

FAST SINGULAR OSCILLATING LIMITS AND GLOBAL  
REGULARITY FOR THE 3D PRIMITIVE EQUATIONS OF  
GEOPHYSICS

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Fast Singular Oscillating Limits of Stably Stratified Three-Dimensional  
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### 3D Euler-Boussinesq Equations in Atmospheric/Ocean Flows:

$$\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{U} + f_0 \mathbf{e}_3 \times \mathbf{U} = -\nabla p - \rho \mathbf{e}_3,$$

$$\partial_t \rho + \mathbf{U} \cdot \nabla \rho = N_0^2 U_3, \quad \nabla \cdot \mathbf{U} = 0$$

$$\mathbf{U}(t, x_1, x_2, x_3)|_{t=0} = \mathbf{U}(0, x_1, x_2, x_3), \quad \rho(t, x_1, x_2, x_3)|_{t=0} = \rho(0, x_1, x_2, x_3).$$

$f_0 \mathbf{e}_3 \times \mathbf{U}$ : Coriolis force (Earth rotation),  $\mathbf{e}_3$  vertical unit vector

$\rho$ : normalized relative density variation w.r.t. density profile

$-\rho \mathbf{e}_3$ : normalized buoyancy (gravity) force on fluid element

$-\rho = \theta$ ,  $\theta$  normalized relative (potential) temp. variation

Equation for  $\rho$  is equivalent to equation for  $\theta$ , the normalized relative potential temperature variation.

$f_0 = 2\Omega_0$  frequency of rotation,  $f_0 \sim 10^{-4} \text{ s}^{-1}$  at midlatitudes

$N_0$  = Brunt-Vaisala frequency of gravity waves

$$N_0 = \left( g \left\langle \frac{1}{\rho_b} \frac{d\rho_b}{dx_3} \right\rangle \right)^{1/2}, \quad \langle \rangle \text{ averaging; } \rho_b \text{- base profile}$$

$N_0 \sim 1.2 \times 10^{-2} \text{ s}^{-1}$  at tropopause

### 3D Navier-Stokes Equations of Geophysics

$$\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla \mathbf{U} + f \mathbf{e}_3 \times \mathbf{U} = -\nabla p + \rho \mathbf{e}_3 + \nu_1 \Delta \mathbf{U} + \mathbf{F},$$

$$\partial_t \rho + \mathbf{U} \cdot \nabla \rho = -N^2 U_3 + \nu_2 \Delta \rho + F_4, \quad \nabla \cdot \mathbf{U} = 0,$$

$$\mathbf{U}(t, \mathbf{x})|_{t=0} = \mathbf{U}(0, \mathbf{x}), \quad \rho(t, \mathbf{x})|_{t=0} = \rho(0, \mathbf{x})$$

Initial Conditions: three-dimensional unprepared (unbalanced) initial data. Smoothness assumptions on initial data are the same as in local existence theorems.

Rotation and mean stratification gradient are aligned parallel to the vertical axis  $x_3$ . Here  $\mathbf{x} = (x_1, x_2, x_3)$ ,  $\mathbf{U} = (U_1, U_2, U_3)$  is the velocity field and  $\rho$  is the buoyancy variable (relative density variation);  $\mathbf{F} = (F_1, F_2, F_3)$ . Here  $\nu_1$  and  $\nu_2$  are the kinematic viscosity and the heat conductivity, respectively; the ratio  $Pr = \nu_1/\nu_2$  is known as the Prandtl number. We consider periodic boundary conditions in a parallelepiped  $[0, 2\pi] \times [0, 2\pi] \times [0, 2\pi]$ , as well as stress-free conditions  $U_3 = 0$ ,  $\partial U_1/\partial x_3 = \partial U_2/\partial x_3 = 0$  at  $x_3 = 0, 2\pi$ .

Let  $U_h$  be a characteristic horizontal velocity scale. Let  $H$  and  $L$  be vertical and horizontal length scales and  $a = H/L$  is the aspect ratio parameter. We define Froude numbers based on horizontal and vertical scales:

$$F_h = U_h/(LN_0) \equiv 1/N, \quad F_v = U_h/(HN_0) = F_h/a.$$

The classical Rossby and anisotropic Rossby numbers are defined as follows

$$Ro = U_h/(f_0L) \equiv 1/f, \quad Ro_a = a Ro, \quad a = H/L.$$

Here  $f = Ro^{-1}$  and  $N = F_h^{-1}$  are dimensionless rotation and stratification parameters. The Burger number characterizes relative importance of the effects of rotation and stratification:

$$Bu = Ro_a^2/F_h^2 \equiv Ro^2/F_v^2 \equiv N^2 a^2 / f^2 = N_0^2 a^2 / f_0^2.$$

