56; each entry in this table is the corresponding entry in Table 10 *plus* the maximum value in the next column of Table 11, where the maximum is taken over all rows below including the entry. Thus, the three entries of the first column of Table 11 are computed as follows:

$750 + \max(2000, 1750, 1500) = 2750$
$875 + \max(1750, 1500) = 2625$
$1000 + 1500 = 2500$

The solution to the arg max problem in equations 57 and 58 is solved by starting from the minimum $\mathbf{p}_1$ value and working sequentially till we reach the maximum $\mathbf{p}_1$ value. The

[3] In effect, the term $\tilde{c}$ enforces the non-decreasing nature of the bidding function by introducing a prohibitive penalty for all bidding functions that do not satisfy the non-decreasing property. In other words, we forbid any transition from a high dispatch amount during a LOW price level to a lower dispatch amount in a HIGH price level by associating a high penalty for such an action.

minimum $\mathbf{p}_1$ value corresponds to the first column of Table 11. The maximum value of this first column is highlighted in **boldface**; the corresponding argmax gives the dispatch level as determined by equation 57). To carry out the steps of equation 58, starting from column 1, we successively find the maximum value of the next column in Table 11 whose dispatch level is not above the current column's argmax dispatch level; this computation is repeated done until all the columns are determined. The maximum values found this way for each column in Table 11 are highlighted in **boldface**. The dispatch levels corresponding to the **boldface** entries in Table 11 form the optimal bidding strategy that satisfies the biddability constraint.

Table 9: Values of $\mathbf{J}_1^*(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1)$ in \$ for $\mathbf{x}_1$ = FULL

	$\mathbf{p}_1$	
$\mathbf{y}_1$ in MW	LOW (\$20/MWh)	HIGH (\$30/MWh)
0	1500	4000
50	1750	3500
100	2000	3000

Table 10: Values of $M = \mathbf{prob}(\mathbf{p}_1)\mathbf{J}_1^*(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1)$ in \$ for $\mathbf{x}_1$ = FULL. The probability that $\mathbf{p}_1$ = LOW is 0.5; the probability that $\mathbf{p}_1$ = HIGH is 0.5

	$\mathbf{p}_1$	
$\mathbf{y}_1$ in MW	LOW (\$20/MWh)	HIGH (\$30/MWh)
0	750	2000
50	875	1750
100	1000	1500

Table 11: Values of V in \$ (from equations 55 to 58)

	$\mathbf{p}_1$	
$\mathbf{y}_1$ in MW	LOW (\$20/MWh)	HIGH (\$30/MWh)
0	**2750**	**2000**
50	2625	1750
100	2500	1500

Therefore, in period 1, the optimal dispatch strategy is:

$$\mathbf{y}_1^*(\text{FULL}, \text{LOW}) = 0 \text{ MW} \tag{59}$$

$$\mathbf{y}_1^*(\text{FULL}, \text{HIGH}) = 0 \text{ MW} \tag{60}$$

Using the above equations and equation 48, the optimal state transition strategy in period 1 is:

$$\mathbf{x}_2^*(\text{FULL}, \text{LOW}) = \text{FULL} \tag{61}$$

$$\mathbf{x}_2^*(\text{FULL}, \text{HIGH}) = \text{FULL} \tag{62}$$

The optimal bidding strategy corresponding to this optimal dispatch strategy is:

- In period 1, do not offer any output (or, equivalently, offer 100 MW at any price greater than HIGH which is \$30/MWh).
- In period 2 offer all output at a price lower than LOW; i.e., at a price lower than \$15/MWh.

The values corresponding to this optimal bidding strategy can be entered into the corresponding positions of Tables 6 and 7. These correspond to the lower two entries of the first column of each table. The remaining entries in these two tables can be obtained by the same methodology, but this time applied to the case when the reservoir is HALF and EMPTY during period 1. Details of this computation are not illustrated.

From equations 42 and 43 and the optimal dispatch strategy in period 1, we can now derive the values for expected optimal cumulative profits $\mathbf{J}_1^*(\mathbf{x}_1 = \text{FULL}, \mathbf{p}_1, \mathbf{y}_1^*(\mathbf{x}_1, \mathbf{p}_1)))$; these are shown as the last 2 entries of column 1 (corresponding to period 1) in Table 8. The remainder of column 1 can be obtained the same way.

We now calculate the expected cumulative profits from following the optimal state transition and dispatch strategies. From Table 8, these values are \$1500 when the period 1 state is FULL and period 1 price is LOW; and \$4000 when the period 1 state is FULL and period 1 price is HIGH.

As before, we can also do an expected profits calculation. Expected profits from following this strategy are $0.5 \cdot 4000 + 0.5 \cdot 1500 = \$2750$ since the probability of price being either HIGH or LOW in period 1 is 0.5.

The optimal bidding strategy for this case is to withhold output in period 1 regardless of the period 1 price *even though the generator would be better off dispatching* 100 *MW if the period* 1 *price is* \$20*/MWh.* This arises because of the auction rule requirement that the generator needs to bid a non-decreasing function.

This optimal strategy differs from the strategy in the previous example where the optimal expected profits were \$3000. The biddability requirement has resulted in not only a different bidding strategy, but also in lower expected profits even when the optimal bidding strategy is followed.

The implications of this example for market design are that restrictions on biddability may simplify market clearing but it comes at the expense of a less efficient market. Furthermore, the decisions that the bidder faces in order to optimize their profits are more complex.

4.3 Energy Limited Hydro: MW Bids Must Be Non-Decreasing Function of Price, Uncorrelated Prices

We consider the same situation as the last example with the added condition that the prices between the two periods are uncorrelated. As before, the forecasted market clearing prices have two price levels, LOW and HIGH, in each period (reproduced for convenience in Table 12). Period 2 price levels are independent of period 1 price levels; each price level in each period has equal probability of occurrence (0.5).

Given this information, what is the hydroelectric generator's optimal bidding strategy?

Table 12: HIGH and LOW values of $\mathbf{p}_k$ in \$/MWh for each period

	Time Period	
$\mathbf{p}_k$	1	2
HIGH	30	40
LOW	20	15

To obtain the optimal dispatch and bidding strategy, we again solve the optimization problem in equation 1. In doing so, we fill Tables 13 to 18, starting with the last column in each table and working our way backwards.

Following the backward DP steps from $K = 2$ as before, it can be verified that:

$$\mathbf{J}_2^*(\mathbf{x}_2, \mathbf{p}_2, \mathbf{y}_2, \mathbf{x}_3) = \mathbf{p}_2\mathbf{y}_2 \tag{63}$$

$$\mathbf{J}_2^*(\mathbf{x}_2, \mathbf{p}_2, \mathbf{y}_2) = \mathbf{p}_2\mathbf{y}_2 \tag{64}$$

$$\mathbf{x}_3^*(\mathbf{x}_2, \mathbf{p}_2, \mathbf{y}_2) = 0 \tag{65}$$

Similarly:

$$\mathbf{y}_2^*(\mathbf{x}_2, \mathbf{p}_2) = \mathbf{x}_2 \tag{66}$$

$$\mathbf{x}_3^*(\mathbf{x}_2, \mathbf{p}_2) = 0 \tag{67}$$

We now fill the last columns (corresponding to period 2) of Tables 13, 14 and 15. These values are identical to those in the previous two examples.

Next we solve equations 9, 10 and 11 for $k = 1$.

$$\mathbf{J}_1^*(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1, \mathbf{x}_2) = \mathbf{p}_1\mathbf{y}_1 + \mathbf{E}\left[\mathbf{J}_2^*\left(\mathbf{x}_2, \mathbf{p}_2, \mathbf{y}_2^*(\mathbf{x}_2, \mathbf{p}_2)\right)|\mathbf{p}_1\right] \tag{68}$$

$$\mathbf{J}_1^*(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1) = \max_{\mathbf{x}_2 \in \mathcal{X}_2 \cap \mathcal{T}_1(\mathbf{x}_1, \mathbf{y}_1)} \mathbf{J}_1^*(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1, \mathbf{x}_2) \tag{69}$$

$$\mathbf{x}_2^*(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1) = \operatorname*{argmax}_{\mathbf{x}_2 \in \mathcal{X}_2 \cap \mathcal{T}_1(\mathbf{x}_1, \mathbf{y}_1)} \mathbf{J}_2^*(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1, \mathbf{x}_2) \tag{70}$$

Using equations 64, 66 and 68, the fact that price transitions from period 1 to period 2 are independent and occur with equal probability, and using the prices from Table 12, we derive the following equation:

$$\mathbf{J}_1^*(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1, \mathbf{x}_2) = \mathbf{p}_1\mathbf{y}_1 + \mathbf{E}\left[\mathbf{p}_2\mathbf{x}_2|\mathbf{p}_1\right] = \mathbf{p}_1\mathbf{y}_1 + (0.5)15\mathbf{x}_2 + (0.5)40\mathbf{x}_2 \tag{71}$$

