FAST SINGULAR OSCILLATING LIMITS AND GLOBAL REGULARITY FOR THE 3D PRIMITIVE EQUATIONS OF GEOPHYSICS

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A. Babin, A. Mahalov and B. Nicolaenko,

Mathematical Models and Methods in Applied Sciences,


J. Guckenheimer and A. Mahalov (1992),
3D Euler-Boussinesq Equations in Atmospheric/Ocean Flows:

$$\partial_t U + U \cdot \nabla U + f_0 e_3 \times U = -\nabla p - \rho e_3,$$

$$\partial_t \rho + U \cdot \nabla \rho = N_0^2 U_3, \quad \nabla \cdot U = 0$$

$$U(t, x_1, x_2, x_3)|_{t=0} = U(0, x_1, x_2, x_3), \quad \rho(t, x_1, x_2, x_3)|_{t=0} = \rho(0, x_1, x_2, x_3).$$

$f_0 e_3 \times U$: Coriolis force (Earth rotation), $e_3$ vertical unit vector

$\rho$: normalized relative density variation w.r.t. density profile

$-\rho e_3$: normalized buoyancy (gravity) force on fluid element

$-\rho = \theta$, $\theta$ normalized relative (potential) temp. variation

Equation for $\rho$ is equivalent to equation for $\theta$, the normalized relative potential temperature variation.

$f_0 = 2\Omega_0$ frequency of rotation, $f_0 \sim 10^{-4} \text{ s}^{-1}$ at midlatitudes

$N_0$: Brunt-Vaisala frequency of gravity waves

$$N_0 = \left( g < \frac{1}{\rho_b} \frac{d\rho}{dx_3} > \right)^{1/2}, \quad < > \text{ averaging}; \rho_b- \text{ base profile}$$

$N_0 \sim 1.2 \times 10^{-2} \text{ s}^{-1}$ at tropopause
3D Navier-Stokes Equations of Geophysics

\[ \partial_t U + U \cdot \nabla U + f e_3 \times U = -\nabla p + \rho \ v_3 + \nu_1 \Delta U + F, \]

\[ \partial_t \rho + U \cdot \nabla \rho = -N^2 U_3 + \nu_2 \Delta \rho + F_4, \nabla \cdot U = 0, \]

\[ U(t, x)|_{t=0} = U(0, x), \ \rho(t, x)|_{t=0} = \rho(0, x) \]

Initial Conditions: three-dimensional unprepared (unbalanced) initial data. Smoothness assumptions on initial data are the same as in local existence theorems.

Rotation and mean stratification gradient are aligned parallel to the vertical axis \( x_3 \). Here \( x = (x_1, x_2, x_3) \), \( U = (U_1, U_2, U_3) \) is the velocity field and \( \rho \) is the buoyancy variable (relative density variation); \( F = (F_1, F_2, F_3) \). Here \( \nu_1 \) and \( \nu_2 \) are the kinematic viscosity and the heat conductivity, respectively; the ratio \( Pr = \nu_1/\nu_2 \) is known as the Prandtl number. We consider periodic boundary conditions in a parallelepiped \([0, 2\pi] \times [0, 2\pi] \times [0, 2\pi]\), as well as stress-free conditions \( U_3 = 0, \partial U_1/\partial x_3 = \partial U_2/\partial x_3 = 0 \) at \( x_3 = 0, 2\pi \).
Let $U_h$ be a characteristic horizontal velocity scale. Let $H$ and $L$ be vertical and horizontal length scales and $a = H/L$ is the aspect ratio parameter. We define Froude numbers based on horizontal and vertical scales:

$$F_h = U_h/(LN_0) \equiv 1/N, \quad F_v = U_h/(HN_0) = F_h/a.$$  

The classical Rossby and anisotropic Rossby numbers are defined as follows

$$Ro = U_h/(f_0L) \equiv 1/f, \quad Ro_a = a \ Ro, \quad a = H/L.$$  

Here $f = Ro^{-1}$ and $N = F_h^{-1}$ are dimensionless rotation and stratification parameters. The Burger number characterizes relative importance of the effects of rotation and stratification:

$$Bu = Ro_a^2/F_h^2 \equiv Ro^2/F_v^2 \equiv N^2a^2/f^2 = N_0^2a^2/f_0^2.$$  

Flows with $Bu \ll 1$ are rotation dominated and $Bu \gg 1$ corresponds to stratification dominated flows. An equivalent measure of the relative importance of stratification vs rotation is the internal radius of deformation $\Lambda$, which compares (stable) density stratification effects with respect to rotation. The internal (Rossby) radius of deformation $\Lambda$ is defined as

$$\Lambda = N_0H/f_0,$$

so that $Bu = (\Lambda/L)^2$. Regimes where $Bu$ is either much greater or much less than one are both important for geophysical flows. Processes which couple rotation and stratification, such as baroclinic instability, have $Bu = O(1)$ as a natural case.
ASYMPTOTIC LIMITS

• High Frequency Inertia-Gravity Wave Limit

• Capture Averaged Effects of Nonlinear Wave Dynamics

• Nonlinear Interactions and Resonances of Fast Waves: Nonlinear Averaging

• Non-Hydrostatic

MATHEMATICAL ISSUES

• Fast Singular Oscillating Limits

• Cancellation of Oscillations and Restricted Interactions

• Strong Solutions and Strong Convergence

• Conservation Laws (Euler)

• Attractors (Navier-Stokes)
3D INITIAL VALUE PROBLEM FOR SMALL $\epsilon = 1/N$

- The nature of the limit resonant equations and the regularity of their solutions.

- The limit equations are genuinely 3-dimensional depending on all three variables $x_1$, $x_2$ and $x_3$ but with restricted wave-number interactions in the nonlinear term. The existence and regularity theory for those limit equations is non-trivial. The proof of global regularity is based on the Littlewood-Paley dyadic decomposition and new estimates for restricted nonlinear interactions.

- Strong convergence of solutions of 3D Navier-Stokes equations to those of the limit equations.

- Bootstrapping from analysis of the first three questions the infinite time regularity of solutions of 3D Navier-Stokes equations for $\epsilon$ small but finite.
We introduce a change of variables \( \rho \to N \rho \) and combine velocity and buoyancy variable in one variable \( \mathbf{U}^\dagger = (\mathbf{U}, \rho) \) after which equations written in non-dimensional variables take more symmetric form \( (\epsilon = 1/N) \):

\[
\partial_t \mathbf{U}^\dagger + \mathbf{U} \cdot \nabla \mathbf{U}^\dagger + \frac{1}{\epsilon} \mathbf{M} \mathbf{U}^\dagger = -\nabla^\dagger \mathbf{p} + \mathbf{\tilde{v}} \Delta \mathbf{U}^\dagger + \mathbf{F}^\dagger, \nabla \cdot \mathbf{U} = 0
\]

\[
\mathbf{U}^\dagger(t, x)|_{t=0} = \mathbf{U}^\dagger(0, x)
\]

where \( \nabla^\dagger \mathbf{p} = (\nabla \mathbf{p}, 0) \), \( \mathbf{F}^\dagger = (\mathbf{F}, F_4) \) (where \( F_4 \) is rescaled),

\[
\mathbf{M} = (\mathbf{S} + \eta \mathbf{R}), \eta = f/N, \mathbf{R} = \begin{pmatrix} \mathbf{J} & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{S} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{J} \end{pmatrix}, \mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
\]

\( \mathbf{\tilde{v}} = \text{diag}(\nu_1, \nu_1, \nu_1, \nu_2) \) is the viscosity matrix, \( \eta \) is fixed.

The linear part of inviscid equations is

\[
\partial_t \mathbf{U}^\dagger + \frac{1}{\epsilon} \mathbf{M} \mathbf{U}^\dagger = -\nabla^\dagger \mathbf{p}, \nabla \cdot \mathbf{U} = 0.
\]

or, equivalently, after applying Leray projection on divergence free fields

\[
\partial_t \Phi + \frac{1}{\epsilon} \mathbf{L} \Phi = 0.
\]

We define (unitary group, preserves every Sobolev norm)

\[
\mathbf{E}(-NLt) = \mathbf{E}(-\frac{1}{\epsilon}Lt) = \exp(-\frac{1}{\epsilon}Lt)
\]

