Cubical Homology Theory and applications in Image Processing and Computer Vision

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Homology theory assigns to any topological space $X$ a sequence of abelian groups $H_0(X), H_1(X), H_2(X), \ldots$.

1. The algebraic structure of these groups depends only on the topological type of $X$.
2. Classification of topological spaces.
3. These groups are computable.
4. Interpretation of homology groups of $X \subset \mathbb{R}^3$:
   
   (a) $\text{Rank}(H_0(X)) = \beta_0 = \# \ 0$-D holes (connected components)
   
   (b) $\text{Rank}(H_1(X)) = \beta_1 = \# \ 1$-D holes (independent tunnels)
   
   (c) $\text{Rank}(H_2(X)) = \beta_2 = \# \ 2$-D holes (independent voids)

5. Applications: Dynamical Systems, Molecular Biology, Robotics, …

6. If there is no torsion in $X$, then $\beta_0, \beta_1, \beta_2, \ldots$ are called the Betti Numbers of $X$. 
The route was started by **Euler** with his early work (1736) on the **Königsberg bridge problem**. Later on (1750), he established his famous formula \( v - e + f = 2 \) for polyhedra. Around 1813, **Antoine-Jean Lhuilier** corrected and generalized the formula for a solid with \( g \) holes: \( v - e + f = 2 - 2g \). **Riemann**, **Möbius**, **Jordan** and **Betti** formulated different methods to study the connectivity of surfaces.

All these ideas were put on a completely rigorous basis by **Henri Poincaré** in a series of papers **Analysis situs** (1895). He introduced the concept of **homology**, gave a more precise definition of the Betti numbers and generalized the Euler’s formula to a general setting of a \( p \)-dimensional manifold. Poincaré’s basic idea is to divide the space up into primitives such as vertices (points), edges (lines), filled triangles, tetrahedra and their higher dimensional analogues then to measure the number and the interrelationships of these parts in a suitable algebraic way.
Definition: basic idea...

Loops with a basepoint can be seen as cycles. Thus, for example, the loops \( A^{-1}DE = -A + D + E \) and \( DEA^{-1} = D + E - A \) become the same object (abelianizing). We can broaden the term cycle to any linear combination of the edges with integer coefficients and form a group \( C_1 \).

Similarly, we form the group \( C_0 \) of the linear combinations of vertices. Define \( \partial_1 : C_1 \to C_0 \) by sending each edge to the vertex at the head minus the vertex at the tail. It follows that cycles are elements of \( \ker \partial_1 \). If we attach a clockwise oriented 2-cell \( T \) along the cycle \( -A + D + E \), this cycle is now a boundary of \( T \) and thus it is homotopically trivial and it no longer encloses a hole. This suggests that we form a quotient of the group of cycles by the group of boundaries that can be defined by introducing \( \partial_2 : C_2 \to C_1 \).
Figure 0.1: *Example of a simplicial complex.*
A $k$-simplex $\sigma$ in $\mathbb{R}^n$ is the convex hull of a set of a $k + 1$ affinely independent points $\{v_0, v_1, \ldots, v_k\}$, i.e.

$$\sigma := \{x \in \mathbb{R}^n : x = \sum_{i=0}^{k} t_i v_i \}.$$ 

Algebraic boundary = geometric boundary where each proper face is visited in a certain order of its vertices $\implies$ orientation.

An orientation of $\sigma$ is a linear ordering on its vertices $\implies$ notation: $\sigma = [v_0, v_1, \ldots, v_k]$.

Boundary operator:

$$\partial_k \sigma = \sum_{i=0}^{k} (-1)^i [v_0, v_1, \ldots, \hat{v}_i, \ldots, v_k]. \quad (1)$$

For each $k$, we define $C_k(X)$ the free abelian group of $k$-chains.

Extend by linearity: $\partial_k : C_k(X) \to C_{k-1}(X)$.

