Robust Quadratically Constrained Programs

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Joint work with Donald Goldfarb

Convex quadratically constrained program

- Generic problem

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{x}^T\mathbf{Q}_i\mathbf{x} + 2\mathbf{q}_i^T\mathbf{x} + \gamma_i \leq 0, \quad i = 1, \ldots, p, \end{array}$$

$\mathbf{x} \in \mathbf{R}^n$, $\mathbf{c} \in \mathbf{R}^n$, $\mathbf{q}_i \in \mathbf{R}^n$ and $\mathbf{Q}_i = \mathbf{V}_i^T\mathbf{V}_i \in \mathbf{R}^{n\times n} \succeq \mathbf{0}$ (positive semidefinite)

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- Convex quadratic constraint $\Leftrightarrow$ *second-order cone constraint*

$$\mathbf{x}^T\mathbf{V}^T\mathbf{V}\mathbf{x} + 2\mathbf{q}^T\mathbf{x} + \gamma \le 0 \Leftrightarrow \left\| \begin{bmatrix} 2\mathbf{V}\mathbf{x} \\ (1 + \gamma + 2\mathbf{q}^T\mathbf{x}) \end{bmatrix} \right\| \le 1 - \gamma - 2\mathbf{q}^T\mathbf{x}$$

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- Convex quadratic constraint $\Leftrightarrow$ *second-order cone constraint*
- This problem is a *second-order cone program (SOCP)*

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \left\| \begin{bmatrix} 2\mathbf{V}_i\mathbf{x} \\ (1 + \gamma_i + 2\mathbf{q}_i^T\mathbf{x}) \end{bmatrix} \right\| \leq 1 - \gamma_i - 2\mathbf{q}_i^T\mathbf{x}, \quad i = 1, \ldots, p. \end{array}$$

Convex quadratically constrained program

- Parameters $\{(\mathbf{Q}_i, \mathbf{q}_i, \gamma_i), i = 1 \ldots, p\}$ not known accurately
  - estimation errors
  - measurement/sensor errors
  - *implementation* errors

Robust quadtrically constrained program

- Parameters $\{(\mathbf{Q}_i, \mathbf{q}_i, \gamma_i), i = 1\ldots, p\}$ not known accurately
  - estimation errors
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  - *implementation* errors

- Uncertain $(\mathbf{Q}_i, \mathbf{q}_i, \gamma_i) \in \mathcal{S}_i$: robust problem

$$\begin{array}{rl} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{x}^T\mathbf{Q}_i\mathbf{x} + 2\mathbf{q}_i^T\mathbf{x} + \gamma_i \leq 0, \forall (\mathbf{Q}_i, \mathbf{q}_i, \gamma_i) \in \mathcal{S}_i \end{array}$$

For a large class of *uncertainty structures* $\mathcal{S}_i$ the robust problem is a *semidefinite program (SDP)* (Nemirovski & Ben-Tal (1998), El Ghaoui et al (1997,1998))

Properties of uncertainty structures

- The uncertainty structures $\mathcal{S}_i$ must be:
  - Flexible: model a large variety of perturbations
  - Parametrizable: parameters defining $\mathcal{S}$ easy to estimate
  - “Optimizable”: resulting robust optimization problem tractable

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Outline of remaining talk

- Three families of uncertainty sets that admit a SOC representation
  - Polytopic uncertainty sets
  - Affine uncertainty sets
  - Factorized uncertainty sets

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- Three families of uncertainty sets that admit a SOC representation
  - Polytopic uncertainty sets
  - Affine uncertainty sets
  - Factorized uncertainty sets
- (Engineering ?) applications of robust quadratically constrained programs

Polytopic uncertainty set

- Uncertainty set

$$\mathcal{S}_a = \left\{ (\mathbf{Q}, \mathbf{q}, \gamma) : \begin{array}{l} (\mathbf{Q}, \mathbf{q}, \gamma) = \sum_{j=1}^{k} \lambda_j (\mathbf{Q}_j, \mathbf{q}_j, \gamma_j), \ \mathbf{Q}_j \succeq \mathbf{0} \\ \mathbf{A} = [\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_k]\boldsymbol{\lambda} = \mathbf{b}, \ \boldsymbol{\lambda} \geq \mathbf{0} \end{array} \right\}$$

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- Robust constraint: $\mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{q}^T\mathbf{x} + \gamma \leq \alpha, \forall (\mathbf{Q}, \mathbf{q}, \gamma) \in \mathcal{S}_a$
  - Define

$$c_j = \mathbf{x}^T\mathbf{Q}_j\mathbf{x} + 2\mathbf{q}_j^T\mathbf{x} + \gamma_j, \quad j = 1, \ldots, k$$

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  - Linear programming duality

$$\boldsymbol{\lambda}^T\mathbf{c} \leq \alpha, \quad \forall \boldsymbol{\lambda} \geq \mathbf{0} : \mathbf{A}\boldsymbol{\lambda} = \mathbf{b} \quad \Leftrightarrow \quad \exists \boldsymbol{\mu} : \ \mathbf{b}^T\boldsymbol{\mu} \leq \alpha, \ \mathbf{A}^T\boldsymbol{\mu} \geq \mathbf{c}$$

