A Two-Phase Model for Compaction and Damage

From Consolidation to Shear Localization
or
From Magmas to Plate Boundaries

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(Original) Motivation (a)

- Clear understanding of two-phase equations (i.e. McKenzie, 1984)
- Incorporate surface tension

Physical Principles

- Material invariance: The equations for the phases **f** and **m** in proportions ϕ and $1 - \phi$ are identical to those of phases **m** and **f** in proportions $1 - \phi$ and ϕ
- Limit case: The equations for the phases **f** and **m** in proportions ϕ and $1 - \phi$ when $\phi \to 0$ or $\phi \to 1$ are identical to those of the single phases **f** or **m**.

McKenzie [1984]

- Fluid:

$$-\phi\left[\boldsymbol{\nabla} P+\rho_f g\hat{\mathbf{z}}\right]+c\Delta\mathbf{v}=0$$

- Matrix:

$$-\boldsymbol{\nabla} P+\bar{\rho} g\hat{\mathbf{z}}+\boldsymbol{\nabla}\cdot\underline{\boldsymbol{\tau}}_m^*=0$$

- Rheology:

$$\underline{\boldsymbol{\tau}}_m^*=\mu_m^*\left(\boldsymbol{\nabla}\mathbf{v}_m+[\boldsymbol{\nabla}\mathbf{v}_m]^t\right)+(\zeta-\frac{2}{3}\mu_m^*)\boldsymbol{\nabla}\cdot\mathbf{v}_m\underline{\mathbf{I}}$$

(Original) Motivation (b)

- Brittle-ductile behavior in lithosphere key to plate generation from mantle flow
- Incorporate essence of brittle/fracture behavior into continuum physics for a simple, usable theory, i.e., compatible with mantle convection theory.

Minimalist Hypothesis

- Cracks, fractures = voids .... implies 2 phases
- Work to make void (crack)

  $\approx$ surface energy on fracture surface

  $\approx$ surface energy on interface between phases

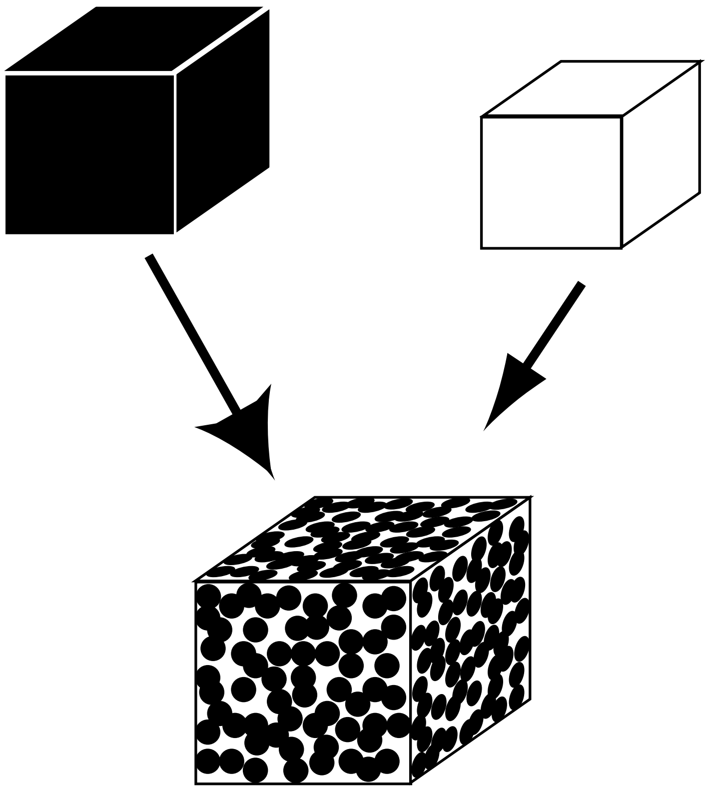
Beware...

Well, for those who like their research to be reasonably well defined physically and to be formulatable in mathematical terms, this is probably not a suitable field. A taste for pioneering and a preparedness to live with confusion are more appropriate qualifications... Underlying this lack of governing dynamical equations is ignorance or uncertainty about the relevant physical processes... Two-phase flow is too important and interesting a subject to be left to those who are obliged to find quick answers to practical questions.

G. K Batchelor 1988.

Approach (mild tutorial)

- Start with two *simple* viscous materials called **matrix** (= host) and **fluid** (= void filler)
  - Basic properties: densities (ρ_m, ρ_f), viscosities (μ_m, μ_f), etc.
- Mix them "simply" (isotropic, interconnected, no phase changes)



Mixture's additional properties

- Location of **fluid pores** and **matrix grains**:

$$\Theta = \begin{cases} 1, \text{in pores} \\ 0, \text{in grains} \end{cases}$$

- $\boldsymbol{\nabla}\Theta$ gives interface location and orientation
- Fluid volume within total volume δV:

$$\delta V_f = \int_{\delta V} \Theta dV$$

- Interface area:

$$\delta A_i = \int_{\delta V} |\boldsymbol{\nabla}\Theta| dV$$

But really...

- Can't track pores, grains and interfaces: must use quantities that are **volume-averaged, continuous** (i.e., exist at all points):
  - Porosity (fluid volume fraction)

$$\phi = \frac{1}{\delta V}\int_{\delta V} \Theta dV$$

  - Interface area per volume

$$\alpha(\phi) = \frac{\delta A_i}{\delta V} = \alpha_0 \phi^a (1-\phi)^b$$

$(\alpha_0 \sim \frac{1}{\mathrm{grain/pore-size}};\ a, b \leq 1)$

  - Interface curvature $\sim d\alpha/d\phi$

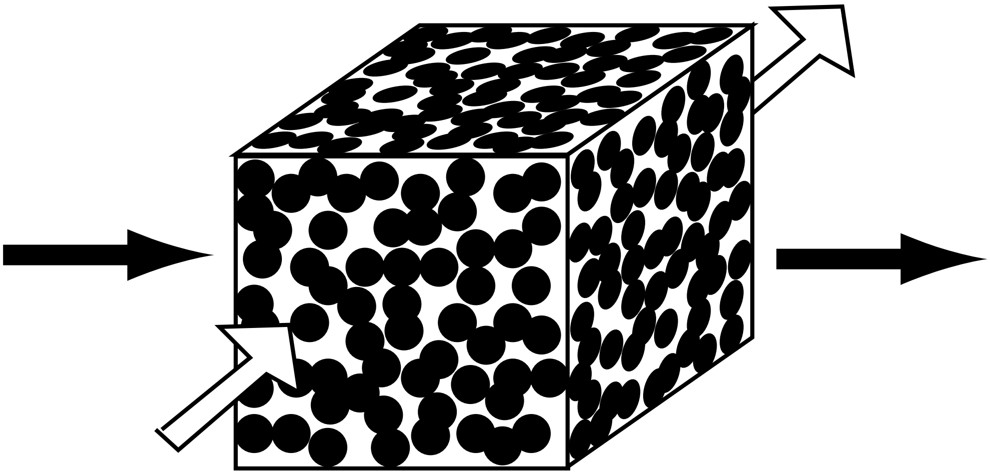
- Get governing equations in terms of **averaged** quantities, e.g., velocities

$$\mathbf{v}_f = \frac{1}{\phi \delta V}\int_{\delta V} \mathbf{v}_f^{true} \Theta dV$$

$$\mathbf{v}_m = \frac{1}{(1-\phi)\delta V}\int_{\delta V} \mathbf{v}_m^{true} (1-\Theta) dV$$