We obtain the structure of a finitely generated free chain complex
Homology of a chain complex

\((C, \partial)\) associated to the given simplicial complex \(X\), i.e.,

\[
0 \xrightarrow{\partial_{n+1}} c_n(x) \cdots \xrightarrow{\partial_{k+1}} c_k(x) \xrightarrow{\partial_k} c_{k-1}(x) \xrightarrow{\partial_{k-1}} \cdots c_0(x) \xrightarrow{\partial_0} 0,
\]

with

\[
\partial_k \circ \partial_{k+1} = 0, \quad \forall k. \tag{2}
\]

- \(Z_k := \ker \partial_k\) : group of \(k\)-cycles.
- \(B_k := \text{Im} \partial_{k+1}\) : group of \(k\)-boundaries.
- \((2) \implies B_k \subset Z_k.
- The \(k\)–th homology group of \(C\) is

\[
H_k(C) := \frac{\text{cycles}}{\text{boundaries}} = \frac{Z_k}{B_k}.
\]

**Remark.** Here, the term simplicial is artificial. The homology can be defined for any given chain complex.
Figure 0.2: *Example of a cubical complex.*
• An elementary interval $I$ is an interval of the form $I = \{k\}$ (degenerate) or $I = [k, k + 1]$ (nondegenerate) for some integer $k$.

• A $q$-elementary cube $\sigma \subset \mathbb{R}^n$ is a product of $n$ intervals, i.e., $\sigma = I_1 \times I_2 \times \cdots \times I_n$, where only $q$ intervals are nondegenerate.

• There is a natural orientation induced on elementary cubes.

• Let $J = \{k_0, k_1, \ldots, k_{q-1}\}$ be the ordered subset of $\{1, 2, \ldots, n\}$ of indices s.t. $I_{k_j} = [a_j, b_j]$ is nondegenerate. Define

\[
A_{k_j} \sigma = I_1 \times \cdots \times I_{k_j-1} \times \{a_j\} \times I_{k_j+1} \times \cdots \times I_n,
\]

\[
B_{k_j} \sigma = I_1 \times \cdots \times I_{k_j-1} \times \{b_j\} \times I_{k_j+1} \times \cdots \times I_n.
\]

• Boundary operator:

\[
\partial_q \sigma = \sum_{i=0}^{q-1} (-1)^i (B_{k_i} \sigma - A_{k_i} \sigma).
\]  

(3)

• Define $C_q(X)$ the free abelian group of $q$-chains.

• Extend by linearity: $\partial_q : C_q(X) \to C_{q-1}(X)$.

• We obtain a finitely generated free chain complex $(\mathcal{C}, \partial)$. 
• Many problems from numerical computations, graphics, computer vision lead naturally to cubical grids subdividing the space into cubes with vertices in an integer lattice.

• Computer vision: discretization via a cubic tessellation leads to a cubical complex of pixels or voxels.

• The simplicial theory would require subdividing each pixel into union of two triangles and hence increasing the complexity of data.

• Natural orientation on cubes.

References:


Smith Normal Form (H.J.S. Smith-1861):

1. \( \partial_n : C_n \to C_{n-1}, \ Z_n = \text{Ker} \partial_n, \)
   \( B_n = \text{Im} \partial_{n+1}. \)

2. Define: \( W_n = \{c \in C_n \mid \exists k \in \mathbb{Z} \setminus \{0\} \text{ such that } kc \in B_n \}. \)

3. Write: \( C_n = U_n \oplus V_n \oplus W_n \) with \( \partial_n(U_n) \subset W_{n-1}, \partial_n(V_n) = 0, \partial_n(W_n) = 0. \)

4. \( \exists B := \{e_1, e_2, \ldots, e_p\}, \ B' := \{e'_1, e'_2, \ldots, e'_q\} \)
   such that

\[
[\partial_n]_{B,B'} := \begin{bmatrix}
  b_1 & 0 & 0 \\
  & \ddots & \\
  0 & b_l & \\
  0 & 0 & \\
\end{bmatrix}
\]

5. \( H_n(C) \cong V_n \oplus (W_n/B_n) \cong \mathbb{Z}^{\beta_n} \oplus T_n(C) \) (free + torsion parts).
Figure 0.3: Geometric view of the reduction.
• Consider a finitely generated free chain complex \((C, \partial)\) given by
\[
\begin{align*}
&\xrightarrow{\partial_n} c_{n+1} \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_{k+2}} c_{k+1} \xrightarrow{\partial_k} c_k \xrightarrow{\partial_k} c_{k-1} \xrightarrow{\partial_k} \cdots \xrightarrow{\partial_0} c_0 \xrightarrow{\partial_0} 0
\end{align*}
\]

• For each \(k\), associate a base \(E_k\) for \(C_k\) (only formed by elementary \(k\)-chains) and define a bilinear form
\[
\langle ., . \rangle : C_k \times C_k \rightarrow \mathbb{Z} \text{ on generators } u, v \in E_k \text{ by }
\]
\[
\langle u, v \rangle = \begin{cases} 
1 & \text{if } u = v \\
0 & \text{otherwise.}
\end{cases}
\]

• Let \(m \in \{1, \ldots, n\}\), \(A \in E_m\) and \(a \in E_{m-1}\) such that \(a\) is a proper face of \(A\), i.e., we can write
\[
\partial_m A = \lambda a + r, \quad \text{where } \lambda = \pm 1. \quad (4)
\]

• We perform a local reduction of the complex, i.e.,
• Remove $A$ from $E_m$ and $a$ from $E_{m-1}$ in such a way the homology of $(\mathcal{C}, \partial)$ is preserved.