Polytopic uncertainty set

- Uncertainty set

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  - Robust constraint equivalent to

$$\begin{array}{rcll} \mathbf{b}^T\boldsymbol{\mu} & \leq & \alpha & \\ \mathbf{x}^T\mathbf{Q}_j\mathbf{x} + 2\mathbf{q}_j^T\mathbf{x} + \gamma & \leq & \mathbf{A}_j^T\boldsymbol{\mu}, & j = 1, \ldots, k \end{array}$$

Affine uncertainty set

- Combined linear and quadratic terms

$$\mathcal{S}_b = \left\{ (\mathbf{Q}, \mathbf{q}, \gamma) : \begin{array}{l} (\mathbf{Q}, \mathbf{q}, \gamma) = (\mathbf{Q}_0, \mathbf{q}_0, \gamma_0) + \sum_{j=1}^{k} u_i (\mathbf{Q}_j, \mathbf{q}_j, \gamma_j) \\ \mathbf{Q}_j \succeq \mathbf{0}, u_j \geq 0, \|\mathbf{u}\| \leq 1 \end{array} \right\}.$$

Problem NP-Hard if u_i unconstrained

Affine uncertainty set

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Problem NP-Hard if u_i unconstrained

- Separate linear and quadratic terms

$$\mathcal{S}_c = \left\{ (\mathbf{Q}, \mathbf{q}, \gamma) : \begin{array}{l} \mathbf{Q} = \mathbf{Q}_0 + \sum_{j=1}^{k} u_i \mathbf{Q}_j, \mathbf{Q}_j \succeq \mathbf{0}, \|\mathbf{u}\| \leq 1 \\ (\mathbf{q}, \gamma) = (\mathbf{q}_0, \gamma_0) + \sum_{j=1}^{k} v_i (\mathbf{q}_j, \gamma_j) \, \|\mathbf{v}\| \leq 1 \end{array} \right\}$$

Affine uncertainty set

- Robust quadratic constraint:

$$\mathbf{x}^T(\mathbf{Q}_0 + \textstyle\sum_j u_j \mathbf{Q}_j)\mathbf{x} \le \beta, \quad \forall \mathbf{u}$$

$$\Updownarrow$$

$$\begin{aligned} \exists \mathbf{f} \in \mathbf{R}^k, \quad \mathbf{x}^T\mathbf{Q}_0\mathbf{x} + \|\mathbf{f}\| &\le \beta, \\ \mathbf{x}^T\mathbf{Q}_j\mathbf{x} &\le f_j, \quad j = 1, \ldots, k. \end{aligned}$$

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- Robust linear constraint:

$$\forall \mathbf{v} : (\mathbf{q}_0 + \textstyle\sum_j v_j \mathbf{q}_j)^T\mathbf{x} + (\gamma_0 + \sum_j v_j \gamma_j) \leq \alpha$$

$$\Updownarrow$$

$$\begin{array}{rcl} \exists \mathbf{g} \in \mathbf{R}^k, \quad \mathbf{q}_0^T\mathbf{x} + \gamma_0 + \|\mathbf{g}\| & \leq & \alpha, \\ g_j & = & \mathbf{q}_j^T\mathbf{x} + \gamma_j, \quad j = 1, \ldots, k. \end{array}$$

Factorized uncertainty set

- Uncertainty set

$$\mathcal{S}_d = \left\{ (\mathbf{Q}, \mathbf{q}, \gamma_0) : \begin{array}{l} \mathbf{Q} = \mathbf{V}^T \mathbf{F} \mathbf{V}, \\ \mathbf{F} = \mathbf{F}_0 + \boldsymbol{\Delta} \succ \mathbf{0}, \boldsymbol{\Delta} = \boldsymbol{\Delta}^T, \|\mathbf{N}^{-\frac{1}{2}} \boldsymbol{\Delta} \mathbf{N}^{-\frac{1}{2}}\| \leq \eta, \\ \mathbf{V} = \mathbf{V}_0 + \mathbf{W} \in \mathbf{R}^{m \times n}, \|\mathbf{W}_i\|_g \leq \rho_i, \ \forall i, \\ \mathbf{q} = \mathbf{q}_0 + \boldsymbol{\zeta}, \|\mathbf{S}^{\frac{1}{2}} \boldsymbol{\zeta}\| \leq \delta. \end{array} \right\}$$

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- Models situations here $\mathbf{Q}$ is not full-dimensional
- Not all perturbations may be present in applications

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- Models situations here $\mathbf{Q}$ is not full-dimensional
- Not all perturbations may be present in applications
- Robust constraint:

$$\mathbf{x}^T \mathbf{Q} \mathbf{x} + \mathbf{q}^T \mathbf{x} + \gamma_0 \leq 0, \quad \forall (\mathbf{Q}, \mathbf{q}, \gamma_0) \in \mathcal{S}_d$$

$$\Updownarrow$$

$$\max_{\mathbf{Q} \in \mathcal{S}_d} \left\{ \mathbf{x}^T \mathbf{Q} \mathbf{x} \right\} + \mathbf{q}_0^T \mathbf{x} + \delta \|\mathbf{S}^{-\frac{1}{2}} \mathbf{x}\| + \gamma_0 \leq 0$$