We now illustrate results for $\mathbf{x}_1$ = FULL; the other cases are done similarly. The three possible state transitions from $\mathbf{x}_1$ = FULL = 100 MWh are either $\mathbf{x}_2$ = FULL = 100 MWh or $\mathbf{x}_2$ = HALF = 50 or $\mathbf{x}_2$ = EMPTY = 0 MWh; these dispatch values correspond to $\mathbf{y}_1 = 0$, $\mathbf{y}_1 = 50$ MW, and $\mathbf{y}_1 = 100$ MW respectively. Therefore, for each value of $\mathbf{y}_1$ there is only one possible state transition. This implies that:

$$\mathbf{J}_1^*(\mathbf{x}_1, \mathbf{p}_1, 0) = 2750 \quad (72)$$
$$\mathbf{J}_1^*(\mathbf{x}_1, \mathbf{p}_1, 50) = 50\mathbf{p}_1 + 1375 \quad (73)$$
$$\mathbf{J}_1^*(\mathbf{x}_1, \mathbf{p}_1, 100) = 100\mathbf{p}_1 \quad (74)$$
$$\mathbf{x}_2^*(\mathbf{x}_1, \mathbf{p}_1, 0) = \text{FULL} \quad (75)$$
$$\mathbf{x}_2^*(\mathbf{x}_1, \mathbf{p}_1, 50) = \text{HALF} \quad (76)$$
$$\mathbf{x}_2^*(\mathbf{x}_1, \mathbf{p}_1, 100) = \text{EMPTY} \quad (77)$$

To find the optimal non-decreasing bidding strategy (equation 12), we use the nested backward DP method of Section 4.2. The analog of Table 9 is Table 16 (corresponding to $\mathbf{x}_1 = \text{FULL}$) representing $\mathbf{J}_1^*(\text{FULL}, \mathbf{p}_1, \mathbf{y}_1)$. From this table, we evaluate the probability weighted values $M(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1) = \mathbf{prob}(\mathbf{p}_1)\mathbf{J}_1^*(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1)$ for $\mathbf{x}_1 = \text{FULL}$; this is displayed in Table 17.

Table 18 shows the entries of V for $\mathbf{x}_1 = \text{FULL}$. The last column of Table 18 has the same values as the last column of Table 17. Each entry in Table 18 is the corresponding entry in Table 17 *plus* the maximum value in the next column of Table 18, where the maximum is taken over all rows below and including the entry; this is needed to meet the non-decreasing biddability requirement. The maximum values found by following equations 57, 58 are highlighted in **boldface** in Table 18. The dispatch levels corresponding to the **boldface** entries in Table 18 form the optimal bidding strategy that satisfies the biddability constraint.

Table 13: Optimal dispatch strategy $\mathbf{y}_k^*(\mathbf{x}_k, \mathbf{p}_k)$ in MW implemented *during* period k

	Time Period	
$(\mathbf{x}_k, \mathbf{p}_k)$	1	2
(EMPTY,LOW)	0	0
(EMPTY,HIGH)	0	0
(HALF,LOW)	0	50
(HALF,HIGH)	50	50
(FULL,LOW)	0	100
(FULL,HIGH)	100	100

Therefore, when the generator state $\mathbf{x}_1 = \text{FULL}$ and the price $\mathbf{p}_1 = \text{LOW}$, the optimal dispatch strategy is to dispatch $\mathbf{y}_1 = 0$ MW; when the price $\mathbf{p}_1 = \text{HIGH}$ the optimal dispatch is $\mathbf{y}_1 = 100$ MW. Hence,

$$\mathbf{y}_1^*(\text{FULL}, \text{LOW}) = 0 \text{ MW} \quad (78)$$
$$\mathbf{y}_1^*(\text{FULL}, \text{HIGH}) = 100 \text{ MW} \quad (79)$$

From equations 13, 75 and 77, we get the following optimal state transition strategies:

$$\mathbf{x}_2^*(\text{FULL}, \text{LOW}) = \mathbf{x}_2^*(\text{FULL}, \text{LOW}, \mathbf{y}_1^*(\text{FULL}, \text{LOW})) = \text{FULL} \quad (80)$$
$$\mathbf{x}_2^*(\text{FULL}, \text{HIGH}) = \mathbf{x}_2^*(\text{FULL}, \text{HIGH}, \mathbf{y}_1^*(\text{FULL}, \text{HIGH})) = \text{EMPTY} \quad (81)$$

Table 14: Optimal state transition strategy $\mathbf{x}_{k+1}^*(\mathbf{x}_k, \mathbf{p}_k)$, implemented at the *end* of period k

	Time Period	
$(\mathbf{x}_k, \mathbf{p}_k)$	1	2
(EMPTY,LOW)	EMPTY	EMPTY
(EMPTY,HIGH)	EMPTY	EMPTY
(HALF,LOW)	HALF	EMPTY
(HALF,HIGH)	EMPTY	EMPTY
(FULL,LOW)	FULL	EMPTY
(FULL,HIGH)	EMPTY	EMPTY

Table 15: Values of $\mathbf{J}_k^*(\mathbf{x}_k, \mathbf{p}_k, \mathbf{y}_k^*(\mathbf{x}_k, \mathbf{p}_k))$ in \$

	Time Period	
$(\mathbf{x}_k, \mathbf{p}_k)$	1	2
(EMPTY,LOW)	0	0
(EMPTY,HIGH)	0	0
(HALF,LOW)	0	750
(HALF,HIGH)	1500	2000
(FULL,LOW)	2700	1500
(FULL,HIGH)	3000	4000

Table 16: Values of $\mathbf{J}_1^*(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1)$ in \$ for $\mathbf{x}_1$ = FULL

	$\mathbf{p}_1$	
$\mathbf{y}_1$ in MW	LOW (\$20/MWh)	HIGH (\$30/MWh)
0	2750	2750
50	2375	2875
100	2000	3000

Table 17: Values of $M = \mathbf{prob}(\mathbf{p}_1)\mathbf{J}_1^*(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1)$ in \$ for $\mathbf{x}_1$ = FULL. The probability that $\mathbf{p}_1$ = LOW is 0.5; the probability that $\mathbf{p}_1$ = HIGH is 0.5

	$\mathbf{p}_1$	
$\mathbf{y}_1$ in MW	LOW (\$20/MWh)	HIGH (\$30/MWh)
0	1375	1375
50	1187.5	1437.5
100	1000	1500

Table 18: Values of V in \$ (from equations 55 to 58)

	$\mathbf{p}_1$	
$\mathbf{y}_1$ in MW	LOW	HIGH
0	**2875**	1375
50	2687.5	1437.5
100	2500	**1500**

This fills the entries in Tables 13 and 14 for period 1 corresponding to the (FULL, LOW) and (FULL, HIGH) cases. Other entries can be similarly filled.

From equations 72 and 74 and the optimal dispatch strategy in period 1, we can now derive the values for expected optimal cumulative profits $\mathbf{J}_1^*(\mathbf{x}_1 = \text{FULL}, \mathbf{p}_1, \mathbf{y}_1^*(\mathbf{x}_1, \mathbf{p}_1)))$; these are shown as the last two entries of column 1 (corresponding to period 1) in Table 15. The remainder of column 1 can be obtained the same way (but it is, strictly speaking, not necessary in this examples since only the FULL case is being considered in period $k = 1$).

The optimal bidding strategy is therefore:

- In period 1, offer 100 MW at any price greater than LOW and lower than HIGH; i.e., offer 100 MW at a price greater than \$20/MWh and lower than \$30/MWh.
- In period 2 offer all remaining output (depending on what output got selected in period 1) at a price lower than LOW; i.e., at a price lower than \$15/MWh.

The lack of price correlation results in a different dispatch and bidding strategy from the previous two cases. It also results in a different expected value of profits, which for this example, is $0.5 \cdot 2700 + 0.5 \cdot 3000 = \$2850$. This value is higher than the value obtained when perfect correlation is assumed (\$2750), but lower than the value when correlation is assumed and there are no restrictions on bidding strategy (\$3000).

The market design implications of the examples so far are that rules make a difference in obtaining the optimal bidding strategy. A change in the bidding rules changes the optimal bids, as illustrated in the previous example. However, assumptions about price forecast uncertainty also make a difference, as illustrated in this example, even if the rules remain the same.

4.4 Thermal Generator with One Reserve Market: Sequential Market Clearing, Forecast Uncertainty, Non-Decreasing Bid Function

We now assume a single-period case and two markets, one for energy and one for reserves. Suppose that a price-taking generator has a capacity of 100 MW, with a minimum power of 0 MW, constant incremental costs of \$30/MWh and no other costs, and can offer reserves up to 40 MW. Suppose that there are no *direct costs* for making reserves available[4]. Suppose that the market design is such that the energy market clears first. *Then*, the reserve bids are accepted and the reserve market clears. All markets have uniform clearing prices. In each

[4]We make the simplifying assumption that the reserves are not called; hence, the generator only gets the reserve availability price.

market, it is required that the generator bid quantity be a non-decreasing function of bid price.

Suppose that the generator forecasts that the energy price will be either \$35/MWh or \$40/MWh, with equal probability. The reserve price is dependent on energy price as follows: if energy price is \$40/MWh, then the reserve availability price is also \$12/MW/h; if energy price is \$35/MWh, then the reserve availability price is \$4/MW/h. For notational convenience, we refer to the low prices in each market to be the LOW price and the high price in each market to be the HIGH price; for example the LOW price in the energy market is \$35/MW, while the LOW price in the reserve market is \$4/MW/h.

What is the generator's optimal bidding strategy in the energy and reserve markets if the objective is to maximize expected profits?

There are two auction rounds; i.e., $K = 2$. Let $k = 1$ refer to the energy market auction and $k = 2$ refer to the reserve market auction. We define the state $\mathbf{x}_k$ as follows: $\mathbf{x}_1$ refers to the capacity left in the energy market and $\mathbf{x}_2$ to be the generation capacity left in the reserve market. The term $\mathbf{x}_3$ refers to the generation capacity left after the energy and reserve markets are cleared. The term $\mathbf{y}_1$ refers to the energy dispatch and $\mathbf{y}_2$ refers to the reserve availability dispatch. Similarly $\mathbf{p}_1$ and $\mathbf{p}_2$ represent the energy price and reserve availability price respectively.