• Define a reduced chain complex $(\overline{\mathcal{C}}, \overline{\partial})$ with new bases

$$E_k := \begin{cases} E_k & \text{if } k \neq m-1, m \\ E_{m-1} \setminus \{a\} & \text{if } k = m-1 \\ E_m \setminus \{A\} & \text{if } k = m \end{cases}$$

and a the reduced boundary map $\overline{\partial}$ by

$$\overline{\partial}_k v := \begin{cases} \partial_k v & \text{if } k \neq m, m + 1 \\ \partial_k v - \lambda^{-1} \langle \partial_k v, a \rangle \partial_k A & \text{if } k = m \\ \partial_k v - \langle \partial_k v, A \rangle A & \text{if } k = m + 1. \end{cases}$$ (5)

**Theorem 0.1** $(\overline{\mathcal{C}}, \overline{\partial})$ is a finitely generated free chain complex and $H(\mathcal{C}) = H(\overline{\mathcal{C}}).$
Input: Basis $E_2$ of 2-cells

Output: # holes and connected components

begin

Create bases $E_1$ and $E_0$;

while $E_2$ nonempty

choose $A$ in $E_2$ and a free face $a$ in $E_1$;

set $E_2 := E_2 \setminus \{A\}$ and $E_1 := E_1 \setminus \{a\}$;

endwhile;

while $E_1 \neq \emptyset$ and $\exists A \in E_1$ and $\exists a \in E_0$ s.t. $a$ is a face of $A$

REDUCE($A, a$);

endwhile;

end;
1. $\forall B \in E_1$, define the set $T(B)$ of its proper faces.

2. $\forall a \in E_0$, attach the set $R(a)$ of its coboundaries.

**Procedure** \texttt{REDUCE}(A, a);

\begin{verbatim}
for each $B$ in $R(a) \setminus \{A\}$ do
  $T(B) := [T(B) \cup T(A)] \setminus \{a\}$;
endfor;

for each $b$ in $T(A) \setminus \{a\}$ do
  $R(b) := [R(b) \cup R(a)] \setminus \{A\}$;
endfor;

$E_1 := E_1 \setminus \{A\}; \quad E_0 := E_0 \setminus \{a\}$;
end REDUCE;
\end{verbatim}
• For a planar complex, the algorithm is linear in the number of generators.
• For an $m$-dimensional complex (with $m \geq 3$), the complexity of the algorithm is about $O(n^3)$ where $n$ is the maximum cardinality among the bases of generators in all dimensions $\leq m$.
• With a suitable choice of data structures, one can reduce the complexity to $O(n^2 \log n)$.

References:


A-Extraction of Image features: 
Connected Components, # Holes, Euler Number.

Figure 0.4: A circuit: 125, 128, -3.
B-3D Structures of Images: # Connected Components, # Tunnels, # Voids, Euler Number.

Figure 0.5: *A slice of the brain.*
Figure 0.6: Rendering of the brain (A view)

Project: (A. Tannenbaum, K. Mischaikow, B. Kalies, M. Allili) Homology computation and localization of holes in a cubical complex; Application in Medical Imaging.
C-Pattern recognition via size functions (with D. Ziou): Let $\varphi$ be a measure function associated to a contour (figure below). For any $\varphi_1, \varphi_2$ s.t. $0 \leq \varphi_1 < \varphi_2$, define $S_1 = \{ x : \varphi(x) < \varphi_1 \}$ and $S_2 = \{ x : \varphi(x) < \varphi_2 \}$. We have

$$l_\varphi(\varphi_1, \varphi_2) = \text{Rank}(H_0(S_2)) - \text{Rank}(H_0(S_2, S_1)).$$

Figure 0.7: A hand contour.
Table 1: Some values of the size function computed with Homology algorithm.

References:


2. A. Verri & C. Uras, Metric-Topological Approach to Shape Representation and Recognition, Image and Vision Computing