Flows with  $Bu \ll 1$  are rotation dominated and  $Bu \gg 1$  corresponds to stratification dominated flows. An equivalent measure of the relative importance of stratification vs rotation is the internal radius of deformation  $\Lambda$ , which compares (stable) density stratification effects with respect to rotation. The internal (Rossby) radius of deformation  $\Lambda$  is defined as

$$\Lambda = N_0 H / f_0,$$

so that  $Bu = (\Lambda/L)^2$ . Regimes where  $Bu$  is either much greater or much less than one are both important for geophysical flows. Processes which couple rotation and stratification, such as baroclinic instability, have  $Bu = O(1)$  as a natural case.

## ASYMPTOTIC LIMITS

- High Frequency Inertio-Gravity Wave Limit
- Capture Averaged Effects of Nonlinear Wave Dynamics
- Nonlinear Interactions and Resonances of Fast Waves:  
Nonlinear Averaging
- Non-Hydrostatic

## MATHEMATICAL ISSUES

- Fast Singular Oscillating Limits
- Cancellation of Oscillations and Restricted Interactions
- Strong Solutions and Strong Convergence
- Conservation Laws (Euler)
- Attractors (Navier-Stokes)

### 3D INITIAL VALUE PROBLEM FOR SMALL $\epsilon = 1/N$

- The nature of the limit resonant equations and the regularity of their solutions.
- The limit equations are genuinely 3-dimensional depending on all three variables  $x_1, x_2$  and  $x_3$  but with restricted wave-number interactions in the nonlinear term. The existence and regularity theory for those limit equations is non-trivial. The proof of global regularity is based on the Littlewood-Paley dyadic decomposition and new estimates for restricted nonlinear interactions.
- Strong convergence of solutions of 3D Navier-Stokes equations to those of the limit equations.
- Bootstrapping from analysis of the first three questions the infinite time regularity of solutions of 3D Navier-Stokes equations for  $\epsilon$  small but finite.

We introduce a change of variables  $\rho \rightarrow N\rho$  and combine velocity and buoyancy variable in one variable  $\mathbf{U}^\dagger = (\mathbf{U}, \rho)$  after which equations written in non-dimensional variables take more symmetric form ( $\epsilon = 1/N$ ):

$$\partial_t \mathbf{U}^\dagger + \mathbf{U} \cdot \nabla \mathbf{U}^\dagger + \frac{1}{\epsilon} \mathbf{M} \mathbf{U}^\dagger = -\nabla^\dagger p + \bar{\nu} \Delta \mathbf{U}^\dagger + \mathbf{F}^\dagger, \quad \nabla \cdot \mathbf{U} = 0$$

$$\mathbf{U}^\dagger(t, x)|_{t=0} = \mathbf{U}^\dagger(0, x)$$

where  $\nabla^\dagger p = (\nabla p, 0)$ ,  $\mathbf{F}^\dagger = (\mathbf{F}, F_4)$  (where  $F_4$  is rescaled),

$$\mathbf{M} = (\mathbf{S} + \eta \mathbf{R}), \eta = f/N, \mathbf{R} = \begin{pmatrix} \mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \mathbf{S} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{J} \end{pmatrix}, \mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$\bar{\nu} = \text{diag}(\nu_1, \nu_1, \nu_1, \nu_2)$  is the viscosity matrix,  $\eta$  is fixed.

The linear part of inviscid equations is

$$\partial_t \mathbf{U}^\dagger + \frac{1}{\epsilon} \mathbf{M} \mathbf{U}^\dagger = -\nabla^\dagger p, \quad \nabla \cdot \mathbf{U} = 0.$$

or, equivalently, after applying Leray projection on divergence free fields

$$\partial_t \Phi + \frac{1}{\epsilon} \mathbf{L} \Phi = \mathbf{0}.$$

We define (unitary group, preserves every Sobolev norm)

$$\mathbf{E}(-N\mathbf{L}t) = \mathbf{E}\left(-\frac{1}{\epsilon}\mathbf{L}t\right) = \exp\left(-\frac{1}{\epsilon}\mathbf{L}t\right)$$

The solutions of this linear equation are (i) 3DQG modes (non-oscillating solutions, zero eigenmodes) and (ii)  $\exp(\pm i \frac{1}{\epsilon} \omega_n t) \mathbf{v}_n$  inerto-gravity waves (oscillating solutions),  $\omega_n^2 = \frac{|n_h|^2}{|n|^2} + \eta^2 \frac{n_3^2}{|n|^2}$ .

