Higher-Order Balance Dynamics and Nonlinear Averaging in GFD

DJOKO WIROSOETISNO

Universiteit Twente

Collaborators:
T. G. Shepherd (Toronto)
R. M. Temam (Orsay & Indiana)

Outline

• Motivation: balance and free gravity waves
• Slow-time problem: balance dynamics
  ◦ Asymptotic accuracy for balance
• Fast-time problem: slow eq’ns for PV and GW
  ◦ Connection with balance dynamics
  ◦ Example: first and second orders
• Discussion
A Prototypical System

The shallow-water equations in terms of

- QG pot'l vorticity \( q := \nabla \times \mathbf{v} - h \)
- divergence \( \delta := \nabla \cdot \mathbf{v} \)
- ageostrophic height \( \eta := h - (\Delta - 1)^{-1}\tilde{q} \)

reads

\[
\frac{\partial q}{\partial t} + \nabla \times (\mathbf{v} \cdot \nabla \mathbf{v}) - \nabla \cdot (\mathbf{v} h) = 0
\]
\[
\frac{\partial \delta}{\partial t} + \frac{1}{\varepsilon}(\Delta - 1)\eta + \nabla \cdot (\mathbf{v} \cdot \nabla \mathbf{v}) = 0 \tag{\ast 1}
\]
\[
\frac{\partial \eta}{\partial t} + \frac{1}{\varepsilon}\delta + \nabla \cdot (\mathbf{v} h) + (\Delta - 1)^{-1} \left[ \nabla \cdot (\mathbf{v} h) - \nabla \times (\mathbf{v} \cdot \nabla \mathbf{v}) \right] = 0
\]

Write as a prototypical system

\[
\frac{dy}{dt} = \mathcal{Y}(x, y) \tag{\ast 2}
\]
\[
\frac{dx}{dt} + \frac{1}{\varepsilon}L_x x = \mathcal{X}(x, y)
\]

or, with \( u = (x, y) \),

\[
\frac{du}{dt} + \frac{1}{\varepsilon}L u = \mathcal{F}(u) \tag{\ast 3}
\]
Slow-time Problem: Balance Dynamics

\[ \frac{dy}{dt} = \mathcal{Y}(x, y) \]
\[ \frac{dx}{dt} + \frac{1}{\varepsilon} L_x x = \mathcal{X}(x, y) \]  \hspace{1cm} (*2)

Solve perturbatively: expand

\[ x = x^0 + \varepsilon x^1 + \cdots \]

but not \( y \).

Solving order-by-order:

\( \mathcal{O}(\varepsilon^{-1}) : L_x x^0 = 0 \)
\[ \Rightarrow x^0 = 0 \]
\( \mathcal{O}(1) : L_x x^1 = \mathcal{X}(0, y) \)
\[ \Rightarrow x^1 = L_x^{-1} \mathcal{X}(0, y) \]

It turns out that \( x(t) \) is “slaved” to \( y(t) \) to all orders:

\[ x = \varepsilon x^1 + \varepsilon^2 x^2 + \cdots \]
\[ = \varepsilon U_1(y) + \varepsilon^2 U_2(y) + \cdots \]
\[ := U(y, \varepsilon) \]
Slow-time Problem: Formal

$U(y; \varepsilon)$ obeys the superbalance equation [Lorenz]

$$U'(y; \varepsilon) \mathcal{Y}(U(y; \varepsilon), y) + \frac{1}{\varepsilon} L_x U(y; \varepsilon) = \mathcal{X}(U(y; \varepsilon), y) \quad (\ast 4a)$$

⇒ equation for the (exact) “slow manifold”

while $y(t)$ obeys

$$\frac{dy}{dt} = \mathcal{Y}(U(y; \varepsilon), y) \quad (\ast 4b)$$

⇒ (∞-order) balance model

Notes:

• No issue of resonances ever arises
• Initial conditions of (\ast 1) must satisfy

$$u(0) \in S_s := \{(x, y) = (U(y; \varepsilon), y)\}$$

i.e., $u(0)$ must be on the slow manifold
• $S_s$ is invariant under (\ast 1)
Slow-time Problem: Asymptotics

Recall

\[
\frac{dy}{dt} = \mathcal{Y}(x, y)
\]

\[
\frac{dx}{dt} + \frac{1}{\varepsilon} L_x x = \mathcal{X}(x, y)
\]  \hspace{1cm} (**2)**

Seek solutions of the form \(x = U(y; \varepsilon)\), which must obey

\[
U'(y; \varepsilon) \mathcal{Y}(U(y; \varepsilon), y) + \frac{1}{\varepsilon} L_x U(y; \varepsilon) = \mathcal{X}(U(y; \varepsilon), y)
\]  \hspace{1cm} (**4a**)

Since solving perturbatively, how far can one go?

**Claim 1.** For \((x, y)\) finite-dimensional and \((\mathcal{X}, \mathcal{Y})\) polynomials of degree \(d\), one can find \(U_*(y; \varepsilon)\) that satisfies (**4a**) up to an error of order \(\exp(-\kappa/\varepsilon^{1/(d+1)})\) as \(\varepsilon \to 0\).