We use the DP formulation of Section 3.

$$\max_{\mathbf{y}_k=\mathbf{y}_k(\mathbf{x}_k,\mathbf{p}_k)} \mathbf{E}\left[\sum_{k=1}^{2} \mathbf{p}_k\mathbf{y}_k - 30\mathbf{y}_1\right] \tag{82}$$

subject to

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \mathbf{y}_k \tag{83}$$

$$\mathbf{y}_1 \le \min(100, \mathbf{x}_1) \tag{84}$$

$$\mathbf{y}_2 \le \min(40, \mathbf{x}_2) \tag{85}$$

$$\mathbf{x}_1 = 100 \tag{86}$$

$$\mathbf{y}_k \text{ is a non-decreasing function of price } \mathbf{p}_k, \quad k = 1, 2 \tag{87}$$

We now solve this problem using the methods described in Section 3. In DP terminology (given in the appendix), we associate the notion of "market" with the notion of "stage." As is normal procedure in the solution of DP problems, we work backwards from the final stage (the reserve market) to the first stage (the energy market), we fill the columns in reverse order from right to left in the Tables below.

Starting with the final stage (the reserve market):

$$\mathbf{J}_2^*(\mathbf{x}_2, \mathbf{p}_2, \mathbf{y}_2, \mathbf{x}_3) = \mathbf{p}_2\mathbf{y}_2 \tag{88}$$

$$\mathbf{J}_2^*(\mathbf{x}_2, \mathbf{p}_2, \mathbf{y}_2) = \mathbf{p}_2\mathbf{y}_2 \tag{89}$$

$$\mathbf{x}_3^*(\mathbf{x}_2, \mathbf{p}_2, \mathbf{y}_2) = \mathbf{x}_2 - \mathbf{y}_2 \tag{90}$$

If $\mathbf{x}_2$ is the capacity left in this market, it is optimal to sell all this possible output (up to the 40 MW limit) into the reserve market at any price. Therefore,

$$\mathbf{y}_2^*(\mathbf{x}_2, \mathbf{p}_2) = \min(\mathbf{x}_2, 40) \ \forall \ \mathbf{p}_2 \ge 0 \tag{91}$$

$$\mathbf{x}_3^*(\mathbf{x}_2, \mathbf{p}_2) = \mathbf{x}_2 - \min(\mathbf{x}_2, 40) \tag{92}$$

We assume that $\mathbf{p}_2 \geq 0$.

Working backwards to the energy market ($k = 1$), there is $\mathbf{x}_1$ capacity left. The next (backward) step of the DP is to solve the following:

$$\mathbf{J}_1^*(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1, \mathbf{x}_2) = (\mathbf{p}_1 - 30)\mathbf{y}_1 + \mathbf{E}\left[\mathbf{J}_2^*\left(\mathbf{x}_2, \mathbf{p}_2, \mathbf{y}_2^*(\mathbf{x}_2, \mathbf{p}_2)\right) | \mathbf{p}_1\right] \quad (93)$$

$$\mathbf{J}_1^*(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1) = \max_{\mathbf{x}_2 \in \mathcal{X}_2 \cap \mathcal{T}_1(\mathbf{x}_1, \mathbf{y}_1)} \mathbf{J}_1^*(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1, \mathbf{x}_2) \quad (94)$$

$$\mathbf{x}_2^*(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1) = \operatorname*{argmax}_{\mathbf{x}_2 \in \mathcal{X}_2 \cap \mathcal{T}_1(\mathbf{x}_1, \mathbf{y}_1)} \mathbf{J}_2^*(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1, \mathbf{x}_2) \quad (95)$$

$\mathbf{p}_2$ and $\mathbf{p}_1$ are perfectly correlated in this example, so when the energy price $\mathbf{p}_1$ is HIGH (LOW), the reserve price $\mathbf{p}_2$ is also HIGH (LOW).

We now want to find a function $\mathbf{y}_1^*(\mathbf{x}_1, \mathbf{p}_1)$ that is a non-decreasing function of price $\mathbf{p}_1$ that maximizes $\mathbf{E}\left[\mathbf{J}_1^*(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1)\right]$ where the expectation $\mathbf{E}$ is over $\mathbf{p}_1$. We did a similar optimization in Section 4.2 (e.g., see Tables 9, 10 and 11). We use the same optimization approach here using discrete values: $\mathbf{y}_1 = 0, 20, 40, 60, 80, 100$ MW. The results are in Tables 19, 20 and 21.

Table 19: Values of $\mathbf{J}_1^*(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1)$ in \$ for $\mathbf{x}_1 = 100$ MW

	$\mathbf{p}_1$	
$\mathbf{y}_1$ in MW	LOW (\$35/MWh)	HIGH (\$40/MWh)
0	160	480
20	260	680
40	360	880
60	460	1080
80	480	1040
100	500	1000

Table 20: Values of $M = \mathbf{prob}(\mathbf{p}_1)\mathbf{J}_1^*(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1)$ in \$ for $\mathbf{x}_1 = 100$ MW. The probability that $\mathbf{p}_1$ = LOW is 0.5; the probability that $\mathbf{p}_1$ = HIGH is 0.5

	$\mathbf{p}_1$	
$\mathbf{y}_1$ in MW	LOW (\$35/MWh)	HIGH (\$40/MWh)
0	80	240
20	130	340
40	180	440
60	230	540
80	240	520
100	250	500

Table 21 shows that $\mathbf{E}\left[\mathbf{J}_1^*(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1)\right]$ is maximized when $\mathbf{y}_1$ is 60 MW, regardless of whether the energy price is LOW or HIGH. Therefore,

$$\mathbf{y}_1^*(100, \mathbf{p}_1) = 60 \quad (96)$$

$$\mathbf{x}_2^*(100, \mathbf{p}_1) = 100 - 60 = 40 \quad (97)$$

Table 21: Values of V in \$

	$\mathbf{p}_1$	
$\mathbf{y}_1$ in MW	LOW (\$35/MWh)	HIGH (\$40/MWh)
0	620	240
20	670	340
40	720	440
60	**770**	**540**
80	760	520
100	750	500

Based on this solution, the optimal strategy is as follows:

1. (Energy market) Bid 60 MW into the energy market at a price lower than the LOW energy price of \$35/MWh (say, at the incremental cost of \$30/MWh);
2. (Reserve market) Bid the remaining 40 MW into the reserve market at a price lower than the LOW reserve price of \$4/MW/h.

4.5 Thermal Generator with One Reserve Market: Sequential Market Clearing, Forecast Uncertainty, Non-Decreasing Bid Function

We use the same assumptions as Example 4.4, except that the energy and reserve prices are uncorrelated, and the reserve price has equal probability of being HIGH or LOW. Tables 22, 23 and 24 give the results of the optimization.

Table 22: Values of $\mathbf{J}_1^*(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1)$ in \$ for $\mathbf{x}_1 = 100$ MW

	$\mathbf{p}_1$	
$\mathbf{y}_1$ in MW	LOW (\$35/MWh)	HIGH (\$40/MWh)
0	320	320
20	420	520
40	520	720
60	620	920
80	560	960
100	500	1000

Table 24 shows that $\mathbf{E}(\mathbf{J}_1^*(\mathbf{x}_1 = 100, \mathbf{p}_1, \mathbf{y}_1)$ is maximized by $\mathbf{y}_1 = 60$ MW when the energy price is LOW; and by $\mathbf{y}_1 = 100$ MW when the energy price HIGH. Therefore,

$$\mathbf{y}_1^*(100, \mathbf{p}_1 = \text{HIGH}) = 100 \quad (98)$$

$$\mathbf{y}_1^*(100, \mathbf{p}_1 = \text{LOW}) = 60 \quad (99)$$

$$\mathbf{x}_2^*(100, \mathbf{p}_1 = \text{HIGH}) = 100 - 100 = 0 \quad (100)$$

$$\mathbf{x}_2^*(100, \mathbf{p}_1 = \text{LOW}) = 100 - 60 = 40 \quad (101)$$

Table 23: Values of $M = \mathbf{prob}(\mathbf{p}_1)\mathbf{J}_1^*(\mathbf{x}_1, \mathbf{p}_1, \mathbf{y}_1)$ in \$ for $\mathbf{x}_1 = 100$ MW. The probability that $\mathbf{p}_1 =$ LOW is 0.5; the probability that $\mathbf{p}_1 =$ HIGH is 0.5

	$\mathbf{p}_1$	
$\mathbf{y}_1$ in MW	LOW (\$35/MWh)	HIGH (\$40/MWh)
0	160	160
20	210	260
40	260	360
60	310	460
80	280	480
100	250	500

Table 24: Values of V in \$

	$\mathbf{p}_1$	
$\mathbf{y}_1$ in MW	LOW (\$35/MWh)	HIGH (\$40/MWh)
0	660	160
20	710	260
40	760	360
60	**810**	460
80	780	480
100	750	**500**

Based on this solution, the optimal strategy is as follows:

1. (Energy Market) Bid two stairs. In the first stair, 60 MW is bid into the energy market at a price lower than the LOW energy price of \$35/MW (say, at the incremental cost of \$30/MW). In the second stair, bid 40 MW into the market at a price that is higher than the first stair price, and lower than the HIGH energy price of \$40/MW. For example, in the first stair, bid 60 MW at \$30/MW, and bid an additional 40 MW at \$38/MW in the second stair.

2. (Reserve Market) Bid whatever is not accepted in the energy market (up to 40 MW) at a price that is lower than the LOW reserve availability price of \$4/MW/h to ensure its acceptance.