The solutions of this linear equation are (i) 3DQG modes (non-oscillating solutions, zero eigenmodes) and (ii) \( \exp(\pm i \frac{1}{\epsilon} \omega_n t)v_n \) inerto-gravity waves (oscillating solutions), \( \omega_n^2 = \frac{|n_n|^2}{|n|^2} + \eta^2 \frac{n_n^3}{|n|^2} \).
NONLINEAR EQUATIONS

Applying the Leray projection $\mathbf{P}$ onto divergence free vector fields, we obtain

$$\partial_t \mathbf{U}^\dagger + \frac{1}{\epsilon} \mathbf{L} \mathbf{U}^\dagger + \nu \mathbf{A} \mathbf{U}^\dagger = \mathbf{B}(\mathbf{U}^\dagger, \mathbf{U}^\dagger) + \mathbf{F}^\dagger,$$

$$\mathbf{U}^\dagger|_{t=0} = \mathbf{U}^\dagger(0)$$

where $\mathbf{A}$ is the Stokes operator and

$$\mathbf{B}(\mathbf{U}^\dagger, \mathbf{U}^\dagger) = (-\mathbf{P}(\mathbf{U} \cdot \nabla \mathbf{U}), -\mathbf{U} \cdot \nabla U_4), \quad \mathbf{U}^\dagger = (\mathbf{U}, U_4) = (\mathbf{U}, \rho);$$

$\mathbf{P}$ is the Leray projection on divergence free vector fields; $\epsilon = 1/N.$

Initial condition: 3D unbalanced unprepared initial data $\mathbf{U}^\dagger_0(x_1, x_2, x_3).$

Note that $\mathbf{U}^\dagger_0 \in \text{Ker}(\mathbf{L}) \equiv 3\text{DQG modes correspond to prepared initial data (balanced regime).}$

We are interested in fully 3D unprepared initial data $\mathbf{U}^\dagger_0(x_1, x_2, x_3)$ (unbalanced regime; no restrictions on resonances, domain aspect ratios etc.).
THE LIMIT RESONANT EQUATIONS, \( N \to +\infty \)

We introduce van der Pol transformation

\[
U^\dagger(t) = E(-Nt)u^\dagger(t)
\]

where \( u^\dagger(t) \) is “slow envelope” variable (\textit{Poincaré variable}): 

\[
\partial_t u^\dagger + \nu A u^\dagger = B(N t, u^\dagger, u^\dagger) + E(N t)F^\dagger,
B(N t, u^\dagger, u^\dagger) = E(N t)B(E(-N t)u^\dagger, E(-N t)u^\dagger)
\]

Limit resonant equations (assume for simplicity \( \nu_1 = \nu_2 = \nu \)): 

\[
\begin{align*}
\partial_t w + \nu A w &= \tilde{B}(w, w) + \tilde{F}, \\
w|_{t=0} &= w(0) = U^\dagger(0).
\end{align*}
\]

Here the operator \( \tilde{B} \) is defined by

\[
\tilde{B}(v, v) = \lim_{N \to +\infty} \frac{1}{T} \int_0^T B(N s, v, v) ds
\]

where arguments \( v \) are \( s \)-independent functions.

Resonant Interactions in \( \tilde{B}_n \):

\[
K = \{ \pm \omega_k \pm \omega_m \pm \omega_n = 0 \}
\]

where \( k = (k_1, k_2, k_3), \ \omega_k^2 = \frac{|k|^2}{|k|^2} + \eta^2 \frac{k^2}{|k|^2}, \ |k|^2 = k_1^2 + k_2^2 + k_3^2, \ 
|k^*_h|^2 = k_1^2 + k_2^2, \ \eta^2 = f^2/N^2 \) (similarly, for \( \omega_m \) and \( \omega_n \)). Also, 
\( n = k + m \) \( (n_1 = k_1 + m_1, n_2 = k_2 + m_2, n_3 = k_3 + m_3) \) from convolution.
Theorem (global regularity of the limit resonant equations):
Let \( \nu_1, \nu_2 > 0, \nu = min(\nu_1, \nu_2), \|w(0)\|_\alpha \leq M_\alpha \) (arbitrary large), \( \alpha > 3/4 \); \( \bar{F} \) satisfies

\[
\sup_T \int_T^{T+1} \|F_i\|_{\alpha-1}^2 dt \leq M_{\alpha F}^2
\]

for \( \alpha > 3/4 \), where \( F_i = (F,F_1) \). Then there exists a unique regular solution \( w(t) \) of the resonant limit Navier-Stokes Equations and \( \|w(t)\|_\alpha \leq M'_1(\nu, M_{\alpha F}, M_\alpha) \) for all \( t \geq 0 \). Moreover, there exists a global attractor.

