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Case Studies in Robust Optimization

Laurent El Ghaoui

Department of EECS, UC Berkeley

A Case Study in Robust Optimization

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A Robust Option Pricing Problem

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Robust optimization

standard form:

$$\min_x \sup_{u \in \mathcal{U}} f_0(x,u) \ : \ \forall\, u \in \mathcal{U}, \ \ f_i(x,u) \le 0, \ \ i = 1,\ldots,m$$

- $x \in \mathbf{R}^n$ is the decision variable
- u is a parameter vector affecting the problem data
- set $\mathcal{U}$ describes uncertainty on u

a semi-infinite optimization problem

Agenda

- option basics
- option pricing problem
- worst-case approach

joint work with: A. d'Aspremont

Options

Options are time bombs. —Warren Buffett, 2003

let $x(T)$ denote the price of an asset at time T

an *European call option* with maturity T and strike price K is a contract with payoff

$$(x(T) - K)_+ = \sup(x(T) - K, 0)$$

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what is the price of the option,
assuming no arbitrage ("free lunch") is possible?

Price of option

fundamental result of finance: assuming a zero risk-free interest rate, the no-arbitrage price of the option is

$$\mathbf{E}_\pi (x(T) - K)_+$$

for some distribution π on the asset price at time T

such a π is called risk-neutral

Forwards

under the risk-neutral distribution π, the current asset price is

$$\mathbf{E}_{\pi} x(T)$$

i.e., π is a martingale

Basket options

let $x(T)$ be a n-vector of prices of assets at time T

a basket option with weight vector w and strike K is the contract with payoff

$$(w^T x(T) - K)_+$$

denote the basket option by (w, K) and its price by

$$C_\pi(w, K) := \mathbf{E}_\pi(w^T x(T) - K)_+$$

(maturity date is implicit here)

Option pricing problem

given

- $w_0, w_1, \ldots, w_m$ in $\mathbf{R}^n_+$ (*basket weights*)
- $K_0, K_1, \ldots, K_m$ in $\mathbf{R}_+$ (*strike prices*)
- $p_1, \ldots, p_m$ in $\mathbf{R}_+$ (*observed option prices*)

determine the price of the basket option with weight w_0 and strike price K_0

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determine the price of the basket option with weight w_0 and strike price K_0

in practice, we are also given the current asset prices themselves, $q \in \mathbf{R}_+^n$ ("forwards")

Challenges & methods

challenges:

- the risk-neutral measure is not the empirical distribution of assets
- hence, we may have to rely on option & forward prices only

two approaches:

- model-based approach
- arbitrage-based approach

Model-based approach

- assume a log-normal diffusion model for the asset prices

$$ds = Asdt + Bsdw,$$

  where $s = \log x$, and w is a multidimensional Brownian motion

- fit the model to observed option prices

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with some approximations, this problem is an SDP (Aspremont, 2000)

Model-based approach

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$$ds = Asdt + Bsdw,$$

where $s = \log x$, and w is a multidimensional Brownian motion

- fit the model to observed option prices

pros and cons:

- very versatile, and "easy" to solve (albeit only recently)
- makes a structural assumption about the risk-neutral measure π
- provides a point estimate for the price of basket (w_0, K_0)

No-arbitrage approach

find bounds on the price of basket (w_0, K_0) by solving semi-infinite LP

$$\begin{array}{ll} \sup_\pi / \inf_\pi & \mathbf{E}_\pi (w_0^T x - K_0)_+ \\ \text{s.t.} & \mathbf{E}_\pi (w_i^T x - K_i)_+ = p_i, \quad i = 1, \ldots, m \end{array}$$

the optimization variable is the risk-neutral probability measure π

No-arbitrage approach

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$$

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pros and cons:

- problem may be difficult
- approach provides only bounds, but ...
- ... makes no assumptions about market dynamics

Upper bound problem

upper bound problem:

$$p^{\sup} := \sup_{\pi \in \mathcal{P}} \int_{\Omega} \phi_0(x)\pi(x)dx \quad : \quad \begin{array}{l} \int_{\Omega} \pi(x)dx = 1, \\ \int_{\Omega} \phi_i(x)\pi(x)dx = p_i, \ i = 1, \ldots, m, \end{array}$$

where

- $\Omega = \mathbf{R}^n_+$
- $\mathcal{P}$ is the set of densities with support in Ω
- $\phi_i(x) := (w_i^T x - K_i)_+, \quad i = 0, 1, \ldots, m$