5 Observations and Applications

Based on our analyses and examples, we conclude that it is critically important to carefully represent costs, cost predictions, operational restrictions and other such considerations when analyzing bidding behavior. Our analyses also helps explain recent market failures.

1. Our DP optimization can be applied to practical problems of interest. For example, in the California PX (and California ISO) day-ahead market, first the energy market would clear for each of the 24 hours for the next day and then the regulation reserve market would clear for each of the 24 hours, followed by spinning reserves, supplemental reserves, and finally backup reserves. This auction structure can be set up as a DP problem. There are a total of $24 \cdot 5 = 120$ auctions. The DP problem can be solved by starting from the final time period ($T = 24$). Within this time period, we start from the final clearing market (backup reserves), and work backwards to the energy market (see Sections 4.4 and 4.5). This gives us the optimal bidding strategy as a function of state for the final time period for each of the 5 markets. We now work backwards from the final time period one time period at a time. Within each time period, we work sequentially backwards from the backup reserve market to the energy market. Carrying out this procedure we get the optimal bidding strategy as a function of state and price for each time period for each of the 5 markets.

2. The optimal bidding strategy is a function of generator state and anticipated price. This is essential to avoid physical operational inconsistencies. For example, suppose that a generator has two states, ON and OFF, and that the generator is constrained to be ON for at least two periods. Let there be two time periods. Suppose that the optimal bidding strategy is that, (for both time periods) when in the ON state, the optimal bid is $35/MWh for its output; of course, in the OFF state the generator makes no bid. If the auction rules are such that the generator is forced to make a bid that is independent of generator state, then it is possible that the bids could lead to inconsistent physical operation conditions. For example, suppose the generator offers positive MW in both time periods and the generator has to be ON for at least two periods. If this bid is accepted in period 1 and rejected in period 2, then we have a physical operational inconsistency.

3. When the auction structure mandates that the generator offer bids that are independent of generator state, then the optimal bidding strategy can be found as follows. First, the generator state dependent optimal bidding strategy (i.e., the optimal dispatch $\mathbf{y}_k^*(\mathbf{x}_k, \mathbf{p}_k)$ as well as optimal state transition $\mathbf{x}_{k+1}^*(\mathbf{x}_k, \mathbf{p}_k)$, is found using the methods in this report). Based on the price distribution probability and the state transition strategy, we can find the probability $\mathbf{prob}(\mathbf{x}_k, k)$ of being in state $\mathbf{x}_k$ in auction round k. Then, we can find the optimal bidding strategy $\widehat{\mathbf{y}_k^*}(\mathbf{p}_k)$ that is independent of state by solving the following least-squares problem:

$$\underset{\widehat{\mathbf{y}_k^*}(\mathbf{p}_k)}{\operatorname{argmax}} \mathbf{E}\left[\left(\mathbf{y}_k^*(\mathbf{x}_k, \mathbf{p}_k)\mathbf{prob}(\mathbf{x}_k, k) - \widehat{\mathbf{y}_k^*}(\mathbf{p}_k)\right)^2\right] \quad \text{for} \quad 1 \le k \le K \tag{102}$$

The solution to this problem may be found by using DP methods. This does not guarantee that the bidding strategy is physically consistent, but it does ensure that the optimal bidding strategy is as close as possible to the optimal[5].

4. The mathematics presented here assumes a one-part bid, where MWs are bid as a function of price. In markets such as the Pennsylvania-Maryland-Jersey and New York systems, three-part bids are allowed for day-ahead markets. Three-part bids include start-up/shut-down costs, no-load costs, as well as MW versus price bids (in addition to all the physical operational constraints). While the mathematics in this report are applicable to three-part bids, if a price-taker costs of operation match the three-part bid structure, a profit-maximizing price-taker simply bids true costs and operational constraints; the DP formulation in our report is not necessary in such cases. Nevertheless, when there are additional costs and constraints that are not directly biddable, the methods described in this report can be applied to find optimal bidding strategies. One particular case where it becomes difficult for a price-taker to fit into the structure of a multi-part bid is the case of hydroelectric generators that have storage capability. Another case is the situation of generators that have operational constraints that extend beyond the time-frame covered by the auction.

[5] In practice, physically inconsistent bids are resolved by buying/selling from an electricity pool in real time to make up for deficits/surpluses.

6 Generator is Not a Price-Taker

In this section, we relax the assumption that the generator is a price-taker. When considering the potential impact of a bid on the price, one alternative is to try to take into consideration the bids of all generators in the market. This can be a complex and daunting task. We model instead the effect of the generator's output on price using the notion of residual demand curve (Samuelson and Nordhaus 1998), which is the market demand minus the output of the other generators.

It is easier to measure the residual demand curve facing a market participant than to simulate the bidding behavior of all the other generators in the market. This is because the use of a residual demand curve decouples the problem. The impact of all other generators (and even the effect of loads) can be aggregated into a single curve. Then, and in a separate step, this residual demand curve is used in the bid analysis.

When a generator bids against a residual demand curve, the mathematics in Section 3 remain unchanged; the only difference is that price is now a function of the output of the generator (rather than being independent of the output), and this must be accounted for.

The following example illustrates the basic approach. It also shows one way to model uncertain residual demand curves.

6.1 Thermal Generator with Market Power Under Some Price Regimes, Single Energy Market, Forecast Uncertainty, Non-Decreasing Bid Function

We assume a single-period case. A thermal generator has constant incremental costs of \$25/MWh, maximum capacity of 100 MW and minimum output of zero MW. The generator has no start-up/shut-down costs or operational constraints. A uniform price auction is assumed.

The generator expects that the price "regime" in the period will be either LOW or HIGH with equal probability. If the LOW price regime happens, the generator has no market power (i.e., it expects a horizontal residual demand curve) and it expects the price will be \$30/MWh.

On the other hand, if the price regime is HIGH, the generator estimates that it has the potential to exercise market power. It estimates that the market price as a function of its output can be expressed as in Table 25.

Table 25: Price as a function of generator output corresponding to the HIGH price regime (for the LOW price regime the price is simply \$30.)

Output $\mathbf{y}_1$ (in MW)	0	25	50	75	100
Price $\mathbf{p}_1$ (in \$/MWh)	50	45	43	41	35

What should be the optimal generator bidding strategy if the generator can only bid non-decreasing costs in the auction? Since there is only one state, we suppress the dependence of

$\mathbf{x}_k$ for notational convenience; the generator has to solve the following problem:

$$\mathbf{J}_1^*(\mathbf{p}_1, \mathbf{y}_1) = \mathbf{p}_1\mathbf{y}_1 - 25\mathbf{y}_1 \quad (103)$$

$$\mathbf{y}_1^*(\mathbf{p}_1) = \underset{\mathbf{y}_1(\mathbf{p}_1)\in\mathcal{Y}_1(\mathbf{p}_1)}{\text{argmax}} \; \mathbf{E}\left[\mathbf{J}_1^*\left(\mathbf{p}_1, \mathbf{y}_1(\mathbf{p}_1)\right)\right] \quad (104)$$

where the expectation operation $\mathbf{E}$ is over the price regimes HIGH and LOW. The allowed set $\mathcal{Y}_1(\mathbf{p}_1)$ is a non-decreasing function of $\mathbf{p}_1$, and must follow Table 25 when the price regime is HIGH.

To satisfy the non-decreasing bid curve requirement, we will use the backward DP method described in Example 4.2 in equations 51 to 58, taking account of the residual demand curve in Table 25 when the price regime is HIGH. The results are given in Tables 26 to 28.

Table 26: Values of $\mathbf{J}_1^*(\mathbf{p}_1, \mathbf{y}_1)$ in \$. In the HIGH price regime, the feasible output $\mathbf{y}_1$ is constrained to be along the diagonal to be consistent with Table 25

	Price Regime (prices $\mathbf{p}_1$ per MWh)					
	LOW	HIGH				
$\mathbf{y}_1$ in MW	\$30	\$35	\$41	\$43	\$45	\$50
0	0					0
25	125				500	
50	250			900		
75	375		1200			
100	500	1000				

Table 27: Values of $M = \mathbf{prob}(\mathbf{p}_1)\mathbf{J}_1^*(\mathbf{p}_1, \mathbf{y}_1)$ in \$. The probability that $\mathbf{p}_1$ is in a LOW regime is 0.5; the probability that $\mathbf{p}_1$ is in a HIGH regime is 0.5

	Price Regime (prices $\mathbf{p}_1$ per MWh)					
	LOW	HIGH				
$\mathbf{y}_1$ in MW	\$30	\$35	\$41	\$43	\$45	\$50
0	0					0
25	62.5				250	
50	125			450		
75	187.5		600			
100	250	500				

Columns 1 and 2 in Table 28 show that the solution to equation 104 is $\mathbf{y}_1^*(\mathbf{p}_1) = 75$ MW (corresponding to the 75 MW row in the table) for both the LOW and HIGH price regimes (**boldface** entries). For the LOW price regime, this corresponds to $\mathbf{p}_1 = \$30$/MWh; for the HIGH price regime, this corresponds to $\mathbf{p}_1 = \$41$/MWh. That is,

$$\mathbf{y}_1^*(\mathbf{p}_1 = \text{LOW}) = 75 \quad (105)$$

$$\mathbf{y}_1^*(\mathbf{p}_1 = \text{HIGH}) = 75 \quad (106)$$

Table 28: Values of V in \$ (see equations 55 to 58)

	Price Regime (prices $\mathbf{p}_1$ per MWh)					
	LOW	HIGH				
$\mathbf{y}_1$ in MW	\$30	\$35	\$41	\$43	\$45	\$50
0	600					0
25	662.5				250	
50	725			450		
75	**787.5**		**600**			
100	750	500				

Since the generator is a price-taker in the LOW price regime, but a potential price setter in the HIGH price regime, the optimal bidding strategy is as to bid *two* stairs. In the first stair, $(75 - \Delta)$ MW is bid into the energy market at a price lower than the LOW energy price of \$35/MWh (perhaps at the incremental cost of \$25/MWh), where Δ is sufficiently small MW quantity. In the second stair, Δ MW is bid into the market at a price of \$41/MWh. For sufficiently small Δ, this is the optimal strategy with expected profits approaching \$787.50 $(= \mathbf{E}(\mathbf{J}_1^*(\mathbf{p}_1, \mathbf{y}_1^*(\mathbf{p}_1)) = 0.5 \cdot \mathbf{J}_1^*(\text{LOW}, 75) + 0.5 \cdot \mathbf{J}_1^*(\text{HIGH}, 75) = 0.5 \cdot 375 + 0.5 \cdot 1200$ from the entries corresponding to the 75 MW row in Table 26). However, the effect of the bidding strategy is that when the price regime is LOW, the generator only produces 75 MW; with perfect hindsight of the LOW price regime, the generator could have produced 100 MW and increased its profits.