Factorized uncertainty set

- FIx $\mathbf{x}$ and $\mathbf{F}$: $\sup_{\mathbf{Q}\in\mathcal{S}_d}\left\{\mathbf{x}^T\mathbf{Q}\mathbf{x}\right\} \leq \beta$

Factorized uncertainty set

- Fix $\mathbf{x}$ and $\mathbf{F}$: $\sup_{\mathbf{Q}\in\mathcal{S}_d}\left\{\mathbf{x}^T\mathbf{Q}\mathbf{x}\right\} \leq \beta$

$$\sup_{\mathbf{Q}\in\mathcal{S}_d}\left\{\mathbf{x}^T\mathbf{Q}\mathbf{x}\right\} = \sup_{\{\mathbf{W}:\|\mathbf{W}_i\|_g\leq\rho_i\}}\|\mathbf{V}\mathbf{x}+\mathbf{W}\mathbf{x}\|_f^2$$

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- Worst case: all $\mathbf{W}_i$ aligned

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- By $\mathcal{S}$-procedure: If and only if $\exists\tau \geq 0$ such that

$$\mathbf{M} = \begin{bmatrix} \beta - \tau - \mathbf{x}^T\mathbf{V}_0^T\mathbf{F}\mathbf{V}_0\mathbf{x} & -r\mathbf{x}^T\mathbf{V}_0^T\mathbf{F}^{\frac{1}{2}} \\ -r\mathbf{F}^{\frac{1}{2}}\mathbf{V}_0\mathbf{x} & \tau\mathbf{G} - r^2\mathbf{F} \end{bmatrix} \succeq \mathbf{0}$$

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- Let $\mathbf{H} = \mathbf{G}^{-\frac{1}{2}}\mathbf{F}\mathbf{G}^{-\frac{1}{2}} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T$. Then $\mathbf{M} \succeq \mathbf{0}$ iff

$$\begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{Q}^T\mathbf{G}^{\frac{1}{2}} \end{bmatrix} \mathbf{M} \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{G}^{\frac{1}{2}}\mathbf{Q} \end{bmatrix} = \begin{bmatrix} \nu - \tau - \mathbf{w}^T\mathbf{w} & -r\mathbf{w}^T\mathbf{\Lambda}^{\frac{1}{2}} \\ -r\mathbf{\Lambda}^{\frac{1}{2}}\mathbf{w} & \tau\mathbf{I} - r^2\mathbf{\Lambda} \end{bmatrix} \succeq \mathbf{0}$$

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- Equivalently $\tau \geq r^2\lambda_{\max}(\mathbf{H})$, and Schur complement $\tau\mathbf{I} - r^2\mathbf{\Lambda}$

$$\beta - \tau - \mathbf{w}^T\mathbf{w} - r^2\Big(\sum_{i:\tau\neq r^2\lambda_i} \frac{\lambda_i w_i^2}{\tau - r^2\lambda_i}\Big) \geq 0.$$

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- Equivalent to $\exists\ \tau, \sigma \geq 0$ and $\mathbf{t} \in \mathbf{R}^m_+$:

$$\begin{array}{rcl} \beta & \geq & \tau + \mathbf{1}^T\mathbf{t}, \\ r^2 & \leq & \sigma\tau, \\ w_i^2 & \leq & (1-\sigma\lambda_i)t_i, \quad i = 1,\ldots,m, \\ \sigma & \leq & \frac{1}{\lambda_{\max}(\mathbf{H})}. \end{array}$$

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- The perturbation in $\mathbf{F}$ can be handled by a simple extension

Some applications of polytopic uncertainty

- Decision problem with scenarios

$$\min_{\mathbf{x}} \left\{ \mathbf{c}^T \mathbf{x} + \max_{\mathbf{p} \in \mathcal{A}} \left\{ \mathbf{x}^T \mathbf{E}^p[\mathbf{Q}] \mathbf{x} \right\} \right\}$$

  - $\mathbf{Q}$: discrete random variable with pmf $\mathbf{p}$
  - $\mathcal{A}$: polyhedral subset of the probability simplex

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- Combining information from different sources
  - Multiple "looks" at the same information

$$y_j = r_j x + n_j, \qquad \mathbf{E}[n_j^2] = \sigma_j^2, j = 1, \ldots, k$$

  - Noise power: $\boldsymbol{\zeta} = (\sigma_1^2, \ldots, \sigma_k^2) \in \mathcal{P}$ a polytope.
  - Output $z = \mathbf{h}^T \mathbf{y} = (\mathbf{h}^T \mathbf{r}) x + \mathbf{h}^T \mathbf{n}$

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  - Optimization problem:

$$\min_{\mathbf{h}} \left\{ \sigma_x^2 (1 - \mathbf{h}^T \mathbf{r})^2 + \max_{\boldsymbol{\zeta} \in \mathcal{P}} \left\{ \sum_{j=1}^{k} \zeta_j h_j^2 \right\} \right\}$$