Upper bound problem

upper bound problem:

$$p^{\text{sup}} := \sup_{f \in \mathcal{F}} \int_\Omega \phi_0(x) f(x) dx \quad : \quad \begin{aligned} &\int_\Omega f(x) dx = 1, \\ &\int_\Omega \phi_i(x) f(x) dx = p_i, \ i = 1, \ldots, m, \end{aligned}$$

Lagrangian:

$$\begin{aligned} \mathcal{L}(\pi, \lambda, \lambda_0) &= \int_\Omega \phi_0(x) \pi(x) dx + \lambda_0 \left(1 - \int_\Omega \pi(x) dx\right) \\ &\qquad + \lambda^T \left(p - \int_\Omega \phi(x) \pi(x) dx\right) \end{aligned}$$

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the dual is the robust linear programming problem

$$\begin{array}{lll} d^{\sup} := & \inf_{\lambda \in \mathbf{R}^m} & \lambda^T p + \lambda_0 \\ & \text{s.t.} & \forall x \in \Omega, \;\; \lambda^T \phi(x) + \lambda_0 \geq \phi_0(x) \end{array}$$

Dual gives a hedging strategy

let λ_0, λ be feasible for the dual:

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strategy: invest λ_i in basket (w_i, K_i), λ_0 in cash

Dual gives a hedging strategy

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strategy: invest λ_i in basket (w_i, K_i), λ_0 in cash

- price of strategy: $\lambda^T p + \lambda_0$
- taking expectations in $(*)$, get

$$\lambda^T p + \lambda_0 \geq \mathbf{E}_\pi \phi_0(x) = p^{\rm sup}$$

(proves weak duality)

A special case

we are given option prices on the $m = n$ individual assets, as well as forward prices

$$\begin{array}{ll} \min_\pi / \sup_\pi & \mathbf{E}_\pi (w_0^T x - K_0)_+ \\ \text{s.t.} & \mathbf{E}_\pi (x_i - K_i)_+ = p_i, \quad i = 1, \ldots, n \\ & \mathbf{E}_\pi x = q \end{array}$$

we may further relax the problem by ignoring the forward price information

Special case : upper bound

- no-arbitrage: problem is feasible iff $0 \le p \le q \le p + K$
- upper bound is given by

$$d^{\sup} = \max_{0 \le j \le n+1} \; w_0^T p + \sum_i w_{0,i} \min(q_i - p_i, \beta_j K_i) - \beta_j K_0,$$

where $\beta_0 = 0 \le \beta_j := (q_j - p_j)/K_j \le 1 = \beta_{n+1}$

- upper bound is attained, hence $d^{\sup} = p^{\sup}$

Ignoring forward prices

ignore the constraints

$$\mathbf{E}_\pi x = q$$

obtain formula by maximizing $d^{\rm sup}$ wrt q:

$$d^{\rm sup} = p^{\rm sup} = w_0^T p + (w_0^T K - K_0)_+$$

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obtain formula by maximizing d^{sup} wrt q:

$$d^{\text{sup}} = p^{\text{sup}} = w_0^T p + (w_0^T K - K_0)_+$$

makes sense:

- concave in p (the RHS of our primal LP)
- convex in (w_0, K_0)
- interpolates p_i at $w_0 = i$-th unit vector of $\mathbf{R}^n$, and $K_0 = K_i$

Sketch of proof

weak duality follows from homogeneity and convexity of $x \to x_+$:

$$\begin{aligned} \mathbf{E}_\pi(w_0^T x - K_0)_+ &= \mathbf{E}_\pi(w_0^T(x-K) + (w_0^T K - K_0))_+ \\ &\leq w_0^T \mathbf{E}_\pi(x-K)_+ + (w_0^T K - K_0)_+ \qquad (w_0 \geq 0) \\ &= w_0^T p + (w_0^T K - K_0)_+ \end{aligned}$$

Sketch of proof

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$$\begin{array}{rcll} \mathbf{E}_\pi(w_0^T x - K_0)_+ & = & \mathbf{E}_\pi(w_0^T(x-K) + (w_0^T K - K_0))_+ & \\ & \leq & w_0^T \mathbf{E}_\pi(x-K)_+ + (w_0^T K - K_0)_+ & (w_0 \geq 0) \\ & = & w_0^T p + (w_0^T K - K_0)_+ & \end{array}$$

strong duality: choose $x = p + K$ with pty 1 if $w_0^T K \geq K_0$, otherwise take a limit of feasible distributions

$$x = \begin{cases} \epsilon^{-1} p + K & \text{with probability } \epsilon, \\ 0 & \text{with probability } 1-\epsilon. \end{cases}$$