Remark:
This theorem establishes global regularity of the limit resonant equations (3D Navier-Stokes equations with nonlinear interactions restricted to the resonant manifold) for arbitrary large initial data and on infinite time interval. The proof of this theorem is based on the Littlewood-Paley dyadic decomposition and new estimates for nonlinear interactions in \( \tilde{B}(w,w) \).
STRONG CONVERGENCE OF SOLUTIONS ON [0, T]

Let $T > 0$ given. Then for every $\delta > 0$ arbitrary small we have

$$\|\mathbf{U}^\dagger(t) - \mathbf{E}(-Nt)\mathbf{w}(t)\|_\alpha = \|\mathbf{E}(-Nt)\mathbf{u}^\dagger(t) - \mathbf{E}(-Nt)\mathbf{w}(t)\|_\alpha \leq \delta, \ 0 \leq t \leq T;$$

for $N$ large enough.

**Theorem:**

Let $\nu_1, \nu_2 > 0$, and

$$\sup_{T} \int_{T}^{T+1} \|\mathbf{F}^\dagger\|_{\alpha-1}^2 \, dt \leq M^2_{\alpha_F}$$

for $\alpha > 3/4$, where $\mathbf{F}^\dagger = (\mathbf{F}, F_4)$. Let $\|\mathbf{U}^\dagger(0)\|_\alpha \leq M_\alpha$ where $\alpha > 3/4$. Then for $N \geq N_1(M_\alpha, M_{\alpha F}, \nu_1, \nu_2)$ solutions of the 3D Navier-Stokes Equations of Geophysics are regular for all $t \geq 0$, and $\|\mathbf{U}^\dagger(t)\|_\alpha \leq M'_\alpha$ for all $t \geq 0$.

Remark:

The proof of this theorem is based on (i) uniform approximation on bounded time intervals in strong norms of $\mathbf{U}^\dagger(t)$ by solutions of the limit resonant equations, and (ii) bootstrapping to infinite time by successive approximations on finite time intervals. Thus, $\mathbf{U}^\dagger(t)$ on infinite time interval is approximated in strong topology by pieces of solutions of the limit resonant equations.
Theorem:
Let $\nu_1, \nu_2 > 0$, $\nu_1 > 0$, $\nu_2 > 0$, $\alpha > 3/4$ and the condition on the force be satisfied. Let $\|U^{\uparrow}(0)\|_0 \leq M_0$, $\hat{T} = \hat{T}(M_0, M_{\alpha F}, \nu)$. Then for every $N \geq N'(\nu, M_{\alpha F})$, $N'$ independent of $M_0$ and for every weak solution $U^{\uparrow}(t, x_1, x_2, x_3)$ of the three-dimensional “primitive” Navier-Stokes equations defined on $[0, \hat{T}]$ which satisfies the classical energy estimates on $[0, \hat{T}]$, the following holds: $U^{\uparrow}(t, x_1, x_2, x_3)$ can be extended to $0 < t < +\infty$ and it is regular for every $t : \hat{T} \leq t < +\infty$; $U^{\uparrow}(t, x_1, x_2, x_3)$ belongs to $H_\alpha$ and $\|U^{\uparrow}(t, x_1, x_2, x_3)\|_\alpha \leq C_1(M_{\alpha F}, \nu)$ for every $t \geq \hat{T}$. If $F^{\uparrow}$ is independent of $t$ then there exists a global attractor for the three-dimensional primitive Navier-Stokes equations bounded in $H_\alpha$; such an attractor has a finite fractal dimension and attracts every weak Leray solution as $t \to +\infty$.

We establish infinite time regularity theorems valid for all domain parameters: for $N$ large but finite. This is obtained by bootstrapping from global regularity of the limit resonant equations and strong convergence theorems.