Linear least squares

- Linear least squares problem:

$$\min_{\mathbf{x}} \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2$$

$$\mathbf{A} = [\mathbf{a}_1 \ \mathbf{a}_2 \ \ldots \ \mathbf{a}_m]^T = [\mathbf{A}_1 \ \mathbf{A}_2 \ \ldots \ \mathbf{A}_n]$$

- Many applications:
  - Regression/Estimation
  - Image reconstruction
  - Output tracking

Linear least squares

- Linear least squares problem:

$$\min_{\mathbf{x}} \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2$$

- Robust linear squares problem

$$\min_{\mathbf{x}} \max_{\mathbf{A} \in \mathcal{S}} \left\{ \|\mathbf{A}\mathbf{x} - \mathbf{y}\|_2 \right\}$$

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- A number of robustness results available
  - $\mathcal{S} = \left\{[\mathbf{A}, \mathbf{b}] : \left\|[\mathbf{A}, \mathbf{b}] - [\bar{\mathbf{A}}, \bar{\mathbf{b}}]\right\| \leq \rho\right\}$: SOCP (El Ghaoui & Lebret)
  - $\mathcal{S} = \left\{[\mathbf{A}, \mathbf{b}] : [\mathbf{A}, \mathbf{b}] = [\mathbf{A}_0, \mathbf{b}_0] + \sum_{i=1}^{n} u_i [\mathbf{A}_i, \mathbf{b}_i], \|\mathbf{u}\| \leq \rho\right\}$: SDP (Ben-Tal & Nemirovski)
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- New: Robust least squares an SOCP when $\mathbf{A}_i \in \left\{ \mathbf{a} : \mathbf{a} = \bar{\mathbf{a}} + \delta\mathbf{a}, \|\delta\mathbf{a}\|_g \leq \rho_i \right\}$

Portfolio selection

- Portfolio selection: $\mathbf{r} = \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ known.

$$\begin{array}{rl} \text{minimize} & \boldsymbol{\phi}^T \boldsymbol{\Sigma} \boldsymbol{\phi}, \\ \text{subject to} & \boldsymbol{\mu}^T \boldsymbol{\phi} \geq \alpha, \\ & \mathbf{1}^T \boldsymbol{\phi} = 1. \end{array}$$

Equivalent reformulations: maximum return problem, Sharpe ratio problem

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- Uncertainty in parameter values leads to poor performance
- Factor model: $\mathbf{r} = \boldsymbol{\mu} + \mathbf{V}^T \mathbf{f} + \boldsymbol{\epsilon}$
  - Mean return vector: $\boldsymbol{\mu}$
  - Factor loadings: $\mathbf{V}$
  - Factor: $\mathbf{f} \sim \mathcal{N}(\mathbf{0}, \mathbf{F})$
  - Error: $\boldsymbol{\epsilon} \sim \mathcal{N}(\mathbf{0}, \mathbf{D})$

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  - Mean return vector: $\boldsymbol{\mu} \in S_m = \{\boldsymbol{\mu} : \boldsymbol{\mu} = \boldsymbol{\mu}_0 + \boldsymbol{\xi}, |\xi_i| \leq \gamma_i\}$
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- Justification and parametrization of the uncertainty structure
  - Sets are implied by confidence regions around the MLE of $(\boldsymbol{\mu}, \mathbf{V}, \mathbf{F})$
  - Parametrized by setting a confidence threshold ω

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$$\begin{array}{rl} \text{min} & \max_{(\mathbf{V},\mathbf{F},\mathbf{D}) \in S_v(\omega) \times S_f(\omega) \times S_d(\omega)} \left\{ \boldsymbol{\phi}^T (\mathbf{V}^T \mathbf{F} \mathbf{V} + \mathbf{D}) \boldsymbol{\phi} \right\}, \\ \text{subject to} & \min_{\boldsymbol{\mu} \in S_m(\omega)} \left\{ \boldsymbol{\mu}^T \boldsymbol{\phi} \right\} \geq \alpha, \\ & \mathbf{1}^T \boldsymbol{\phi} = 1. \end{array}$$