Lower bound

- dual problem reduces to a finite LP
  (with $O(n)$ variables and constraints)

Lower bound

- dual problem reduces to a finite LP
  (with $O(n)$ variables and constraints)
- if we ignore forward prices

$$p^{\inf} = d^{\inf} = \sum_{i\,:\,K_i w_i \geq K_0} p_i w_i + \max_{j\,:\,K_j w_j < K_0} \left(\sum_{i\,:\,K_i w_i < K_0} p_i w_i \min(1, \frac{K_0 - K_j w_j}{K_0 - K_i w_i}) - K_0 + w_j K_j \right)_+$$

in which case, direct proof of perfect duality

General case: integral transform approach

the option price function

$$C(w, K) = \mathbf{E}_\pi (w^T x - K)_+$$

is an integral transform of measure π, with kernel the payoff function $(w^T x - K)_+$

General case: integral transform approach

the option price function

$$C(w, K) = \mathbf{E}_\pi (w^T x - K)_+ \qquad (1)$$

is an integral transform of measure π, with kernel the payoff function $(w^T x - K)_+$

make C the variable, and solve interpolation problem

$$\sup_C \ / \ \inf_C \ C(w_0, K_0) \ : \quad \begin{array}{l} C(w_i, K_i) = p_i, \ i = 1, \ldots, m, \\ C \text{ of the form } (1) \end{array}$$

LP relaxation

conditions under which C is of form

$$C(w, K) = \mathbf{E}_\pi (w^T x - K)_+$$

for some measure π exist, but seem hard to check

semi-infinite LP relaxation:

$$\begin{array}{ll} \sup_C / \inf_C & C(w_0, K_0) \\ \text{s.t.} & C(w, K) \text{ convex in } (K, w) \\ & C(w, K) \text{ homogeneous of degree } 1 \\ & -1 \leq \partial C(w, K)/\partial K \leq 0 \text{ and } C(w, K) \text{ nondecreasing in } w \\ & C(w_i, K_i) = p_i, \; i = 1, \ldots, m \end{array}$$

Finite LP formulation

optimal C of semi-infinite LP is piecewise affine

hence the semi-infinite LP can be reduced exactly to a finite LP

$$
\begin{array}{ll}
\sup / \inf & p_0 \\
\text{subject to} & \langle g_i, (w_j, K_j) - (w_i, K_i) \rangle \leq p_j - p_i, \quad i, j = 0, ..., m+n+1 \\
& g_{i,j} \geq 0, -1 \leq g_{i,n+1} \leq 0, \quad i = 0, ..., m+n+1, \quad j = 1, ..., n \\
& \langle g_i, (w_i, K_i) \rangle = p_i, \quad i = 0, ..., m+n+1,
\end{array}
$$

where the variables g_i are subgradients of C^{opt}

(for upper bound, in special case, LP is exact)

Robustness

in practice we have uncertainty:

- basket weights may change, or not exactly known at present time
- price information may be noisy (*e.g.*, bid-ask)
- strike prices also may vary
- hedging strategies may be implemented with errors

we address the upper bound problem, in the special case

Bid-ask spread

bid-ask spread corresponds to a box uncertainty for the vector of observed option prices

$$\underline{p} \leq p \leq \overline{p}$$

the worst-case value of upper (lower) bound is attained at $p^{\mathrm{worst}} = \underline{p}$ (or $p^{\mathrm{worst}} = \overline{p}$)

we may want to introduce "correlation" in the model ... ellipsoidal uncertainty on p is as easy