Mass conservation

- Growth in fluid volume governed by influx/efflux of fluid through surface exposure of pores on control volume; likewise for matrix volume:



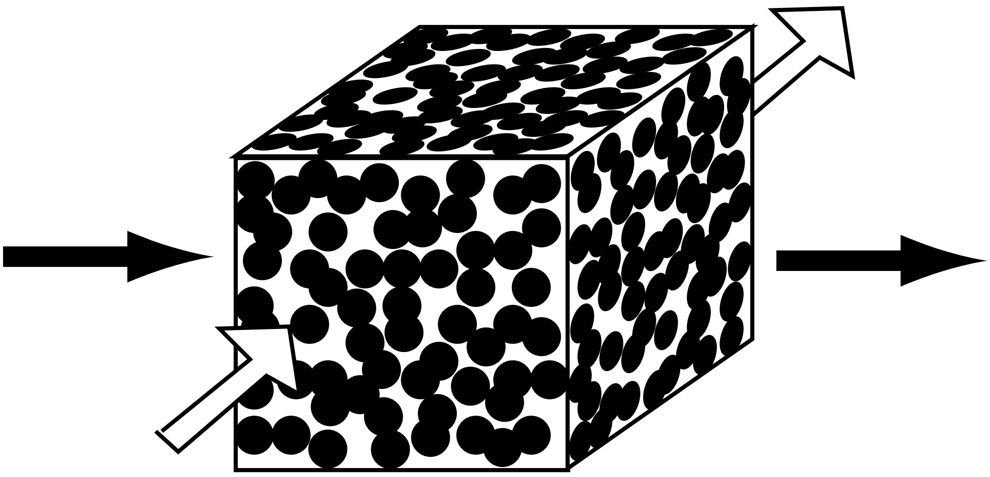
- Result: equations for volume-fraction of pores and grains:

$$\frac{\partial \phi}{\partial t} + \nabla \cdot [\phi \mathbf{v}_f] = 0$$

$$\frac{\partial (1-\phi)}{\partial t} + \nabla \cdot [(1-\phi) \mathbf{v}_m] = 0$$

Momentum conservation (force balance)

- Fluid and matrix pressures $\boxed{P_f, P_m}$ and stresses $\boxed{\underline{\tau}_f, \underline{\tau}_m}$ act on surface exposures of pores and grains
- Body force, e.g., gravity $\boxed{g}$, acts on pores and grains
- Surface tension $\boxed{\sigma}$ acts as line force on intersection of interface with surface



And... the interaction (body) force

- Forces acting on fluid through interface (by matrix + interface)
- ... and on matrix through interface (by fluid + interface).
- Includes:
  - Common pressure force
  - Common viscous drag: $\boxed{\pm c(\mathbf{v}_m - \mathbf{v}_f)}$ where $c \sim \frac{\text{viscosity}}{\text{permeability}}$
  - Interface surface tension

Momentum equations

- Fluid:

$$0 = -\phi\left[\boldsymbol{\nabla} P_f + \rho_f g\hat{\mathbf{z}}\right] + \boldsymbol{\nabla}\cdot\left[\phi\underline{\boldsymbol{\tau}}_f\right]$$

$$+c\Delta\mathbf{v} + \omega\left[\Delta P\boldsymbol{\nabla}\phi + \boldsymbol{\nabla}(\sigma\alpha)\right]$$

- Matrix

$$0 = -(1-\phi)\left[\boldsymbol{\nabla} P_m + \rho_m g\hat{\mathbf{z}}\right] + \boldsymbol{\nabla}\cdot\left[(1-\phi)\underline{\boldsymbol{\tau}}_m\right]$$

$$-c\Delta\mathbf{v} + (1-\omega)\left[\Delta P\boldsymbol{\nabla}\phi + \boldsymbol{\nabla}(\sigma\alpha)\right]$$

- where stress is

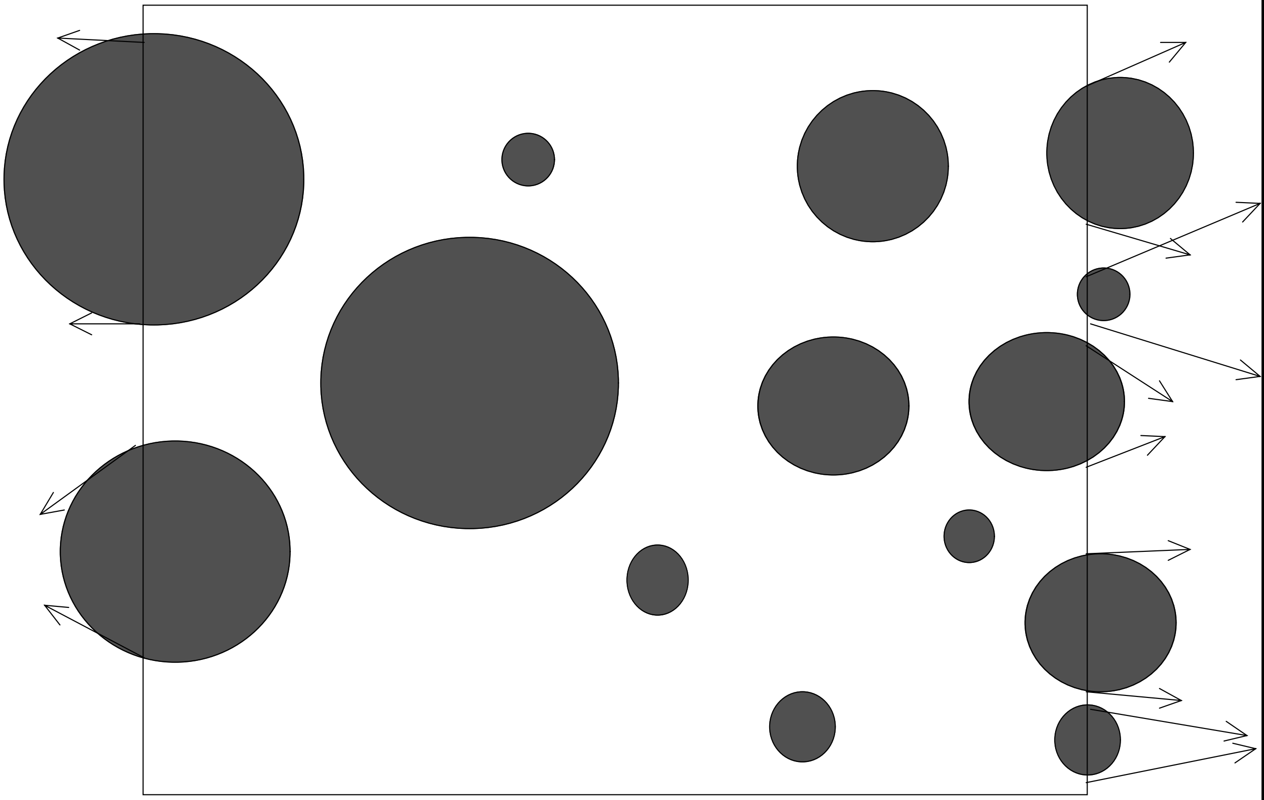
$$\underline{\boldsymbol{\tau}}_j = \mu_j\left(\boldsymbol{\nabla}\mathbf{v}_j + [\boldsymbol{\nabla}\mathbf{v}_j]^t - \frac{2}{3}(\boldsymbol{\nabla}\cdot\mathbf{v}_j)\underline{\mathbf{I}}\right)$$

  with $j = f$ or m; NOTE: **no** bulk viscosity.