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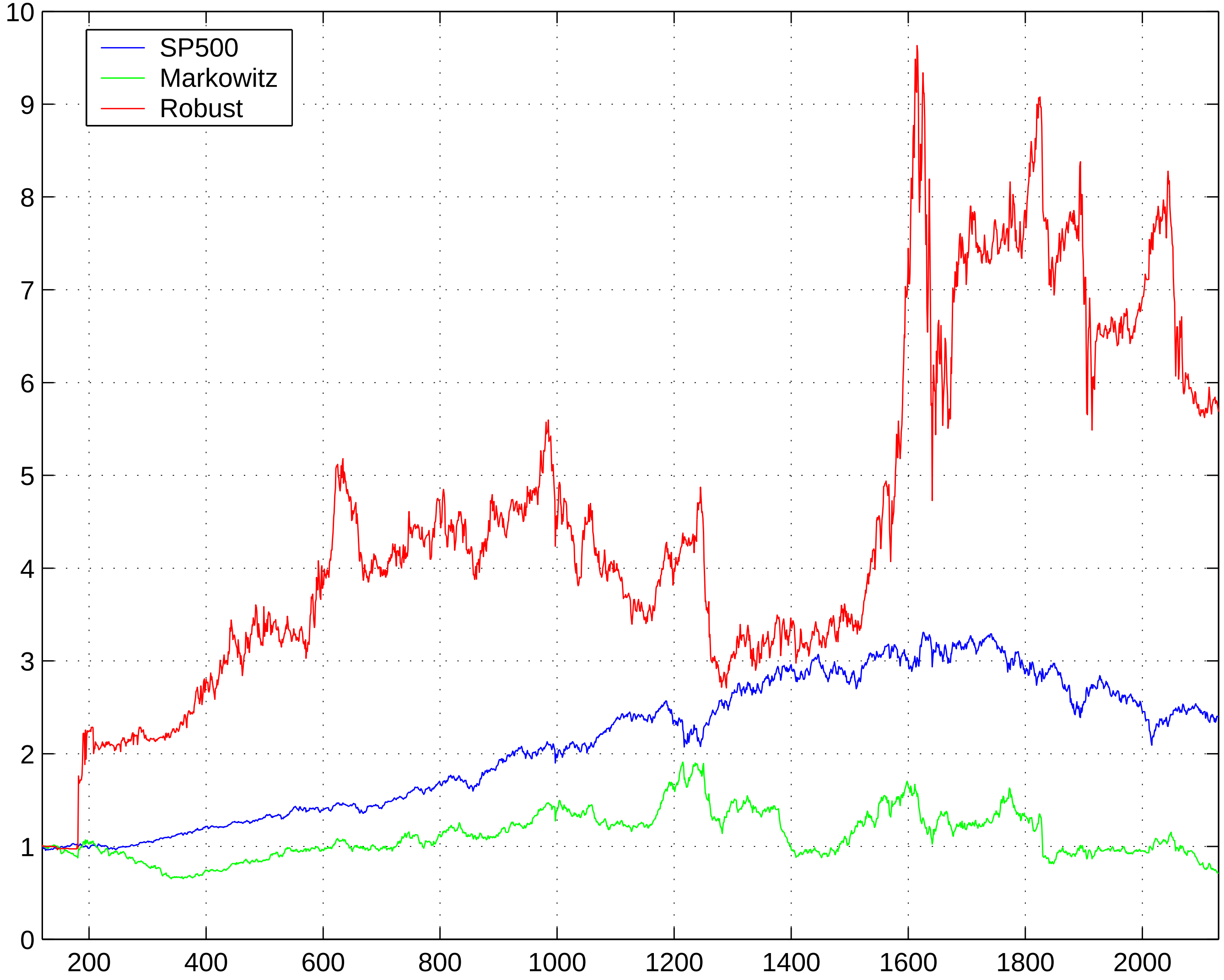
- $S_m(\omega), S_v(\omega), S_f(\omega), S_d(\omega)$ is a combination of S_a–S_d ... problem SOCP
- Translates ω into a confidence on the performance of the optimal portfolio $\boldsymbol{\phi}^*$