## NONLINEAR EQUATIONS

Applying the Leray projection  $\mathbf{P}$  onto divergence free vector fields, we obtain

$$\begin{aligned}\partial_t \mathbf{U}^\dagger + \frac{1}{\epsilon} \mathbf{L} \mathbf{U}^\dagger + \nu \mathbf{A} \mathbf{U}^\dagger &= \mathbf{B}(\mathbf{U}^\dagger, \mathbf{U}^\dagger) + \mathbf{F}^\dagger, \\ \mathbf{U}|_{t=0}^\dagger &= \mathbf{U}^\dagger(0)\end{aligned}$$

where  $\mathbf{A}$  is the Stokes operator and

$$\mathbf{B}(\mathbf{U}^\dagger, \mathbf{U}^\dagger) = (-\mathbf{P}(\mathbf{U} \cdot \nabla \mathbf{U}), -\mathbf{U} \cdot \nabla U_4), \quad \mathbf{U}^\dagger = (\mathbf{U}, U_4) = (\mathbf{U}, \rho);$$

$\mathbf{P}$  is the Leray projection on divergence free vector fields;  
 $\epsilon = 1/N$ .

Initial condition: 3D unbalanced unprepared initial data  $\mathbf{U}_0^\dagger(x_1, x_2, x_3)$ .

Note that  $\mathbf{U}_0^\dagger \in \text{Ker}(\mathbf{L}) \equiv \text{3DQG}$  modes correspond to prepared initial data (balanced regime).

We are interested in fully 3D *unprepared* initial data  $\mathbf{U}_0^\dagger(x_1, x_2, x_3)$  (unbalanced regime; no restrictions on resonances, domain aspect ratios etc.).

## THE LIMIT RESONANT EQUATIONS, $N \rightarrow +\infty$

We introduce van der Pol transformation

$$\mathbf{U}^\dagger(t) = \mathbf{E}(-Nt)\mathbf{u}^\dagger(t)$$

where  $\mathbf{u}^\dagger(t)$  is “slow envelope” variable (*Poincaré variable*):

$$\begin{aligned} \partial_t \mathbf{u}^\dagger + \nu \mathbf{A} \mathbf{u}^\dagger &= \mathbf{B}(Nt, \mathbf{u}^\dagger, \mathbf{u}^\dagger) + \mathbf{E}(Nt) \mathbf{F}^\dagger, \\ \mathbf{B}(Nt, \mathbf{u}^\dagger, \mathbf{u}^\dagger) &= \mathbf{E}(Nt) \mathbf{B}(\mathbf{E}(-Nt) \mathbf{u}^\dagger, \mathbf{E}(-Nt) \mathbf{u}^\dagger) \end{aligned}$$

Limit resonant equations (assume for simplicity  $\nu_1 = \nu_2 = \nu$ ):

$$\begin{aligned} \partial_t \mathbf{w} + \nu \mathbf{A} \mathbf{w} &= \tilde{\mathbf{B}}(\mathbf{w}, \mathbf{w}) + \tilde{\mathbf{F}}, \\ \mathbf{w}|_{t=0} &= \mathbf{w}(0) = \mathbf{U}^\dagger(0). \end{aligned}$$

Here the operator  $\tilde{\mathbf{B}}$  is defined by

$$\tilde{\mathbf{B}}(\mathbf{v}, \mathbf{v}) = \lim_{N \rightarrow +\infty} \frac{1}{T} \int_0^T \mathbf{B}(Ns, \mathbf{v}, \mathbf{v}) ds$$

where arguments  $\mathbf{v}$  are  $s$ -independent functions.

Resonant Interactions in  $\tilde{\mathbf{B}}_n$ :

$$K = \{\pm\omega_k \pm \omega_m \pm \omega_n = 0\}$$

where  $k = (k_1, k_2, k_3)$ ,  $\omega_k^2 = \frac{|k_h|^2}{|k|^2} + \eta^2 \frac{k_3^2}{|k|^2}$ ,  $|k|^2 = k_1^2 + k_2^2 + k_3^2$ ,  $|k_h|^2 = k_1^2 + k_2^2$ ,  $\eta^2 = f^2/N^2$  (similarly, for  $\omega_m$  and  $\omega_n$ ). Also,  $n = k + m$  ( $n_1 = k_1 + m_1, n_2 = k_2 + m_2, n_3 = k_3 + m_3$ ) from convolution.