- and average and difference quantities are:

$$\bar{q} = \phi q_f + (1-\phi)q_m \quad \text{and} \quad \Delta q = q_m - q_f$$

Surface tension



$$\nabla(\sigma\alpha) = volumic\ \ force$$

Surface energy partitioning

$$\omega = \frac{\mu_f \phi}{\mu_f \phi + \mu_m (1 - \phi)}$$

McKenzie [1984]

- Fluid:

$$-\phi\left[\nabla P + \rho_f g \hat{\mathbf{z}}\right] + \nabla \cdot \left[\phi \underline{\boldsymbol{\tau}}_f^*\right] + c\Delta\mathbf{v} = 0$$

- Matrix:

$$-(1-\phi)\left[\nabla P + \rho_m g \hat{\mathbf{z}}\right] + \nabla \cdot \left[(1-\phi)\underline{\boldsymbol{\tau}}_m^*\right] - c\Delta\mathbf{v} = 0$$

These equations differ by the absence of surface tension ($\sigma = 0$) and and the assumption of a single pressure field, $P_m = P_f = P$. They are augmented by the rheological relations

$$\phi \underline{\boldsymbol{\tau}}_f^* \approx 0$$

$$(1-\phi)\underline{\boldsymbol{\tau}}_m^* = \mu_m^* \left(\nabla \mathbf{v}_m + [\nabla \mathbf{v}_m]^t\right) + (\zeta - \frac{2}{3}\mu_m^*)\nabla \cdot \mathbf{v}_m \underline{\mathbf{I}}$$

The fluid phase is assumed inviscid relative to the matrix, and the viscosities μ_m^* and ζ are effective viscosities.

Another Equation?

- So far, 9 unknowns: ϕ, $\mathbf{v}_m$, $\mathbf{v}_f$, P_m and P_f
- But, only 8 equations: 2 mass eqns, 6 momentum eqns
- Although system might be isothermal, still need another equation, nominally to relate P_m and P_f. Analogous full stress *jump condition* inappropriate (since we don't know location and orientation of interface in mixture)

Simple: Laplace's equilibrium surface tension condition:

$$P_f - P_m = \sigma \frac{d\alpha}{d\phi}$$

where $\frac{d\alpha}{d\phi}$ is interface curvature.
But incorrect:

$\rightarrow$ assumes inviscid fluids!

$\rightarrow$ assumes reversibility and equilibrium!

Energy Equation-a)

- Consider all input and growth of energy in fluid and matrix, and on interface:
  - energy flux, energy sources, mechanical work
- We get:

$$\overline{\rho c}\frac{\overline{\overline{D}}T}{Dt} - T\frac{\widetilde{D}}{Dt}\left(\alpha\frac{d\sigma}{dT}\right) - T\alpha\frac{d\sigma}{dT}\boldsymbol{\nabla}\cdot\tilde{\mathbf{v}} =$$

$$Q - \boldsymbol{\nabla}\cdot\mathbf{q} + \Psi - \Delta P\frac{\widetilde{D}\phi}{Dt} - \sigma\frac{\widetilde{D}\alpha}{Dt}$$

where the deformational work is Ψ

$$\Psi = c\Delta v^2 + \phi\boldsymbol{\nabla}\mathbf{v}_f : \underline{\boldsymbol{\tau}}_f + (1-\phi)\boldsymbol{\nabla}\mathbf{v}_m : \underline{\boldsymbol{\tau}}_m$$

- where T = temperature of the phases, Q = internal heat source, $\mathbf{q}$ = energy flux vector (e.g., heat conduction)

Energy Equation-b)

$$\overline{\rho c}\frac{\overline{\overline{D}}T}{Dt} - T\frac{\widetilde{D}}{Dt}\left(\alpha\frac{d\sigma}{dT}\right) - T\alpha\frac{d\sigma}{dT}\boldsymbol{\nabla}\cdot\tilde{\mathbf{v}} =$$

$$Q - \boldsymbol{\nabla}\cdot\mathbf{q} + \Psi - \Delta P\frac{\widetilde{D}\phi}{Dt} - \sigma\frac{\widetilde{D}\alpha}{Dt}$$

where

$$\overline{\rho c} = \phi\rho_f c_f + (1-\phi)\rho_m c_m$$

$$\frac{\widetilde{D}}{Dt} = \frac{\partial}{\partial t} + \tilde{\mathbf{v}}\cdot\boldsymbol{\nabla} = \omega\frac{D_f}{Dt} + (1-\omega)\frac{D_m}{Dt}$$

$$\frac{\overline{\overline{D}}}{Dt} = \frac{1}{\overline{\rho c}}\left(\phi\rho_f c_f\frac{D_f}{Dt} + (1-\phi)\rho_m c_m\frac{D_m}{Dt}\right)$$

in which

$$\frac{D_f}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}_f\cdot\boldsymbol{\nabla}, \qquad \frac{D_m}{Dt} = \frac{\partial}{\partial t} + \mathbf{v}_m\cdot\boldsymbol{\nabla}.$$

Interface Work and Damage

- Equilibrium:

$$P_m - P_f + \sigma \frac{d\alpha}{d\phi} = 0$$

- Quasi-equil.:

$$P_m - P_f + \sigma \frac{d\alpha}{d\phi} = -B \frac{\widetilde{D}\phi}{Dt}$$

- Far from equil.:

$$(\Delta P + \sigma \frac{d\alpha}{d\phi}) \frac{\widetilde{D}\phi}{Dt} = -B \left(\frac{\widetilde{D}\phi}{Dt} \right)^2 + f\Psi$$

- Partitioning argument:

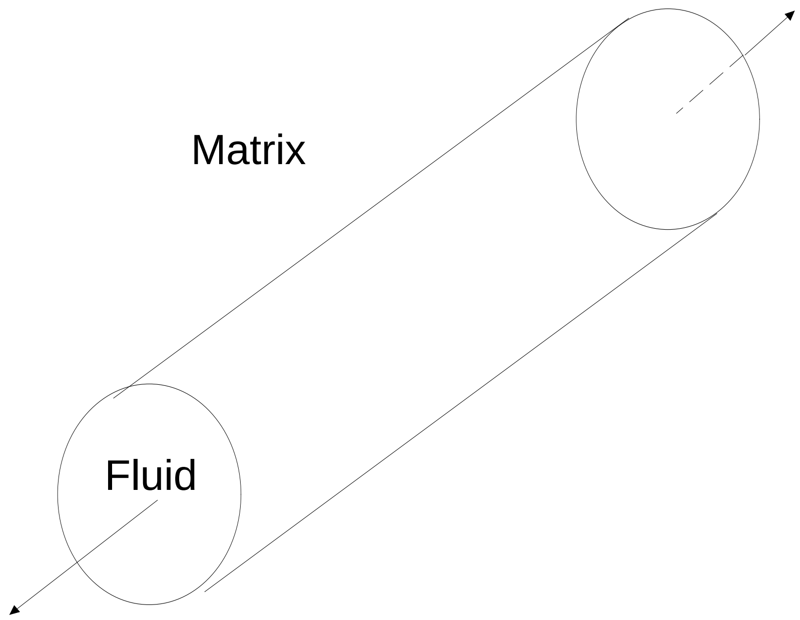
  $1 - f =$ fraction of deformational work going into dissipative heating.