Historical performance of robust strategy



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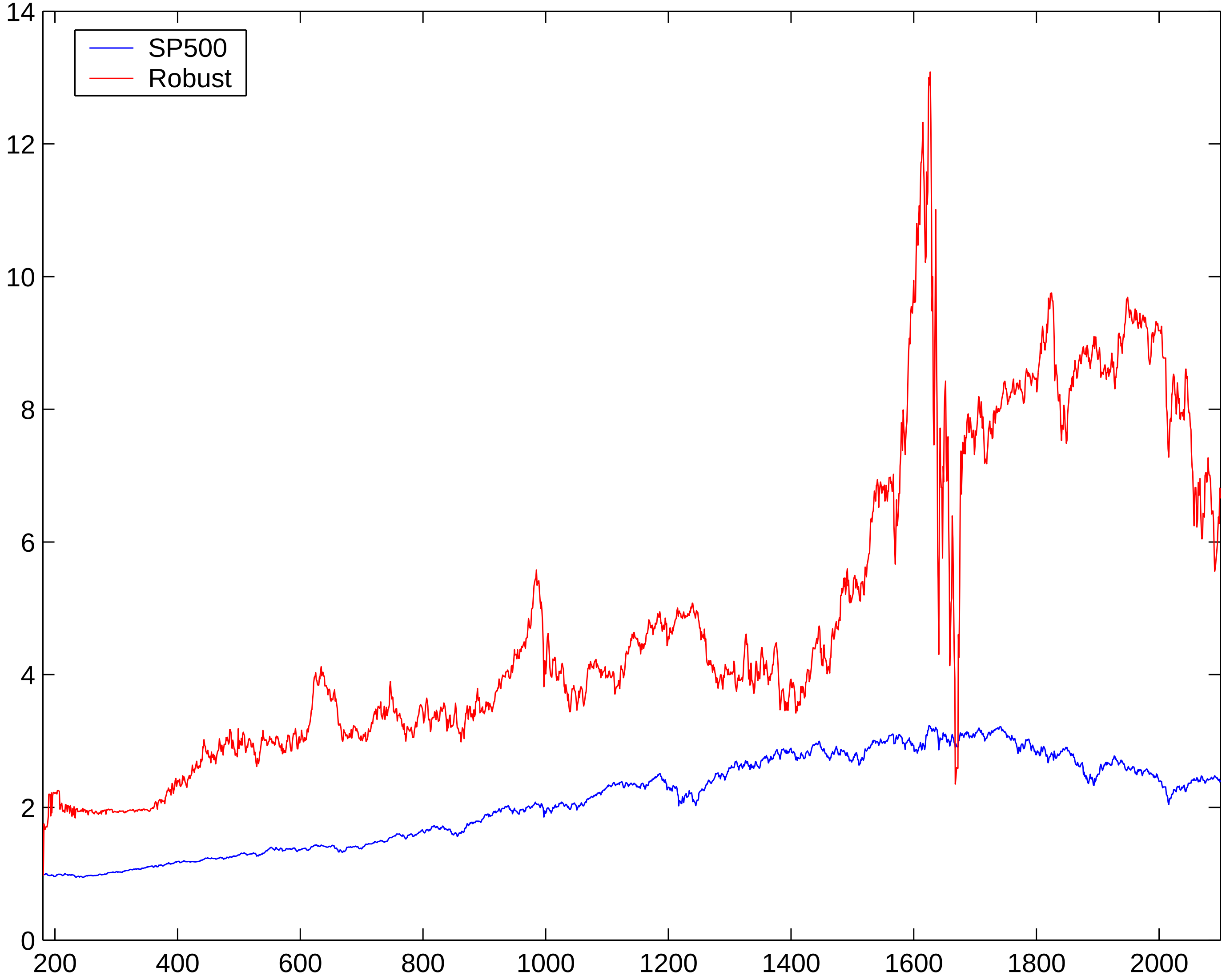
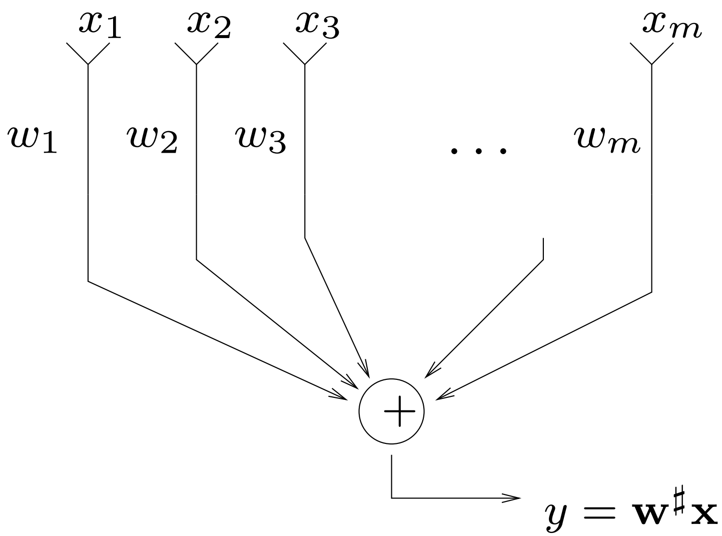


Antenna design problem (Tom Luo)

- System description: m antennae



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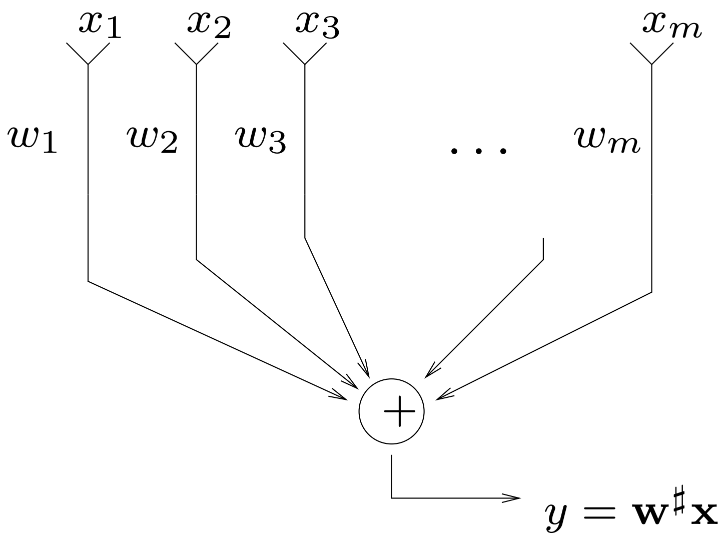