**Theorem (global regularity of the limit resonant equations):**

Let  $\nu_1, \nu_2 > 0$ ,  $\nu = \min(\nu_1, \nu_2)$ ,  $\|\mathbf{w}(0)\|_\alpha \leq M_\alpha$  (arbitrary large),  $\alpha > 3/4$ ;  $\tilde{\mathbf{F}}$  satisfies

$$\sup_T \int_T^{T+1} \|\mathbf{F}^\dagger\|_{\alpha-1}^2 dt \leq M_{\alpha F}^2$$

for  $\alpha > 3/4$ , where  $\mathbf{F}^\dagger = (\mathbf{F}, F_4)$ . Then there exists a unique regular solution  $\mathbf{w}(t)$  of the resonant limit Navier-Stokes Equations and  $\|\mathbf{w}(t)\|_\alpha \leq M'_1(\nu, M_{\alpha F}, M_\alpha)$  for *all*  $t \geq 0$ . Moreover, there exists a global attractor.

Remark:

This theorem establishes global regularity of the limit resonant equations (3D Navier-Stokes equations with nonlinear interactions restricted to the resonant manifold) for arbitrary large initial data and on infinite time interval. The proof of this theorem is based on the Littlewood-Paley dyadic decomposition and new estimates for nonlinear interactions in  $\tilde{\mathbf{B}}(\mathbf{w}, \mathbf{w})$ .

## STRONG CONVERGENCE OF SOLUTIONS ON $[0, T]$

Let  $T > 0$  given. Then for every  $\delta > 0$  arbitrary small we have

$$\|\mathbf{U}^\dagger(t) - \mathbf{E}(-Nt)\mathbf{w}(t)\|_\alpha = \|\mathbf{E}(-Nt)\mathbf{u}^\dagger(t) - \mathbf{E}(-Nt)\mathbf{w}(t)\|_\alpha \leq \delta, \quad 0 \leq t \leq T;$$

for  $N$  large enough.

### **Theorem:**

Let  $\nu_1, \nu_2 > 0$ , and

$$\sup_T \int_T^{T+1} \|\mathbf{F}^\dagger\|_{\alpha-1}^2 dt \leq M_{\alpha F}^2$$

for  $\alpha > 3/4$ , where  $\mathbf{F}^\dagger = (\mathbf{F}, F_4)$ . Let  $\|\mathbf{U}^\dagger(0)\|_\alpha \leq M_\alpha$  where  $\alpha > 3/4$ . Then for  $N \geq N_1(M_\alpha, M_{\alpha F}, \nu_1, \nu_2)$  solutions of the 3D Navier-Stokes Equations of Geophysics are regular for all  $t \geq 0$ , and  $\|\mathbf{U}^\dagger(t)\|_\alpha \leq M'_\alpha$  for all  $t \geq 0$ .

### **Remark:**

The proof of this theorem is based on (i) uniform approximation on bounded time intervals in strong norms of  $\mathbf{U}^\dagger(t)$  by solutions of the limit resonant equations, and (ii) bootstrapping to infinite time by successive approximations on finite time intervals. Thus,  $\mathbf{U}^\dagger(t)$  on infinite time interval is approximated in strong topology by pieces of solutions of the limit resonant equations.

**Theorem:**

Let  $\nu_1, \nu_2 > 0$ ,  $\nu_1 > 0$ ,  $\nu_2 > 0$ ,  $\alpha > 3/4$  and the condition on the force be satisfied. Let  $\|\mathbf{U}^\dagger(0)\|_0 \leq M_0$ ,  $\hat{T} = \hat{T}(M_0, M_{\alpha F}, \nu)$ . Then for every  $N \geq N'(\nu, M_{\alpha F})$ ,  $N'$  independent of  $M_0$  and for every weak solution  $\mathbf{U}^\dagger(t, x_1, x_2, x_3)$  of the three-dimensional “primitive” Navier-Stokes equations defined on  $[0, \hat{T}]$  which satisfies the classical energy estimates on  $[0, \hat{T}]$ , the following holds:  $\mathbf{U}^\dagger(t, x_1, x_2, x_3)$  can be extended to  $0 < t < +\infty$  and it is regular for every  $t : \hat{T} \leq t < +\infty$ ;  $\mathbf{U}^\dagger(t, x_1, x_2, x_3)$  belongs to  $H_\alpha$  and  $\|\mathbf{U}^\dagger(t, x_1, x_2, x_3)\|_\alpha \leq C_1(M_{\alpha F}, \nu)$  for every  $t \geq \hat{T}$ . If  $\mathbf{F}^\dagger$  is independent of  $t$  then there exists a global attractor for the three-dimensional primitive Navier-Stokes equations bounded in  $H_\alpha$ ; such an attractor has a finite fractal dimension and attracts every weak Leray solution as  $t \rightarrow +\infty$ .

We establish infinite time regularity theorems valid for all domain parameters: for  $N$  large but finite. This is obtained by bootstrapping from global regularity of the limit resonant equations and strong convergence theorems.