6.2 Thermal Generator with Market Power Under All Price Regimes, Single Energy Market, Forecast Uncertainty, Non-Decreasing Bid Function

Now we consider the same problem as in Section 6.1, with the exception that the price regimes are different. As before, the generator expects that the price "regime" in the period will be either LOW or HIGH with equal probability. But the generator forecasts that it has the potential to exercise market power in both regimes.

If the LOW price regime happens, it estimates that the market price (as a function of its output) is expressed in Table 29. On the other hand, if the price regime is HIGH, the generator estimates that the market price (as a function of its output) is expressed in Table 30.

Table 29: Price as a function of generator output corresponding to the LOW price regime

Output $\mathbf{y}_1$ (in MW)	0	25	50	75	100
Price $\mathbf{p}_1$ (in \$/MWh)	48	42	39	33	26

What should be the optimal generator bidding strategy if it can only bid non-decreasing costs in the auction, which is assumed to be uniform-price?

Table 30: Price as a function of generator output corresponding to the HIGH price regime

Output $\mathbf{y}_1$ (in MW)	0	25	50	75	100
Price $\mathbf{p}_1$ (in \$/MWh)	100	77	43	41	35

As before, we get

$$\mathbf{J}_1^*(\mathbf{p}_1, \mathbf{y}_1) = \mathbf{p}_1\mathbf{y}_1 - 25\mathbf{y}_1 \tag{107}$$

$$\mathbf{y}_1^*(\mathbf{p}_1) = \underset{\mathbf{y}_1(\mathbf{p}_1)\in\mathcal{Y}_1(\mathbf{p}_1)}{\text{argmax}} \; \mathbf{E}\left[\mathbf{J}_1^*\left(\mathbf{p}_1, \mathbf{y}_1(\mathbf{p}_1)\right)\right] \tag{108}$$

where the expectation operation $\mathbf{E}$ is over the price regimes HIGH and LOW. Again, the allowed set $\mathcal{Y}_1(\mathbf{p}_1)$ is a non-decreasing function of $\mathbf{p}_1$ and must follow Tables 29 and 30 when the price regime is LOW and HIGH respectively.

To satisfy the non-decreasing bid curve requirement, we will use the backward DP method described in Example 4.2 in equations 51 to 58, taking account of the residual demand curves in Tables 29 and 30. Tables 28 are Tables 31 to 33 give the results.

Table 31: Values of $\mathbf{J}_1^*(\mathbf{p}_1, \mathbf{y}_1)$ in \$. Note that the HIGH and LOW price regimes are interspersed; we use *italics* to display entries corresponding to the LOW price regime

	Price Regime (prices $\mathbf{p}_1$ per MWh)									
	LOW and HIGH									
$\mathbf{y}_1$ in MW	*\$26*	*\$33*	\$35	*\$39*	\$41	*\$42*	\$43	*\$48*	\$77	\$100
0								*0*		0
25						*425*			1300	
50				*700*			900			
75		*600*			1200					
100	*100*		1000							

Table 32: Values of $M = \mathbf{prob}(\mathbf{p}_1)\mathbf{J}_1^*(\mathbf{p}_1, \mathbf{y}_1)$ in \$. The probability that $\mathbf{p}_1$ is in a LOW regime is 0.5; the probability that $\mathbf{p}_1$ is in a HIGH regime is 0.5. The HIGH and LOW price regimes are interspersed; we use *italics* to display entries corresponding to the LOW price regime.

	Price Regime (prices $\mathbf{p}_1$ per MWh)									
	LOW and HIGH									
$\mathbf{y}_1$ in MW	*\$26*	*\$33*	\$35	*\$39*	\$41	*\$42*	\$43	*\$48*	\$77	\$100
0								*0*		0
25						*212.5*			650	
50				*350*			450			
75		*300*			600					
100	*50*		500							

Table 33: Values of V in \$ (see equations 55 to 58). The HIGH and LOW price regimes are interspersed; since in both regimes the feasible output $\mathbf{y}_1$ is constrained to be along the diagonal, we only need to pick one argmax value per regime in equations 57 and 58

	Price Regime (prices $\mathbf{p}_1$ per MWh)									
	LOW and HIGH									
$\mathbf{y}_1$ in MW	*\$26*	*\$33*	\$35	*\$39*	\$41	*\$42*	\$43	*\$48*	\$77	\$100
0								650		0
25						862.5			650	
50				**950**			450			
75		900			**600**					
100	550		500							

Columns 1 and 2 of Table 33 show that the solution to equation 108 is $\mathbf{y}_1^*(\mathbf{p}_1 = \text{LOW}) = 50$ MW for the LOW price regime, and $\mathbf{y}_1^*(\mathbf{p}_1 = \text{HIGH}) = 75$ MW for the HIGH price regime. For the LOW price regime, this corresponds to $\mathbf{p}_1 = \$39/\text{MWh}$; for the HIGH price regime, this corresponds to $\mathbf{p}_1 = \$41/\text{MWh}$. That is,

$$\mathbf{y}_1^*(\mathbf{p}_1 = \text{LOW}) \;=\; 50 \tag{109}$$

$$\mathbf{y}_1^*(\mathbf{p}_1 = \text{HIGH}) \;=\; 75 \tag{110}$$

Since the generator is a price-setter in both the HIGH and LOW price regimes, the optimal bidding strategy is to bid *two* stairs. In the first stair, 50 MW is bid into the energy market at the LOW energy price of \$39/MW. In the second stair, 25 MW is bid into the market at a price of \$41/MW. This is the optimal strategy with expected profits equal to \$950 $(= \mathbf{E}(\mathbf{J}_1^*(\mathbf{p}_1, \mathbf{y}_1^*(\mathbf{p}_1)) = 0.5 \cdot \mathbf{J}_1^*(\text{LOW}, 50) + 0.5 \cdot \mathbf{J}_1^*(\text{HIGH}, 75) = 0.5 \cdot 700 + 0.5 \cdot 1200$ from the entries corresponding to the 50 MW and 75 MW rows in Table 31).

Not having perfect foresight of the price regime is costly for the generator. If the generator knows with perfect foresight that the price regime will be HIGH, the generator will maximize profits by only bidding and producing 25 MW.

6.3 Observations

From an analysis of the cases when the market participant is not a price-taker, we make the following observations:

1. The objective function for profit-maximization must include the profits due to *all* the market participant's physical positions as well as pure financial positions. This is because the market participant can manipulate the spot market to increase profits from financial positions[6].

2. In general, it makes more sense to model the residual demand curve as one that faces a market participant's N_G generators rather than an *individual* generator. In such a case,

[6] The converse is not true: if the market participant does not have any control of physical assets in the spot market, then the market participant is a price-taker and cannot affect prices.

the residual demand curve is a function of the aggregate output produced (or withheld) by the market participant's N_G generators. The general way to solve this problem is by "relaxing" the aggregate output constraint using Lagrangian relaxation techniques (i.e., introducing a penalty factor) and then decomposing the problem into N_G subproblems; each of the N_G subproblems can be solved using methods similar to Example 6.1. The penalty factor can then be varied to ensure that the aggregate constraint is met; if not, then the primal-dual Lagrangian steps can be repeated. However, it may be possible to do better.

(a) All other things being equal, a market participant would recognize that withholding Y MW from slightly "in-the-money" generator whose incremental costs are close to the market price is less expensive (in exercising market power) than withholding the same Y MW from generators with cheaper incremental costs. This implies that a smaller set of generators in the market participant's fleet can be modeled as having an influence on price; for example, it would typically not be profit-maximizing to withhold output from a baseload power plant such as a nuclear plant.

(b) As a reasonable approximation, it may be possible to construct a residual demand curve for each individual generator; this curve would typically be more elastic the cheaper the generator's incremental costs. We recommend this approach for most problems.

(c) In certain cases, the structure of the generation cost curves and operational constraints may admit analytical solutions to the optimization problems. For these cases, the optimal strategy could be represented as analytical formulas.

3. Transmission congestion could complicate the analysis, especially if the market participant has generators on both sides of a transmission constraint and therefore could influence both the occurrence of the constraint as well as the price of the constraint. In such a case, modeling prices (or residual demand curves) as exogenous is still the "best" way to solve the problem. Prices are natural candidates for relaxation:

   (a) They allow separability of multi-generator optimization problems into decoupled single generator optimization problems.