**PROOF:** construct

\[
U^N(y; \varepsilon) = \sum_{n=1}^{N} \varepsilon^n U_n(y)
\]

and take \(N(\varepsilon) \sim \varepsilon^{-1/d+1}\).
Slow-time Problem: Error Bounds

Let $Y(t)$ be the solution of

$$
\frac{dY}{dt} = \mathcal{V}(U^N(Y; \varepsilon), Y).
$$

where

$$(U^N)'(Y; \varepsilon)\mathcal{V}(U^N(Y; \varepsilon), y) + \frac{1}{\varepsilon} L_x U^N(Y; \varepsilon) - A(U^N(Y; \varepsilon), Y)
= \varepsilon^{N+1}\text{Rem}(Y; \varepsilon)$$

Let $w(t) := (x(t) - U^N(Y(t); \varepsilon), y(t) - Y(t))$, it satisfies

$$
\frac{dw}{dt} = \varepsilon^{N+1}\text{Rem}(Y(t); \varepsilon) + (D\mathcal{F}).w(t) + \text{h.o.t.}
$$

Taking $w(0) = 0$, by Gronwall’s inequality

$$
|w(t)| \leq \varepsilon^{N+1}|\text{Rem}| \exp(t|D\mathcal{F}|)
$$

Back to the optimal truncation,

**Claim 2.** *One has*

$$
|x(t) - U_*(Y(t); \varepsilon)| + |y(t) - Y(t)| \leq \exp(-\kappa/\varepsilon^{1/d+1})
$$

for $0 \leq t \leq T(y_0)$. 
Slow-time Problem: PDE Asymptotics

Consider a weak-wave model [Nore & Shepherd]

\[
\frac{\partial q}{\partial t} + \partial (q, \psi) = 0 \\
\frac{\partial \delta}{\partial t} + \frac{1}{\varepsilon}(\Delta - 1)\eta = 0 \\
\frac{\partial \eta}{\partial t} + \frac{1}{\varepsilon}\delta + (\Delta - 1)^{-1}\partial (q, \psi)
\]

where \(\Delta \psi = [1 + (\Delta - 1)^{-1}]q + \eta\).

Claim 3. Suppose that \(q(t) \in H^s(T^2), s \geq 2\). Then one can find \(U^N[q; \varepsilon]\) that satisfies (*4a) up to a remainder

\[
|R[q]|_s \leq C[q]\exp(-\kappa[q]/\varepsilon^{1/3})
\]

as \(\varepsilon \to 0\).
The Problem Revisited

Recall, with \( u = (x, y) \),

\[
\frac{du}{dt} + \frac{1}{\varepsilon} Lu = \mathcal{F}(u) \\
u(0) = u_0
\]  

(*3)

In the fast time \( s = t/\varepsilon \)

\[
\frac{du}{ds} + Lu = \varepsilon \mathcal{F}(u)
\]

Rewrite this with

\[
v(s) := e^{sL}u(s) \\
F(v, s) := e^{sL}\mathcal{F}(e^{-sL}v)
\]

to get

\[
\frac{dv}{ds} = \varepsilon F(v, s) \\
v(0) = u_0
\]  

(*5)
Fast-time Problem: Formal

With \( u(s) = e^{-sL}v(s) \),

\[
\frac{dv}{ds} = \varepsilon F(v, s) \quad (*)5
\]
\[v(0) = u_0\]

Invoke the averaging ansatz:
[Krylov–Bogolyubov–Mitropol’skii, 1930s–]

\[
v(s) = R(\tilde{v}, s; \varepsilon) \quad (*6a)
\]

where

\[
\frac{d\tilde{v}}{ds} = W(\tilde{v}; \varepsilon) \quad (*6b)
\]

Sub (*6a) and (*6b) into (*5):

\[
\varepsilon F(R(\tilde{v}, s; \varepsilon), s) = \frac{\partial R}{\partial \tilde{v}}(\tilde{v}, s; \varepsilon)W(\tilde{v}; \varepsilon) + \frac{\partial R}{\partial s}(\tilde{v}, s; \varepsilon) \quad (*6c)
\]
Fast-time Problem (cont’d)

To solve, expand in $\varepsilon$:

$$R(\bar{v}, s; \varepsilon) = \bar{v}(s) + \varepsilon R_1(\bar{v}, s) + \varepsilon^2 R_2(\bar{v}, s) + \cdots$$

$$W(\bar{v}; \varepsilon) = \varepsilon W_1(\bar{v}) + \varepsilon^2 W_2(\bar{v}) + \cdots$$

At the first two orders:

$${\mathcal{O}}(1): \quad W_0(\bar{v}) = 0$$

$$R_0(\bar{v}, s) = \bar{v}$$

$${\mathcal{O}}(\varepsilon): \quad \frac{\partial R_1(\bar{v}, s)}{\partial s} + W_1(\bar{v}) = F(\bar{v}, s)$$

which gives

$$W_1(\bar{v}) = \{ F(\bar{v}, s) \}_{\text{res}}$$

$$R_1(\bar{v}, s) = R_1(\bar{v}, 0) + \int_0^s F(\bar{v}, \sigma) - W_1(\bar{v}) \, d\sigma$$