  $f =$ remainder "stored" on interface, leads to **damage**

$$f = f^* \frac{(\widetilde{D}\phi/Dt)^2}{\gamma + (\widetilde{D}\phi/Dt)^2}$$

Jump Condition

- Micro-mechanical model:



$$B = K\frac{(\mu_f + \mu_m)}{\phi(1-\phi)}$$

where K is a dimensionless constant of $O(1)$.

Geological Case $\mu_f << \mu_m$

- Fluid mass conservation
  $\frac{\partial \phi}{\partial t} + \boldsymbol{\nabla} \cdot [\phi \mathbf{v}_f] = 0$
- Matrix mass conservation
  $\frac{\partial (1-\phi)}{\partial t} + \boldsymbol{\nabla} \cdot [(1-\phi)\mathbf{v}_m] = 0$
- Fluid force balance
  $0 = -\phi \left[\boldsymbol{\nabla} P_f + \rho_f g \hat{\mathbf{z}}\right] + c\Delta \mathbf{v}$
- Matrix force balance
  $0 = -(1-\phi)\left[\boldsymbol{\nabla} P_m + \rho_m g \hat{\mathbf{z}}\right] + \boldsymbol{\nabla} \cdot [(1-\phi)\underline{\boldsymbol{\tau}}_m]$
  $-c\Delta \mathbf{v} + \Delta P \boldsymbol{\nabla} \phi + \boldsymbol{\nabla}(\sigma\alpha)$
- Pressure jump
  $(\Delta P + \sigma \frac{d\alpha}{d\phi}) \frac{D_m \phi}{Dt} =$
  $-K \frac{(\mu_f + \mu_m)}{\phi(1-\phi)} \left(\frac{D_m \phi}{Dt}\right)^2 + f\Psi$
- Heat
  $\overline{\rho c} \frac{\overline{\overline{D}} T}{Dt} - T \frac{D_m}{Dt}\left(\alpha \frac{d\sigma}{dT}\right) - T\alpha \frac{d\sigma}{dT} \boldsymbol{\nabla} \cdot \mathbf{v}_m =$
  $Q - \boldsymbol{\nabla} \cdot \mathbf{q} + (1-f)\Psi + K(1-\phi)\frac{\mu_m}{\phi}(\boldsymbol{\nabla} \cdot \mathbf{v}_m)^2$

Previous Equations

In the special case where $\mu_f << \mu_m$, the mixture close to equilibrium and the interface without surface tension, our equations become

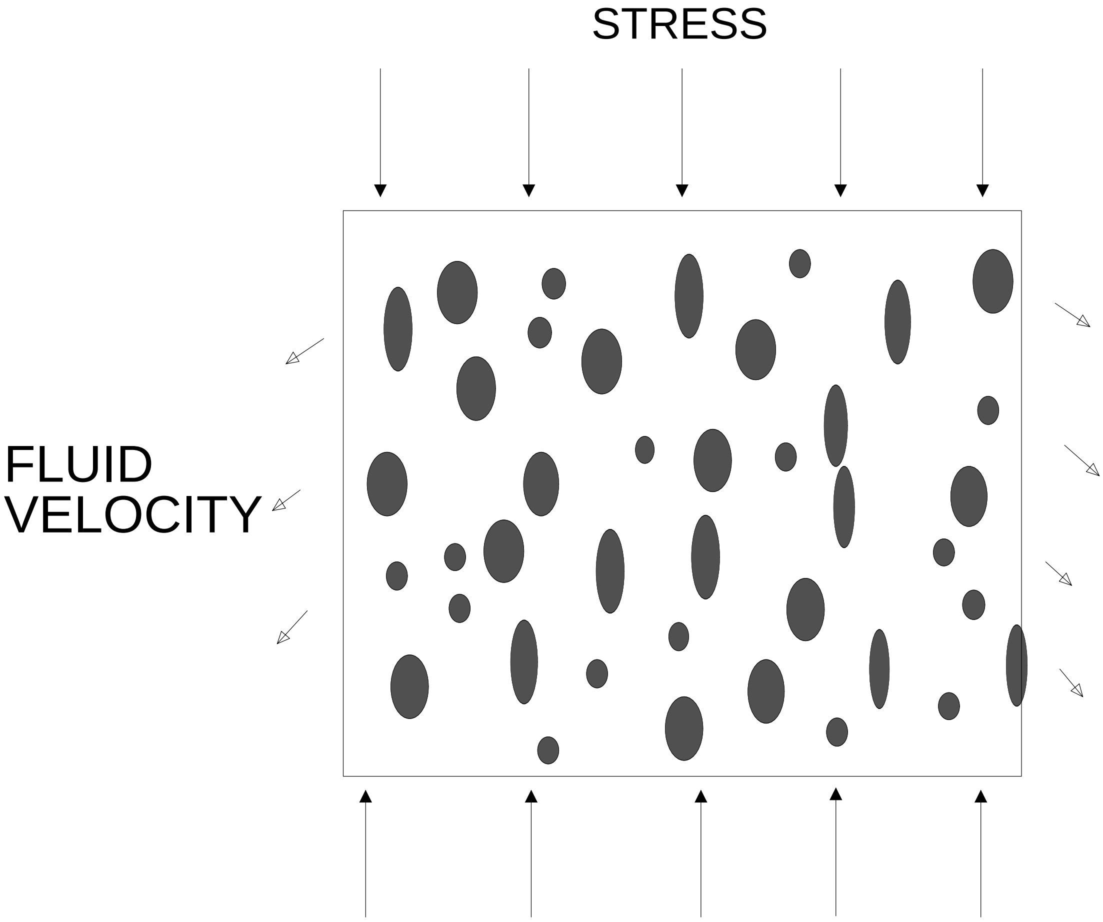
$$-\phi\left[\boldsymbol{\nabla} P_f + \rho_f g \hat{\mathbf{z}}\right] + c\Delta \mathbf{v} = 0,$$

$$-\left[\boldsymbol{\nabla} P_f + \bar{\rho} g \hat{\mathbf{z}}\right] +$$

$$\boldsymbol{\nabla}\cdot\left[(1-\phi)(\underline{\boldsymbol{\tau}}_m + \frac{K}{\phi}\mu_m \boldsymbol{\nabla}\cdot\mathbf{v}_m \underline{\mathbf{I}})\right] = 0.$$

These equations are identical to those of McKenzie (1984) when $\bar{P}$ is identified with P_f and $\zeta = K_0\mu_m/\phi$. They also correspond to those of Scott and Stevenson (1984)