- User i uses steering vector $\mathbf{a}_i$, $i = 0, \ldots, N-1$

$$y = \mathbf{w}^\sharp \Big(\sum_{i=0}^{N-1} x_i \mathbf{a}_i + \mathbf{n})$$

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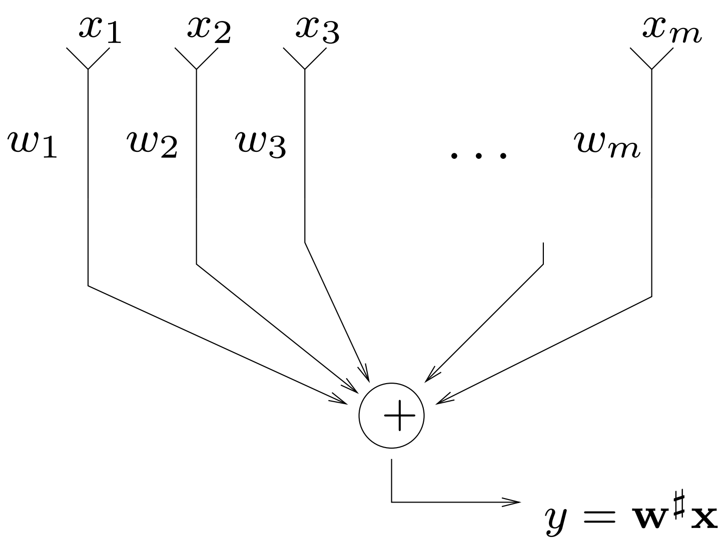

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- Signal power: $P_s = \sigma_0^2 |\mathbf{w}^\sharp \mathbf{a}_0|^2$ ... Interference power $P_i = \mathbf{w}^\sharp(\mathbf{A}^\sharp \mathbf{\Sigma} \mathbf{A} + \mathbf{R})\mathbf{w}$
  - $\mathbf{A} = [\mathbf{a}_1 \ \mathbf{a}_2 \ \ldots \ \mathbf{a}_{N-1}]$, $\mathbf{\Sigma} = \mathbf{diag}(\sigma_1^2, \ldots, \sigma_{N-1}^2)$, $\mathbf{R}$ is the noise covariance

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- Optimization problem:

$$\max_{\mathbf{w}} \left\{ \frac{P_s}{P_i} \right\} = \max_{\mathbf{w}} \left\{ \frac{\sigma_0^2 |\mathbf{w}^\sharp \mathbf{a}_0|^2}{\mathbf{w}^\sharp (\mathbf{A}^\sharp \boldsymbol{\Sigma} \mathbf{A} + \mathbf{R}) \mathbf{w}} \right\}$$

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- Equivalent problem: . . . can convexify using phase symmetry

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- Robust antenna: robust quadratically constrained problem

$$\begin{array}{ll} \min_{\mathbf{w}} & \max_{\mathbf{Q} \in \mathcal{S}} \left\{ \mathbf{w}^\sharp (\mathbf{Q}) \mathbf{w} \right\} \\ \text{s. t.} & \min_{\mathbf{a}_0 \in \mathcal{S}_0} \left\{ \sigma_0^2 |\mathbf{w}^\sharp \mathbf{a}_0|^2 \right\} \geq 1 \end{array}$$

  - Uncertainty set $\mathcal{S} = \left\{ \mathbf{Q} = \mathbf{A}^\sharp \boldsymbol{\Sigma} \mathbf{A} + \mathbf{R} : \|\mathbf{a}_i - \bar{\mathbf{a}}_i\| \leq \epsilon \right\}$

Hyperplane separation

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$$\begin{array}{rll} \text{minimize} & \frac{1}{2}\|\mathbf{w}\|^2 + C\left(\sum_{i=1}^{l} \xi_i\right), & \\ \text{subject to} & \mathbf{w}^T\mathbf{x}_i + b \geq 1 - \xi_i, & \text{if } y_i = +1, \\ & \mathbf{w}^T\mathbf{x}_i + b \geq 1 + \xi_i, & \text{if } y_i = -1, \\ & \xi_i \geq 0, & i = 1, \ldots, l. \end{array}$$

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- In practice, one solves the dual:

$$\begin{array}{ll} \text{maximize} & \mathbf{1}^T\boldsymbol{\alpha} - \frac{1}{2}\sum_{i,j=1}^{l}\alpha_i\alpha_j(y_i\mathbf{x}_i)^T(y_j\mathbf{x}_j), \\ \text{subject to} & \sum_{i=1}^{l}\alpha_i y_i = 0, \\ & 0 \leq \alpha_i \leq C, \quad i = 1, \ldots, l. \end{array}$$

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where the uncertainty set

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- The norm $\|\cdot\|$ and ρ can be chosen to “match” $\mathcal{S}$ to confidence regions

Estimation in linear models

- Parameter: $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$
  - $\boldsymbol{\mu}$ unknown ... apriori estimate $\bar{\boldsymbol{\mu}}$
  - $\boldsymbol{\Sigma} \in \mathcal{S}_1 = \left\{ \boldsymbol{\Sigma} : \boldsymbol{\Sigma}^{-1} = \boldsymbol{\Sigma}_0^{-1} + \boldsymbol{\Delta} \succeq \mathbf{0}, \boldsymbol{\Delta} = \boldsymbol{\Delta}^T, \|\boldsymbol{\Sigma}_0^{\frac{1}{2}} \boldsymbol{\Delta} \boldsymbol{\Sigma}_0^{\frac{1}{2}}\| \leq \eta \right\}$