   (b) They (unlike competitors' behavior) are directly observable.

4. To account properly for transmission congestion, we must model locational prices that are consistent with the structure of the transmission congestion and the transmission network. Even in a large network, this can be done by modeling the behavior of relatively few price variables (Rajaraman and Alvarado 1998). For example, consider two nodes (West and East) with a single link connecting them. Assume that a market participant has a generator at each node. Generally, the pattern of transmission congestion is West to East; the prices at the West node are typically lower than that at the East node. Then, under the right circumstances, it is possible for the market participant to profitably cause transmission congestion in the link by overproducing (at a loss) at West and recovering this loss by other financial positions, e.g., holding a transmission congestion contract between West and East (Joskow and Tirole 2000). One could then model two price regimes: an UNCONGESTED regime, and a CONGESTED regime,

each having a certain probability of occurrence. In the UNCONGESTED regime, the prices at West and East are perfectly correlated (and differ only due to losses). In this regime, if the market participant does not have market power, then one would choose prices as exogenous inputs; if the market participant potentially has market power, then one would choose (as a reasonable approximation) residual demand curves (each for West and East) as exogenous inputs. In the CONGESTED regime, the congestion price of the link between West and East could be affected by the bidding behavior of the market participant; one could then model residual demand curves at both West and East facing the market participant at both nodes. The problem is now separable into independent problems for West and East; it can then be solved by constructing the equivalent of Tables 26 to 28 for both West and East.

5. There will be practical constraints on the market participant's bidding strategy. For example, the generator bids may not be allowed to change too frequently. As another example, regulatory penalties and legal hurdles may be sufficient disincentive to exercise market power.

6. In Example 6.1, we modeled different price regimes. For multiple periods, price regimes could be modeled as a Markov chain; e.g., if the price regime is HIGH in one period, then there is a certain probability that it will continue to be HIGH (or LOW, CONGESTED, etc.) in the next period.

7. The problem of maximizing profits when the market participant has the potential to exercise market power is a highly complex problem, and there may be a temptation to represent this complexity by modeling all the "bells and whistles" in the optimization. Our opinion is that it is much more important to properly model all the parameters of the bidder properly, but to otherwise simplify the optimization. For example, we recommend the decoupling of the problem by the use of a residual demand curve. Because there is considerable uncertainty about data, it is better to encapsulate this uncertainty into the more easily observed price uncertainty that to attempt a concurrent detailed model of all generators at once. In order to accomplish this, the market participant can begin by assuming that he/she is a price-taker, and later introduce residual demand curves; the expected profits can be plotted as a function of the demand elasticities by solving problems similar to Examples 6.1 and 6.2. It is then a matter of using empirical studies to find out the actual elasticity of the residual demand curve that is facing the market participant[7].

[7] These residual demand curves depend not only on the cost but also the possible bidding behavior of other suppliers, demand response, contingencies and outages, and other effects.

7 Computational Issues

In the DP formulation in this report, we optimized expected profits. The rationale behind this is that electricity markets clear repeatedly every day/hour. If these time periods are homogeneous (i.e., the profit distribution is the same across such periods), then in such cases maximizing expected profits makes sense because the law of large numbers will ensure that over sufficiently long time periods, profits will be maximized; indeed, adding a risk premium term to the profit term only leaves "money on the table" over long periods. However, electricity prices show tremendous variation. While there may be many "typical spring days" in a year, a peak summer period may occur only once or twice. In such cases, it may be necessary to add a risk premium term to penalize for possible losses during these conditions. One possibility would be to penalize financial losses in a period by using a penalty multiplier; this formulation preserves the problem structure and has the added flexibility of penalizing different types of losses differently[8]. Another possibility is to modify the exogenous prices and/or their probabilities of occurrence to make the price forecast more "pessimistic".

Another consideration is the size of the problem, particularly, the number of auction rounds and/or time periods. For generators such as hydroelectric generators that have significant storage capacity, the operational cycle time-frame could exceed a year. In such a case, solving the bidding problem for each hour over the whole year could become computationally intractable. Such problems can be handled using nested optimizations; e.g., the yearly problem is decomposed into a problem with twelve time periods and solved, with each time period representing one month, then the monthly problem for the first time period is decomposed into (say) weekly time periods and solved, and so on. Thus, the sequential optimizations allocate water first by month, and then by week, etc.

We have not discussed how to deduce the exogenous prices or residual demand curves that are needed as inputs in the optimization problem. While estimating these exogenous prices is not a trivial problem, good near-term price forecasts (day-ahead or weekly) can generally be obtained with modest effort.[9] A full discussion is beyond the scope of this report; one possibility is to use historical data and construct near-term price forecasts based on system loads, transmission outages, and other known generator outages.

[8]For example, losses due to no-load costs in a period may be treated differently from start-up/shut-down costs incurred in that period.

[9]It is far more difficult to obtain longer term forecasts; moreover, the quality of the long-term forecasts will tend to be poorer.

8 Conclusions

This report has presented the mathematics for finding the multi-period optimal bidding strategy for a generator in electricity markets under *exogenous* uncertain energy and reserve prices. Our treatment covers thermal, hydroelectric as well as energy-limited generators. We have demonstrated how operational constraints can interact with market design rules, price uncertainty, and non-convexity of costs to affect "allowed" bidding strategies; we feel that this is a neglected feature in work on bidding strategies. We have shown that even if a generator has convex costs, the effect of price uncertainty may be such that the optimal bidding strategy — if there were no restrictions on the bidding function — is to bid a declining bid curve of MW versus price. However, since electricity market auctions typically disallow such bidding behavior (usually the MWs offered must be a non-decreasing function of the bid price), we must change the bidding strategy to satisfy this constraint. We have shown that this optimal bidding strategy problem can be examined by solving multiple-level nested backward DP problems. Moreover, the data needs to find optimal generator bidding strategies are modest and attainable; in particular, the generator does not need to have detailed information about competitors' generators. Only price forecasts and detailed information about one's own costs and operational characteristics are sufficient. The price forecasts may incorporate correlation and uncertainty.

We have also shown how the formulation can be used to find profit-maximizing bidding strategies for generators that have the potential to exercise market power; for this problem, we use a residual demand curve instead of price as an exogenous input.

Knowledge of optimal bidding strategies under both the price-taking as well as price-influencing conditions is useful to generators because they can improve profits using these strategies. On the other hand, it is also useful to a regulator who is testing for the exercise of market power by a generator; knowing the optimal bidding strategy, a regulator can compare the actual generator bids with its optimal price-taking bidding strategy to judge whether a generator behaved anti-competitively. Thus, our approach has important implications for market monitoring and market power determination decisions.

The examples in this report show that, in some sense, there is a tradeoff between market simplicity (evidenced, for example, by allowing only one-part non-declining bids) which can lead to less efficient outcomes, and complexity (for example, allowing multi-part and possibly declining cost bids with possible inter-period restrictions) which can lead to more efficient outcomes. However, a potentially more efficient design that offers greater complexity is likely to negate some of the gains in design due to inefficiencies associated with complexity. The report has suggested that it is important to permit market participants to become self-schedulers in cases where their cost structure and operational constraints do not readily fit the market design.

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Appendix: An Overview of Dynamic Programming

This appendix discusses in slightly more general terms the issues associated with the implementation of Dynamic Programming (DP) and its application to the problems described in this report. It starts with a description of a deterministic DP case, followed by the consideration of uncertainty, and then with a description of nested DP problems.

We assume that the reader has at least a basic familiarity with the DP problem. An excellent treatment of the DP problem is given in the graduate-level textbook by Bertsekas (Bertsekas 1987).

The examples in this appendix are larger than those in the body of the report and they are intended to illustrate DP issues, not bidding concepts in the context of this report. Thus, in most of the examples randomly generated numbers are used. For the sake of expository simplicity, in this appendix we do not concern ourselves directly with issues of optimal dispatch $\mathbf{y}_k^*$. It is sufficient for us to determine the optimal state transition $\mathbf{x}_{k+1}^*$.

Deterministic DP

Problems in DP correspond to optimization problems over a sequence of stages. Each stage often (but not always) corresponds to a time frame. In a DP, the optimization is done sequentially in stages.

For all deterministic DP problems, assume that there is a set of K stages. Examples of stages include (a) time periods, (b) rounds in an auction, or (c) price level.

At each stage k ($k \in \{1 \dots K\}$) of the optimization, the system can be in any of N discrete *states* $\mathcal{X}_k$; that is, $\mathbf{x}_k \in \mathcal{X}$. We formulate the DP problem as a profit maximization problem. For simplicity, we assume that costs $C_k(\mathbf{x}_k) = 0$ for all states $\mathbf{x}_k$ for all stages k. The revenues $R_k(\mathbf{x}_k)$ associated with operating state $\mathbf{x}_k$ is known and given by $R_k(\mathbf{x}_k)$. These values can be arranged as a matrix R of dimension N by K.

We also define the state transition matrix $c_k(\mathbf{x}_k, \mathbf{x}_{k+1})$. For simplicity, we let the transition costs be independent of the stage; i.e., $c(\mathbf{x}_k, \mathbf{x}_{k+1}) \equiv c_k(\mathbf{x}_k, \mathbf{x}_{k+1})$. The transition costs may be represented by a transition cost matrix c with dimension N by N.