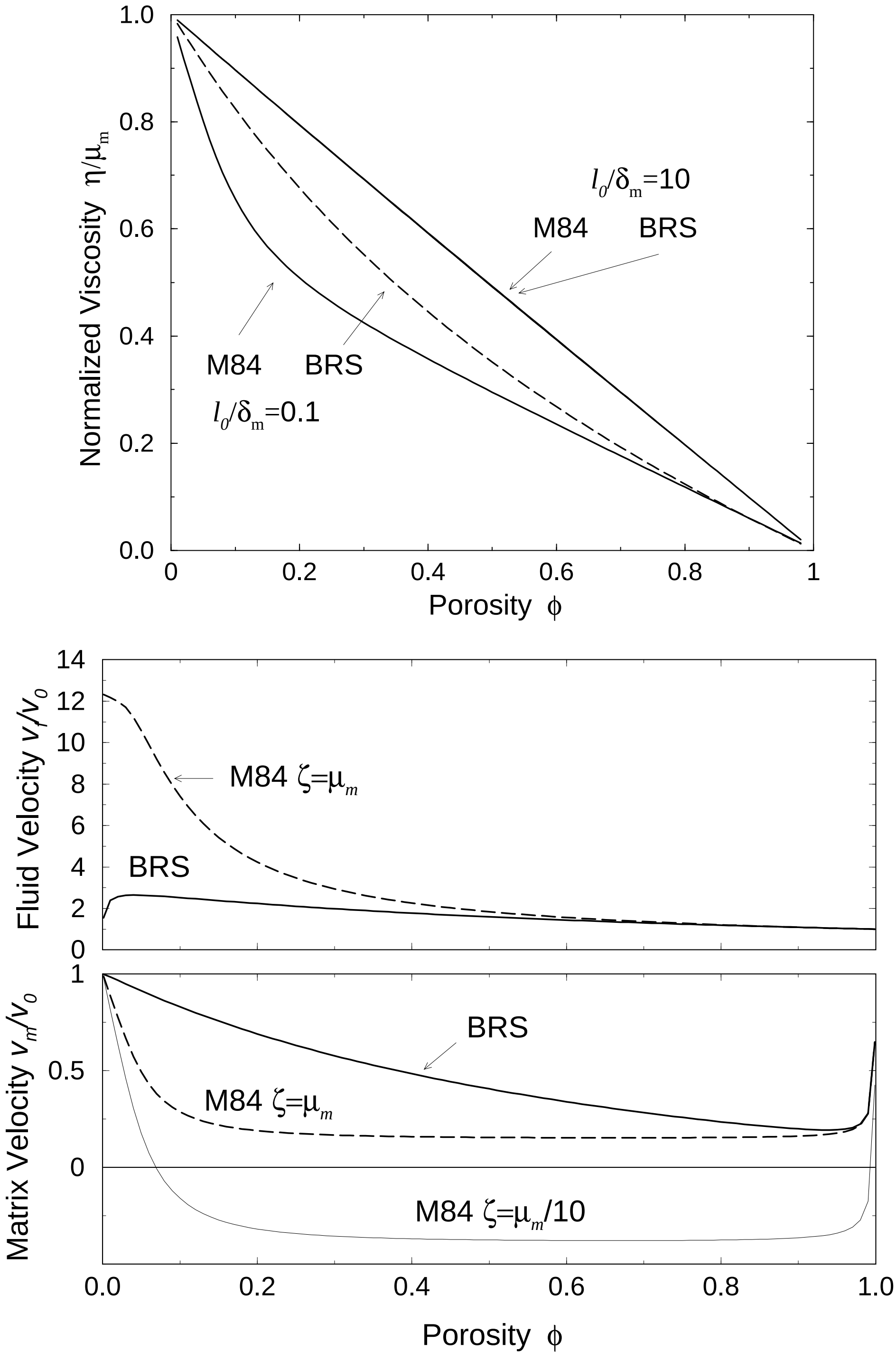
2D uniaxial compression (a)



$$\mu_f = 0$$

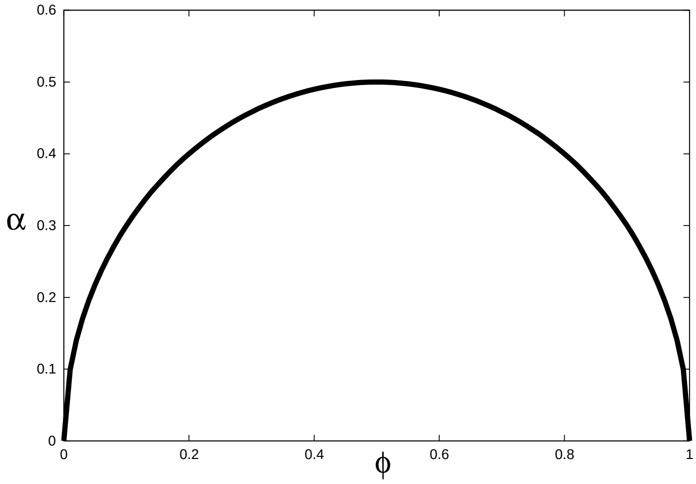
$$\delta_m = \sqrt{\frac{4\mu_m}{3c}}$$

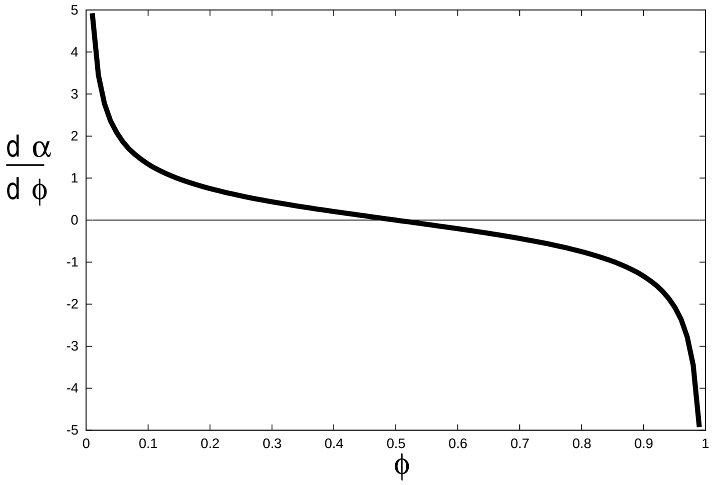
2D uniaxial compression (b)