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- Measurement: $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{d}$, $\mathbf{d} \sim \mathcal{N}(\mathbf{0}, \mathbf{D})$,

$$\mathbf{D} \in \mathcal{S}_2 = \left\{ \mathbf{D} : \begin{array}{l} \mathbf{D} = \mathbf{V}^T \mathbf{F} \mathbf{V}, \mathbf{F} = \mathbf{F}_0 + \boldsymbol{\Delta} \succeq \mathbf{0}, \|\mathbf{N}^{-\frac{1}{2}} \boldsymbol{\Delta} \mathbf{N}^{-\frac{1}{2}}\| \leq \eta, \\ \mathbf{V} = \mathbf{V}_0 + \mathbf{W},\ \|\mathbf{W}_i\| \leq \rho_i, i = 1, \ldots, m \end{array} \right\}$$

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- Measurement: $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{d}$, $\mathbf{d} \sim \mathcal{N}(\mathbf{0}, \mathbf{D})$,

$$\mathbf{D} \in \mathcal{S}_2 = \left\{ \mathbf{D} : \begin{array}{l} \mathbf{D} = \mathbf{V}^T \mathbf{F} \mathbf{V}, \mathbf{F} = \mathbf{F}_0 + \boldsymbol{\Delta} \succeq \mathbf{0}, \|\mathbf{N}^{-\frac{1}{2}} \boldsymbol{\Delta} \mathbf{N}^{-\frac{1}{2}}\| \leq \eta, \\ \mathbf{V} = \mathbf{V}_0 + \mathbf{W},\ \|\mathbf{W}_i\| \leq \rho_i, i = 1, \ldots, m \end{array} \right\}$$

- Unbiased estimator: $\widehat{\boldsymbol{\mu}} = (\mathbf{I} - \mathbf{K}\mathbf{C})\bar{\boldsymbol{\mu}} + \mathbf{K}\mathbf{y}$

Estimation in linear models

- Parameter: $\mathbf{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$
  - $\boldsymbol{\mu}$ unknown ... apriori estimate $\bar{\boldsymbol{\mu}}$
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- $\mathbf{P} = \mathbf{E}[(\widehat{\boldsymbol{\mu}} - \boldsymbol{\mu})(\widehat{\boldsymbol{\mu}} - \boldsymbol{\mu})^T] = (\mathbf{I} - \mathbf{K}\mathbf{C})^T \boldsymbol{\Sigma} (\mathbf{I} - \mathbf{K}\mathbf{C}) + \mathbf{K}^T \mathbf{D} \mathbf{K}$

Estimation in linear models

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- Goal: Choose $\mathbf{K}$ to minimize $\max_{1 \le i \le m} \left\{ \mathbf{Tr}(\mathbf{P}\mathbf{v}_i\mathbf{v}_i^T) \right\}$, $\mathbf{v}_i$ given vectors

Estimation in linear models

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- Measurement: $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{d}$, $\mathbf{d} \sim \mathcal{N}(\mathbf{0}, \mathbf{D})$, $\mathbf{D} \in \mathcal{S}_2$
- Unbiased estimator: $\widehat{\boldsymbol{\mu}} = (\mathbf{I} - \mathbf{K}\mathbf{C})\bar{\boldsymbol{\mu}} + \mathbf{K}\mathbf{y}$
- $\mathbf{P} = \mathbf{E}[(\widehat{\boldsymbol{\mu}} - \boldsymbol{\mu})(\widehat{\boldsymbol{\mu}} - \boldsymbol{\mu})^T] = (\mathbf{I} - \mathbf{K}\mathbf{C})^T \boldsymbol{\Sigma} (\mathbf{I} - \mathbf{K}\mathbf{C}) + \mathbf{K}^T \mathbf{D} \mathbf{K}$
- Goal: Choose $\mathbf{K}$ to minimize $\max_{1 \leq i \leq m} \left\{ \mathbf{Tr}(\mathbf{P}\mathbf{v}_i \mathbf{v}_i^T) \right\}$, $\mathbf{v}_i$ given vectors
- Robust optimization problem:

$$\min_{\mathbf{K}} \max_{\{(\boldsymbol{\Sigma} \in \mathcal{S}_1, \mathbf{D} \in \mathcal{S}_2)\}} \max_{\{1 \leq j \leq m\}} \left\{ \mathbf{v}_j^T (\mathbf{I} - \mathbf{K}\mathbf{C})^T \boldsymbol{\Sigma} (\mathbf{I} - \mathbf{K}\mathbf{C}) \mathbf{v}_j + \mathbf{v}_j^T \mathbf{K}^T \mathbf{D} \mathbf{K} \mathbf{v}_j \right\}$$

Estimation in linear models

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- $\mathcal{S}_s$, $\mathcal{S}_d$ are factorized uncertainty sets ... problem SOCP

The punchline !

- Three classes of tractable uncertainties: SOCPs instead of SDPs
  - Polytopic uncertainty
  - Affine uncertainty
  - Factorized uncertainty

The punchline !

- Three classes of tractable uncertainties: SOCPs instead of SDPs
  - Polytopic uncertainty
  - Affine uncertainty
  - Factorized uncertainty
- These arise quite naturally in disparate application areas