Depending on the specific problem at hand, the states can represent (a) amount of energy in a hydro reservoir, (b) for a thermal unit total emissions constraints, the energy associated with the remaining allowable emissions, (c) the level of readiness of a thermal unit, including possibly the amount of energy or reserves, (d) a price level from among a set of possible price levels, (e) an amount of power bid into an auction, or (f) some Cartesian product or combination of one or more of the above or some other attributes sufficient to describe the complete state of affairs at any given stage. By definition, for a properly defined state, no information from any other stage shall be required to completely describe the status of the optimization process during any stage.

Every state can evolve from one state during one stage to another state during the next stage. The objective of DP is to determine the optimal state transition, or rather, the optimal state transition strategy. Some state transitions may not permissible.

The principle of optimality upon which DP is based simply states that an optimal strategy contains optimal sub-strategies. More precisely, if a state in an intermediate stage is part of the optimal strategy, then the remaining strategy (starting from this state) is an optimal strategy for the corresponding sub-problem (from this stage onwards). In other words, once we are at a given state in a given stage, we only have to determine the optimal strategy from this state (and stage) onwards for the remaining stages; how we *got to this state in this stage is not relevant*. Therefore, optimal strategies from stage 1 to stage K are found by finding sequentially the optimal sub-strategies from stage k to stage K, starting from $k = K - 1$ and working backwards to the stage $k = 1$. The optimal sub-strategies from stage k to stage K is found by finding the optimal transition from stage k to stage $k + 1$ that takes account of the optimal sub-strategy found (in the previous iterative step) from stage $k + 1$ to stage K.

For our deterministic DP example, consider a system with four states ($N = 4$) and six stages $K = 6$. The profits associated with each state for each stage (the stage profit matrix R) is illustrated in Table 34 (in the deterministic version these profits are fixed and known beforehand):

Table 34: Stage profits R from operating in a given state at a given stage. Values are randomly generated numbers between 1 and 100.

	Stages					
States	1	2	3	4	5	6
1	96	90	83	93	94	6
2	24	77	45	74	92	36
3	61	46	62	18	42	82
4	49	2	80	41	90	1

Likewise, in our example the state transition profits matrix c (assumed the same for all stages) is given according to the Table 35. Forbidden transitions have transition costs equal to ∞.

Table 35: State Transition cost matrix c. Values are randomly generated numbers between -1 and -100 with a few random "forbidden" transitions thrown in (∞ values)

	x_{k+1}			
x_k	1	2	3	4
1	-7	-14	-23	∞
2	-11	-10	-47	-27
3	-10	-1	∞	-11
4	-31	∞	-21	-34

Let $\mathbf{J}^*_{k+1}$ be the vector of cumulative optimal profits for stage $k+1$ onward, and let R_k be the profits associated with stage k for each of the states (R_k corresponds to the k^{th} column of the stage profit matrix R illustrated above). To determine the possible profits from each possible state transition from stage k to stage $k + 1$ we add the profits from stage k to the cumulative profits from stage $k + 1$ stage and to the profits from the state transition itself. Mathematically we can determine each entire new columns of $\mathbf{J}^*$ by first constructing a matrix $\mathbf{J}^*_k$ that determines all possible cumulative state transition profits (starting with $k = K - 1$)

and working backwards):

$$\mathbf{J}_k^* = \text{rowmax}\left(R_k e' - c + e(\mathbf{J}_{k+1}^*)'\right) \tag{111}$$

where e is a vector of ones of length N and the subscript k when applied to a matrix denotes the selection of the k^{th} column. The matrix $\mathbf{J}^*$ is of dimension N by K and initially its last column is equal to the last columns of R. For the example at hand, the matrix $\mathbf{J}_k^*$ for period $k = K - 1$ is illustrated in Table 36. The operator rowmax operator finds the maximum value by row.

Thus, only the optimal values for each state $\mathbf{x}_k$ need be considered (the maximum values in each row). Thus, we only need to select the optimal values as obtained in equation 111 (the maximum values within each row of Table 36). These values are highlighted in the table.

Table 36: All transition profits for the transition from stage 5 ($k = K - 1$) to stage 6 obtained from application of equation 111. Optimal values for each row are highlighted

	x_6			
x_5	1	2	3	4
1	107	144	**199**	$-\infty$
2	109	138	**221**	120
3	58	**79**	$-\infty$	54
4	127	$-\infty$	**193**	125

If we apply this procedure to the values from Tables 34 and 35 proceeding backwards for all values of k we obtain the cumulative profits matrix $\mathbf{J}^*$ illustrated in Table 37. It is also useful to illustrate a matrix of optimal state transitions, and this is done in Table 38.

Table 37: Cumulative profits matrix $\mathbf{J}^*$ obtained from repeated application of equation 111 and selection of optimal profits

	Stage					
State	1	2	3	4	5	6
1	653	515	418	328	199	6
2	577	543	384	305	221	36
3	605	496	400	240	79	82
4	595	475	439	271	193	1

Table 38: Optimal state transitions corresponding to the optimal profits

	Stage k				
x_{k+1}	1	2	3	4	5
1	2	1	1	2	3
2	2	4	1	2	3
3	2	4	1	2	2
4	1	4	1	1	3

The optimal transition values are shown in the table. The set of these values represents the newly determined vector $\mathbf{J}_k^*$, which can then be used to perform the transition from the previous stage using a similar procedure.

Figure 6 illustrates the solution of our randomly generated DP example. An interesting observations from this graphic representation of the optimal strategy is that there is a "convergence" in the optimal sequence; that is, after a few stages, and regardless of initial state, the optimal strategy converges to a single optimal thread. Thus, there are numerous states and state transitions that are not reachable. This can save considerably on computational effort. This type of observation gives rise, however, to many heuristic improvements in straightforward DP solvers.

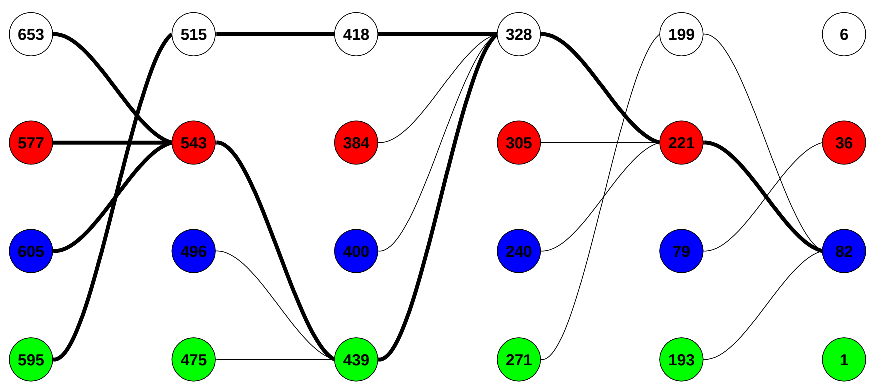

Figure 6: Optimal strategy and optimal profits obtained by backward DP for the randomly generated problem described in the text. Observe the typical "convergence" of the optimal thread. This example considers no multiple levels, no uncertainty.

In the body of this report to solve a DP problem involving time we associate the notion of "stage" with "time period." To solve a DP problem involving a sequence of decisions over auction rounds, we associate the notion of "stage" with the notion of "auction round." Thus, optimization over time periods or over auction rounds can be mapped into a DP problem. We also use DP to solve the biddability requirement problem. This is discussed next.

DP with Restrictions on Permissible Transitions

As we have seen, it is possible to use DP to characterize the bidding of sequential amounts of any quantity into a market. Assume that we have N blocks of, say, energy that we can offer, and that we can offer each block at any price that optimizes our profits, but with the restriction that the offer price for the second block cannot be lower than the offer price for the first block. The problem of making optimal offers that are compatible with a non-decreasing strategy, we can solve a DP problem involving a sequence of decisions with increasing values of the variable of interest (for example, the amount of energy the is being offered into an auction as part of the offer blocks). We let the amount being offered to the auction be the "stage." The "state" variable for each stage corresponds to the price state of the system, which of course can vary according to time period.

This use of DP to ensure biddability deserves additional explanation. Consider a table of profits (or expected profits, in the case of a probabilistic situations) for the amount bid versus the price levels (stages). A specific situation involving three possible price levels and 5 possible dispatch amounts is illustrated in Table 39. The rows in this Table (states) correspond to values of a bid quantity (the values in this table and in the corresponding figures are sorted from smaller to larger). The columns (stages) correspond to the price level scenarios. The first column corresponds to the lowest price level. Thus, the biddability rule requiring non-decreasing bids simply means that bids must be chosen so there is never an upward move from the state (amount bid) on one stage to the state (amount bid) in the next stage (or the next larger quantity bid into the market). The optimal bid for any level is the bid that maximizes the profits.

Table 39: Assumed random expected profits for a given bid as a function of price level.

States	Stages		
	1	2	3
1	20	70	50
2	69	38	90
3	31	87	83
4	55	86	65
5	16	60	82

Any DP problem must incorporate restrictions in the ability to transfer from one state to another as we transition from one stage to another of the DP process. Forbidden transitions, as we described earlier, are represented expediently by resulting in ∞ costs. In particular, using DP to determine optimal bids under biddability rule restrictions can be represented by creating a DP problem with stages associated with price levels and states associated with amounts bid, with the additional restriction that there shall be no "upward transition" from a greater quantity bid during a lower price level to a smaller valued bid at a higher price level. Thus, in order to require non-decreasing prices in bids can be elegantly represented by the state transition cost matrix in Table 40.[10]

Table 40: Matrix of state transition costs. This matrix has been designed to forbid downward state transitions.