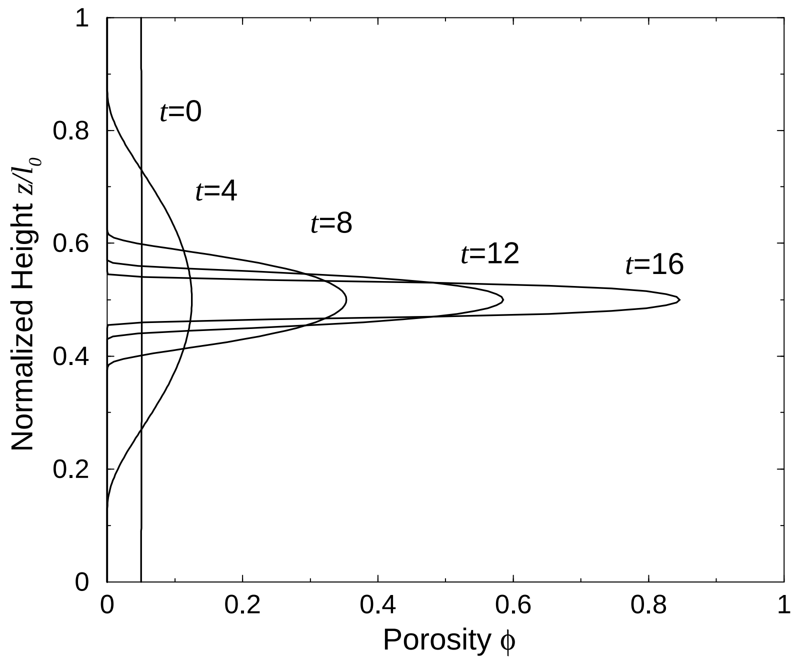
Self-Separation

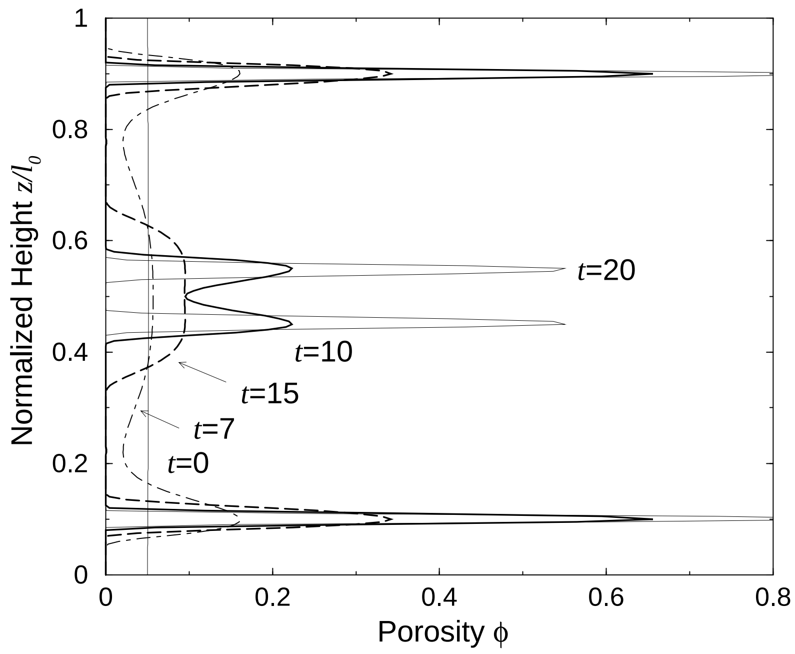
- Surface tension driven segregation ($f = g = 0$)
- Note: complete saturation, no grain-grain boundaries
- Consider interface area density α and curvature $d\alpha/d\phi$:





Self-separation:Nonlinear Solutions

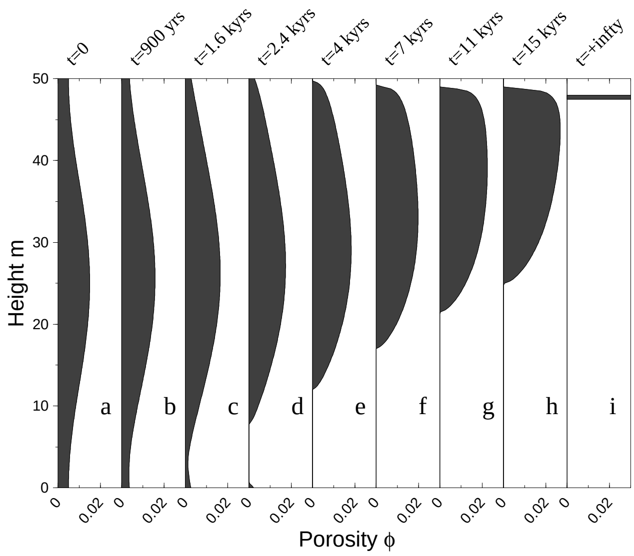


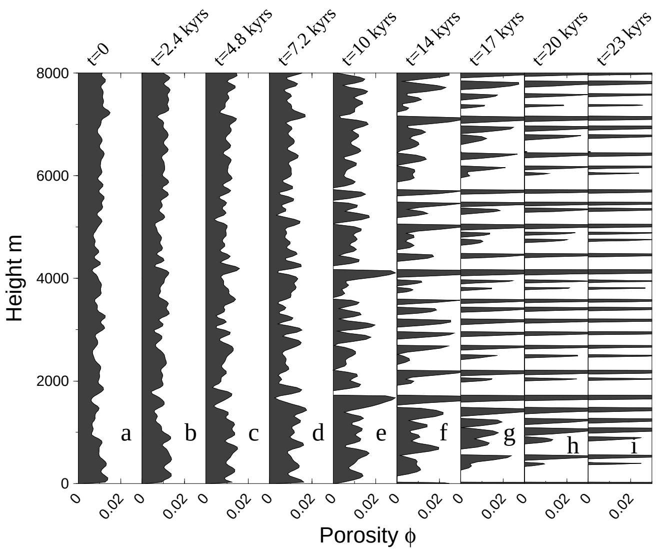


Gravitational Compaction

Solitons and Sills

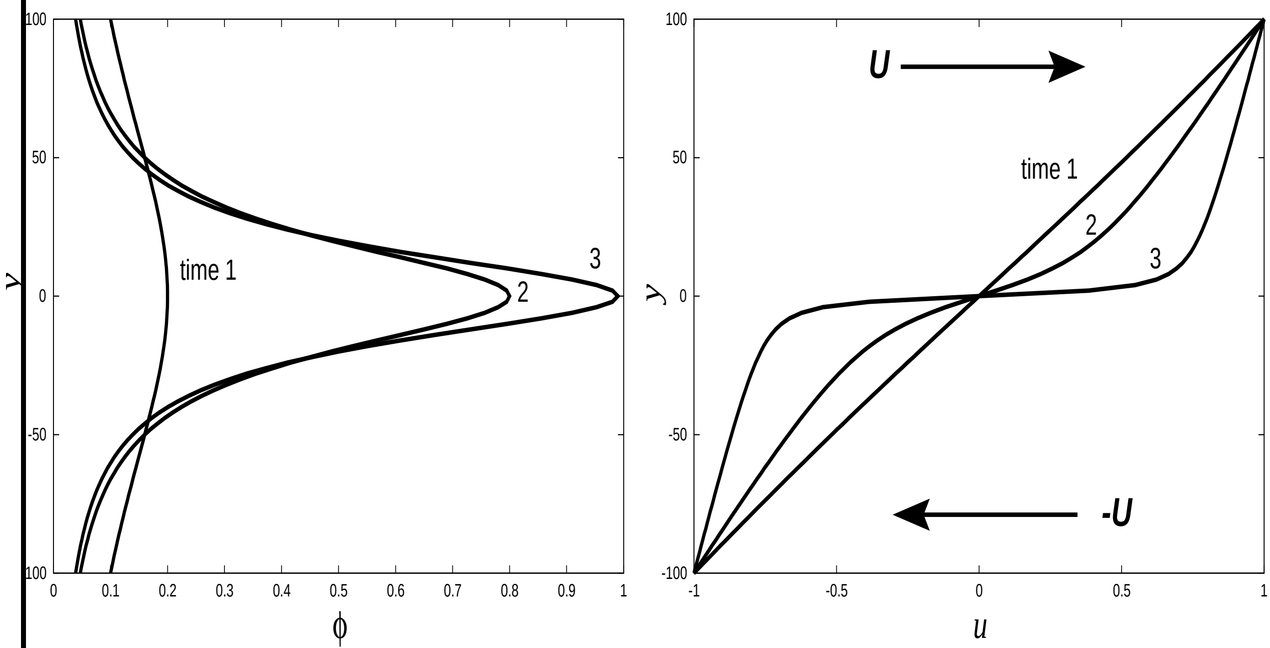
- Add gravity; matrix heavier than fluid