Ellipsoidal uncertainty in basket weights

worst-case upper bound under weight uncertainty is

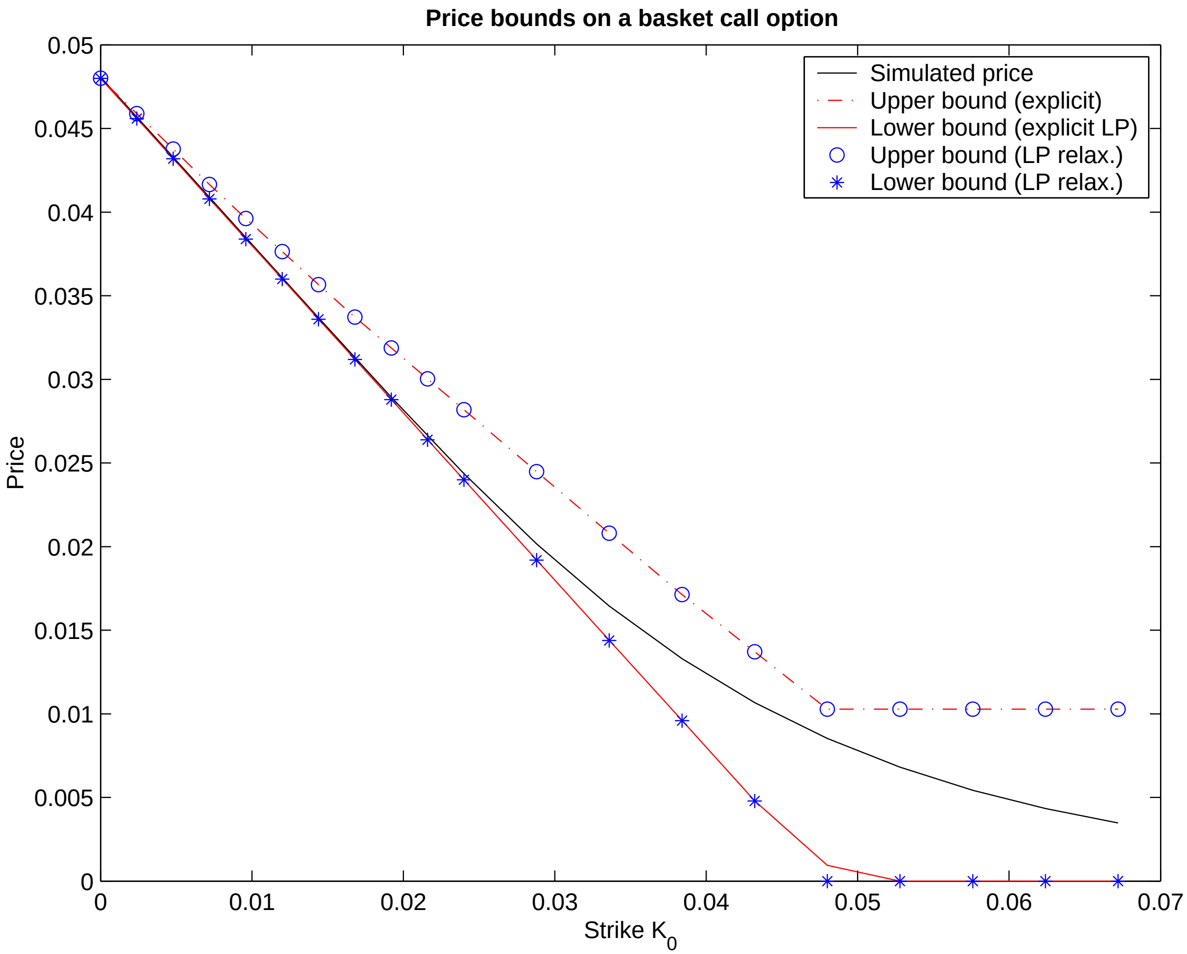
$$p^{\sup} = \max_{w \in \mathcal{E}} w^T p + (w^T K - K_0)_+$$

where $\mathcal{E} = \{\hat{w} + Ru \ : \ \|u\|_2 \leq 1\}$ is a given ellipsoid

we have

$$\begin{aligned} p^{\sup} &= \max_{w,t} w^T p + t(w^T K - K_0) \ : \ w \in \mathcal{E}, \ \ 0 \leq t \leq 1 \\ &= \max_{t=0,1} (tK + p)^T w + \|R^T(tK + p)\|_2 - tK_0 \end{aligned}$$

Example



Price bounds on a basket call option
Simulated price
Upper bound (explicit)
Lower bound (explicit LP)
Upper bound (LP relax.)
Lower bound (LP relax.)
Price
Strike K0
0.05
0.045
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
0
0.01
0.02
0.03
0.04
0.05
0.06
0.07

Robustness for the general case

i have no idea about this

Summary

- general problem important in practice
- special cases yields easy-to-compute bounds
- developed an LP relaxation for general case
- relaxation exact in some special cases

Further research

- imposing smoothness constraints
- multi-period problem (Bertsimas, 2003)
- link with model-based approaches

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for more info: d'Aspremont & El Ghaoui, Static arbitrage bounds for basket option prices, submitted to *Oper. Res.*, 2003