$\mathbf{x}_k$	$\mathbf{x}_{k+1}$			
	10	20	30	40
10	0	0	0	0
20	∞	0	0	0
30	∞	∞	0	0
40	∞	∞	∞	0

Once the stage expected profit matrix has been determined and the transition cost matrix has been established as indicated, the optimization process is pretty much the same as before,

[10] If we wish to remove this restriction and allow all state transitions, we have the option of simply setting all entries of this matrix equal to zero.

except that now there are many transitions that are not permitted. We illustrate the nature of this type of optimization with a randomly generated example. The same problem is solved twice, once without restriction on transitions, and the second time with restricted transitions. Figure 7 illustrates the example with all transitions allowed and Figure 8 illustrates the case for the exact same problem but with decreasing transitions not allowed. As expected, the optimal prices are either the same or higher for the problem with non-decreasing transitions as a result of the additional constraint.

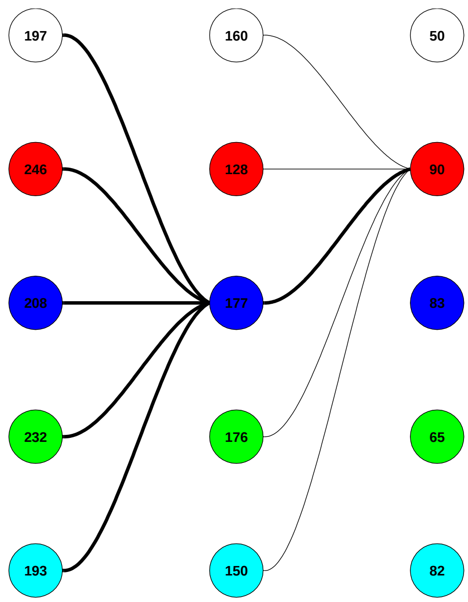

Figure 7: Optimal solution of a random multi-state multi-stage problem when with all transitions allowed.

Probabilistic DP

In the presence of uncertainty, an optimal strategy typically requires that expected values be optimized. The transition from one state in one stage to another state in the following stage may not be fully deterministic. Or rather, there is only partial control over the state transitions. We assume that there is a state $\mathbf{x}_k$ for each stage that characterizes the status of the generator, and that there is also a price level "state" $\mathbf{p}_k$ that characterizes the price regime. The complete state at any time period is the composite, or cross-product, of these two components to the composite state characterization $(\mathbf{x}_k, \mathbf{p}_k)$. We then transfer from one composite state $(\mathbf{x}_k, \mathbf{p}_k)$ to a new composite state $(\mathbf{x}_{k+1}, \mathbf{p}_{k+1})$, where $\mathbf{x}_{k+1}$ is known deterministically but $\mathbf{p}_{k+1}$ is only known only statistically.

To illustrate the DP process with uncertainty, we consider a case with 3 states and two random price levels (HIGH and LOW), for a total of $3 \cdot 2 = 6$ composite states, over 5 stages ($K = 5$). The profit matrix R associated with operation within each composite state is illustrated in Table 41.

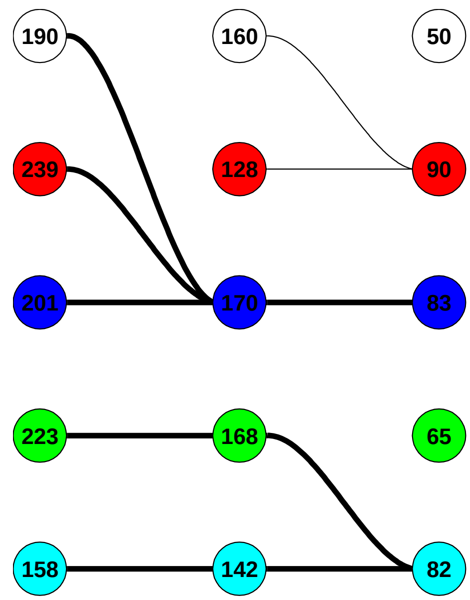

Figure 8: Optimal solution of the same random multi-state multi-stage problem when with transitions restricted to being non-decreasing. That is, no upward transitions to lower values of the state variable are permitted.

The expected state transition cost matrix c (also randomly generated) for this example is illustrated in Table 42.

Finally, it is important to know the statistics of the random process. Since these apply only to the price level sub-states, the dimension of the probability of price transitions is, in our example, two by two. For the example at hand, the expected transition probability matrix is illustrated in Table 43.

To solve this problem, the DP methods of the previous section can be applied directly if expected values are used. Table 44 illustrates the optimal profits. The solution to this problem is illustrated in Figure 9. In this particular case, there is no "convergence" of the optimal transitions to a single state at an intermediate stage. Even with a constant transition cost matrix, it is possible that the optimal sequence thread will re-split after some steps as a result of the randomness of outcome.

Nested DP

As part of the DP optimization process, it is necessary to determine the profits associated with each possible transition out of that state into any other state, or the profits associated with operating in a given state given the value of some exogenous random variable. Assume, however, that these profits are not readily available. Assume instead that to determine these profits or expected profits requires, in turn, an optimization. Depending on the nature of this optimization, one may need DP methods to solve it. In such a case, the original (top level)

Table 41: Matrix R of assumed random expected profits for a given bid as a function of price level.

	Stage				
Composite State	1	2	3	4	5
(1,HIGH)	63	10	17	46	61
(1,LOW)	82	25	50	62	87
(2,HIGH)	62	36	41	8	18
(2,LOW)	96	43	84	96	77
(3,HIGH)	65	19	47	15	45
(3,LOW)	71	38	84	32	63

Table 42: Transition costs associated with each state transition. In this example the state transition costs are associated only with the state transition, not with the price level random transition.)

	(x_{k+1}, p_{k+1})					
(x_k, p_k)	(1,HIGH)	(1,LOW)	(2,HIGH)	(2,LOW)	(3,HIGH)	(3,LOW)
(1,HIGH)	-13	-13	∞	∞	-18	-18
(1,LOW)	-13	-13	∞	∞	-18	-18
(2,HIGH)	-30	-30	∞	∞	-21	-21
(2,LOW)	-30	-30	∞	∞	-21	-21
(3,HIGH)	-26	-26	-48	-48	-16	-16
(3,LOW)	-26	-26	-48	-48	-16	-16

DP problem turns into a two-level nested DP problem. The inner DP problem may, in turn, require some form of optimization in order to determine some of the stage profits. If this innermost optimization itself required the solution of a DP problem, this leads to a three-level nested DP.

In our work, there are at least three levels of nesting in some applications. For one of the more typical problems, the following would be the levels of interest:

- We wish to optimize over a sequence of time periods. In order to do so, we must be able to estimate the profits associated with any state, even when price exogenous uncertainty may occur.
- The profits that we anticipate for any state will necessarily depend on the outcome of

Table 43: Price level transition probability matrix illustrating the probability of transition from one price regime in stage k to another in stage $k+1$. (For the case of perfectly correlated prices, this matrix would be the identity matrix.)

	k+1	
k	LOW	HIGH
LOW	0.76	0.24
HIGH	0.21	0.79

Table 44: Optimal profits for when following an optimal state transition policy. (These values are not precise but are rounded.)

	Stage				
(x_k, p_k)	1	2	3	4	5
(1,HIGH)	383	275	164	126	61
(1,LOW)	438	341	213	157	87
(2,HIGH)	385	304	205	105	18
(2,LOW)	455	362	264	208	77
(3,HIGH)	431	286	225	108	45
(3,LOW)	469	352	318	145	63

some auction rounds in which we will be selling our various capabilities, for example, our generation output and our reserve capability(ies). Since there are interactions between these, the allocation of amounts to bid into each of these markets will itself involve a DP problem.

- When we offer our capabilities into an auction, if there are rules regarding biddability, it becomes necessary to perform another DP optimization to determine the optimal bid structure. To do this, each price at which an amount is bid can be viewed as a stage in a DP problem. Optimizing the amounts to be offered at each price level for optimal profitability involves, therefore, another DP problem with a restriction on transitions between the stages.

Figure 5 in Section 4 of this report illustrates the nested nature of these DP problems.

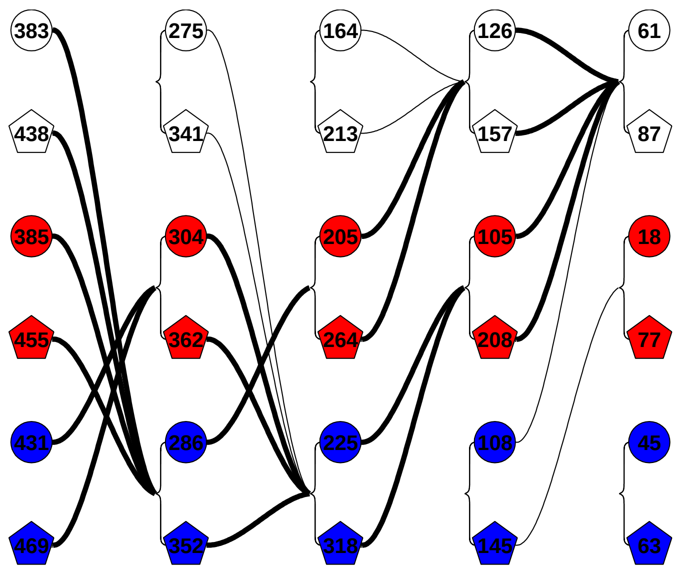

Figure 9: Optimization of a five stage three state and two random price regimes. Circles denote HIGH price level conditions; pentagons illustrate LOW price level conditions. There are, in effect, six states for each stage, and transitions from one stage to the next can only be deterministic to within the generator state but not the price level state.