Damage and Shear localization

- Impose simple shear across layer; boundaries move at velocity $\boxed{U}$. (Also, $g = 0; \;\; f \neq 0$)
- Damage due to shear accelerates void growth
- Area of high porosity is weak, concentrates shear: **feedback**



One dimensional equations

- Force balance in the y direction

$$0 = \phi \frac{\partial}{\partial y} \left[\sigma\alpha - (1-\phi)\Delta P + \frac{4}{3}\mu_m \Theta \right] - c\frac{v_{m_y}}{\phi}.$$

- Pressure jump

$$\Delta P = -\sigma \frac{d\alpha}{d\phi} - K \frac{\mu_m}{\phi(1-\phi)} \Theta + \frac{f^* \Theta}{\gamma + \Theta^2} \left[c\frac{v_{m_y}^2}{\phi^2} + \frac{\mu_m}{1-\phi} \left(\Omega^2 + \frac{4}{3}\Theta^2 \right) \right].$$

- where

$$\Omega = (1-\phi)\frac{\partial v_{m_x}}{\partial y} = \text{ const.}$$

$$\frac{D_m \phi}{Dt} = \Theta = (1-\phi)\frac{\partial v_{m_y}}{\partial y}$$

Linear stability

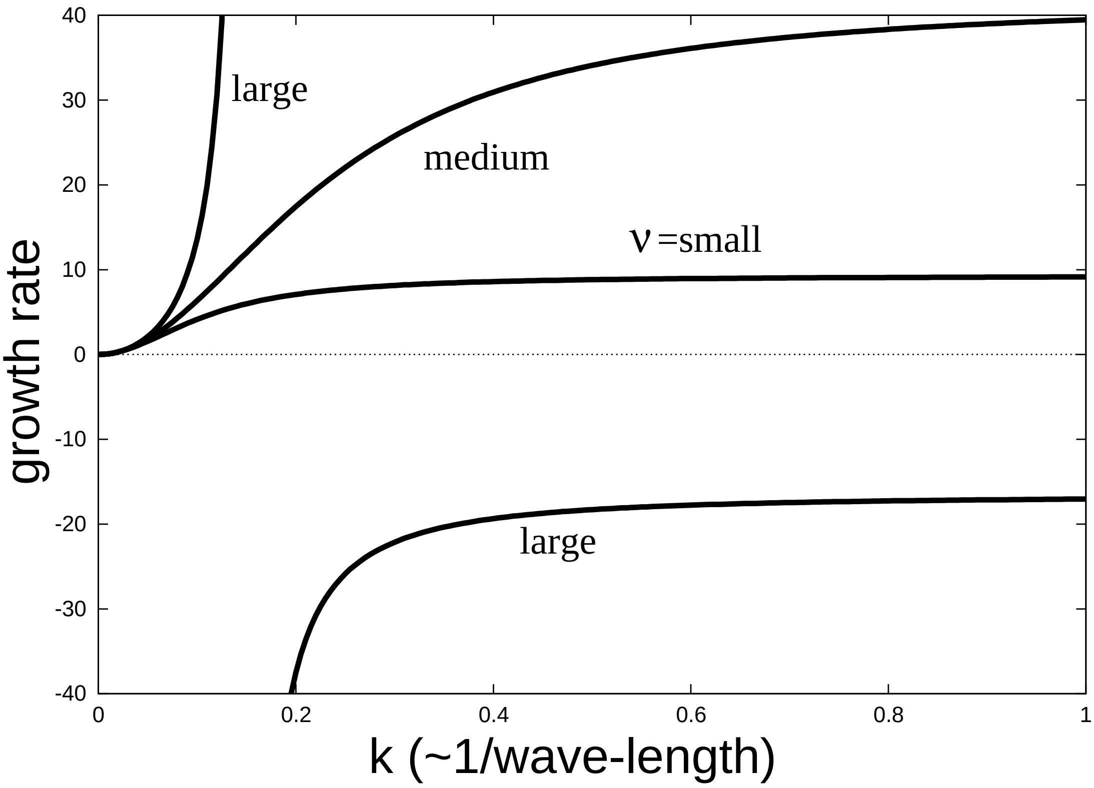
For negligible Darcy resistance ($\lambda \sim c/\mu_m \approx 0$)

$$\text{growth rate} = \text{self sep. rate} \frac{K + 4/3\phi}{K + \phi(4/3 - \nu)}$$

where

$$\nu \sim \frac{f^*}{\gamma} U^2$$

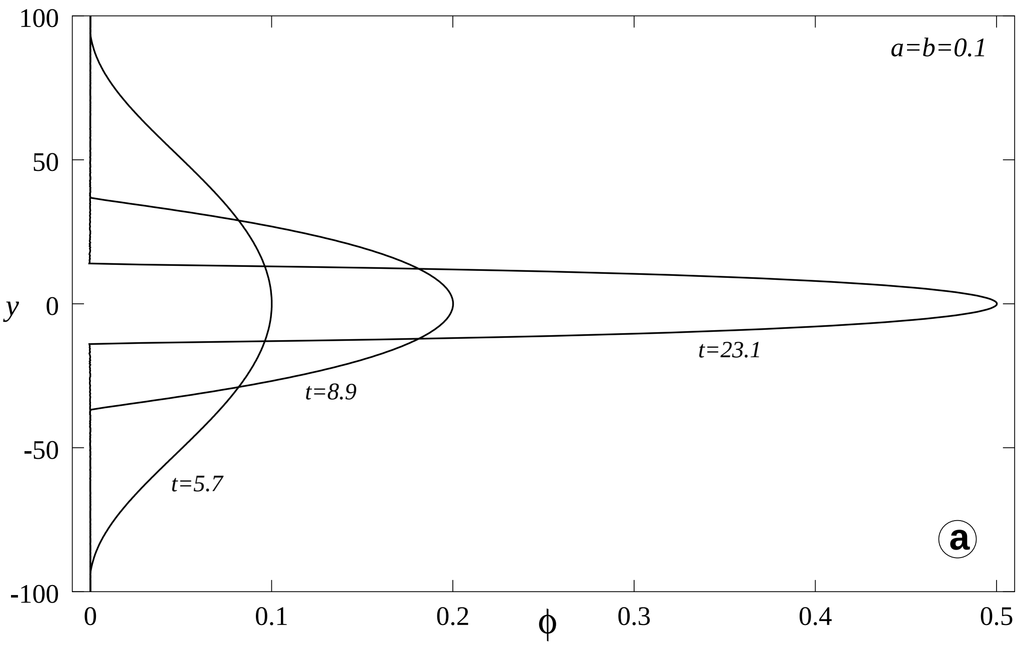
In general...