$\Rightarrow$ solution is not unique

Choices:

- normal form: $R_1(\bar{v}, 0) \neq 0$
  $\rightarrow$ simpler expansion

- ours: $R_1(\bar{v}, 0) = 0$
  $\rightarrow$ close connection with slow expansion
Fast-time Problem: Slow Solutions

Continuing to higher orders,

\[ u(0) = v(0) = \bar{v}(0) = R(\bar{v}(0), 0; \varepsilon) \]  \hspace{1cm} (#7)

Imposing slowness on the original variable

\[ u(s) = e^{-sL}v(s) = e^{-sL}R(\bar{v}(s), s; \varepsilon), \]

\[ \frac{\partial}{\partial s}[e^{-sL}R(\bar{v}(s), s; \varepsilon)] = 0 \]  \hspace{1cm} (#8)

Integrating (#8) using (#7) as initial conditions gives

\[ e^{-sL}R(\bar{v}(s), s; \varepsilon) = \bar{v}(s) \]

\[ \Rightarrow \text{Slow solutions are fixed points of } e^{-sL}R(\bar{v}, s; \varepsilon) \]

Denote \( S_f := \{ \bar{v} | \bar{v} = e^{-sL}R(\bar{v}, s; \varepsilon) \forall s \} \)

Claim 4. This fixed point is the solution of the slow-time expansion. In other words, \( S_s = S_f \) (formally). Moreover, \( S_s \) is (formally) invariant under the averaged dynamics,

\[ \frac{d\bar{v}}{ds} = \varepsilon W(\bar{v}; \varepsilon). \]
Example: Leading Order

Return to the weak-wave model

\[
\frac{\partial q}{\partial t} + \partial (q, \psi) = 0 \\
\Delta \psi = [1 + (\Delta - 1)^{-1}]q + \eta \\
\frac{\partial \delta}{\partial t} + \frac{1}{\varepsilon}(\Delta - 1)\eta = 0 \\
\frac{\partial \eta}{\partial t} + \frac{1}{\varepsilon}\delta + (\Delta - 1)^{-1}\partial (q, \psi)
\]

The averaged variables \((\bar{q}, \bar{\delta}, \bar{\eta})\) are

\[
q(t) = \bar{q}(t) + \varepsilon R_1^{(q)}(\bar{q}, \bar{\delta}, \bar{\eta}, t/\varepsilon) + \mathcal{O}(\varepsilon^2) \\
\begin{bmatrix} \delta(t) \\ \eta(t) \end{bmatrix} = e^{-(t/\varepsilon)L} \begin{bmatrix} \bar{\delta}(t) \\ \bar{\eta}(t) \end{bmatrix} + \varepsilon R_1^{(x)}(\bar{q}, \bar{\delta}, \bar{\eta}, t/\varepsilon) + \mathcal{O}(\varepsilon^2)
\]

valid for \(t \leq T \sim \mathcal{O}(1)\) or \(\mathcal{O}(- \log \varepsilon)\), and obeys

\[
\frac{\partial \bar{q}}{\partial t} + \partial (\bar{q}, \bar{\psi}) = 0 \\
\Delta \psi = [1 + (\Delta - 1)^{-1}]\bar{q} \\
\frac{\partial}{\partial t} \left( \frac{\bar{\delta}}{\bar{\eta}} \right) + M(\bar{q}) \left( \frac{\bar{\delta}}{\bar{\eta}} \right) = 0
\]

which is decoupled: the “modulation” operator \(M(\bar{q})\)

- is skew-hermitian
- only couples \((\bar{\delta}, \bar{\eta})\) within each energy shell
Example: Second Order

The averaged variables \((\bar{q}, \bar{\delta}, \bar{\eta})\) obey

\[
\frac{\partial \bar{q}}{\partial t} + \partial (\bar{q}, \bar{\psi}) = \varepsilon W_2^{(q)}(\bar{q}, \bar{\delta}, \bar{\eta})
\]

\[
\frac{\partial}{\partial t} \left( \frac{\bar{\delta}}{\bar{\eta}} \right) + M(\bar{q}) \left( \frac{\bar{\delta}}{\bar{\eta}} \right) = \varepsilon W_2^{(x)}(\bar{q}, \bar{\delta}, \bar{\eta})
\]

\((^9\ast)\)

which is no longer decoupled:

- GW interactions between energy shells possible
- GW–PV energy exchange possible

but only when the free GW is non-zero:

The slow manifold \((\bar{\delta}, \bar{\eta}) = \varepsilon U_1(\bar{q})\) is a fixed point of \((^9\ast)\)
Summary

Objective: obtain slow evolution equations

Balance Models:
• only describe “potential vorticity dynamics”
• can go to very high orders
• resonances seem to play no rôle

Averaged Models:
• include “free gravity wave” amplitude
• have balance models as a “fixed point”
• sensitive to resonances
• higher-order models still problematic?