Nonlinear solutions

REGIME 1

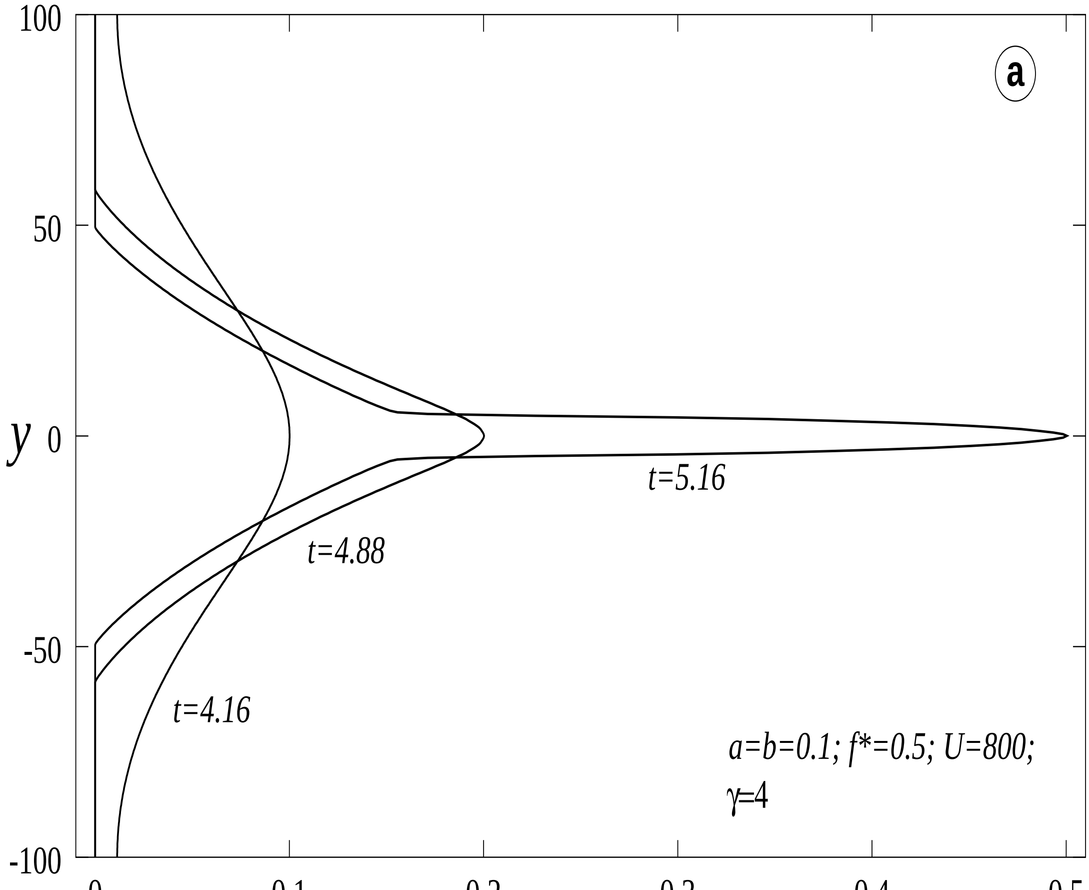
- Low U or ν: **accelerated self separation**



Nonlinear solutions

REGIME 2

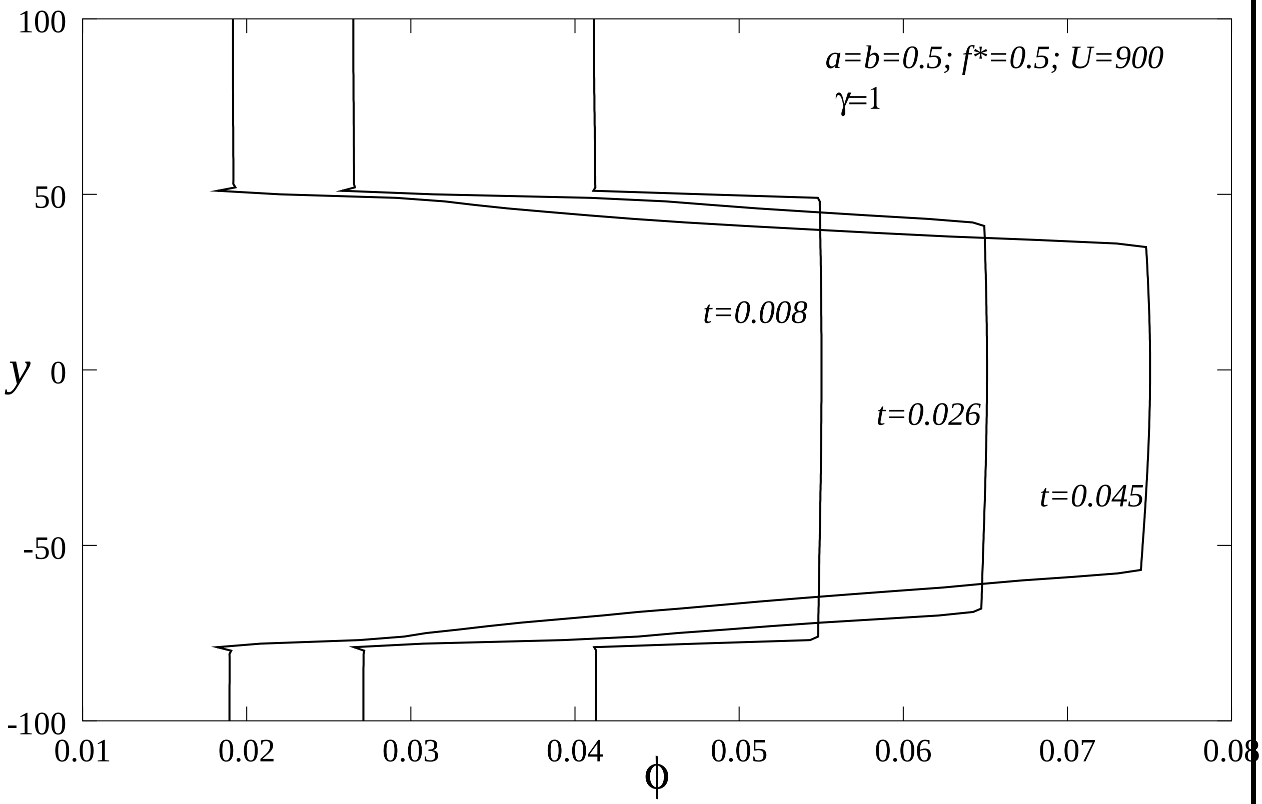
- Larger U and ν (but $\nu < 4/3 + \kappa/\phi_0$): **tear instability**



Nonlinear solutions

REGIME 3

- Large U and $\nu > 4/3 + \kappa/\phi_0$: **arrested void growth; distributed damage**



Conclusions

- Self-separation is a basic underlying mechanism; example, oil on the surface of water
- Buoyancy: surface tension affects magma percolation at 100m to 1km scale; turns *magmons* into *sill*
- Damage and shear localization: The three regimes:
  - Regimes 1 and 2: Brittle and brittle-ductile: low deformation and stress leads to distributed microcracking and at higher deformation and stress get localization instabilities and *macro*-cracks.
  - Regime 3: At very high deformation rates: *crack or branching instabilities* (distributed damage)

Closing

- Simple 2-phase theory with surface tension: idealized but rich
  - Significant affect on magma dynamics at moderate to small scale
  - Appears to capture complex physics of brittle and brittle-ductile behavior
- Future directions:
  - comparison to experiments!
  - healing (phase reactions, e.g., hydration)
  - phase changes (melting)
  - non-isothermal damage and apparent partitioning
  - anisotropy
  - 2D and 3D geometries (e.g., uniaxial compression and failure?)
  - *and much...*

much...

more....