Cyclic Group and Knapsack Facets

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Motivation

General Pure Integer Program

$$\min\{cx \mid Bx_B + Nx_N = b, \quad x_B, x_N \geq 0, \text{ integer}\}$$

⇓

Group Problem (Relaxation)

$$\min\{cx \mid Nx_N \equiv b \pmod{B}, \quad x_N \geq 0, \text{ integer}\}$$

⇓

Cyclic Group Problem (Relaxation)

$$x_1 + 2x_2 + 3x_3 + \ldots + (n-1)x_{n-1} \equiv r \pmod{n}$$

$$x_j \geq 0, \text{ integer } \forall j, \ x \neq 0$$

Topics

1. Master cyclic group and knapsack polyhedra
    - Classes of facets
    - Mappings between problems
2. Gomory’s shooting experiment
3. Generation of classes of facets

Problem Definitions

- Master Cyclic Group Problem C(n,r)

$$x_1 + 2x_2 + 3x_3 + \ldots + (n-1)x_{n-1} \equiv r \pmod{n}$$

$$x_j \geq 0, \text{ integer } \forall j,\ x \neq 0$$

- Master Knapsack Partitioning Problem $K^{=}(n)$

$$x_1 + 2x_2 + 3x_3 + \ldots + nx_n = n$$

$$x_j \geq 0, \text{ integer } \forall j$$

- Master Knapsack Covering Problem $K^{>}(n)$

$$x_1 + 2x_2 + 3x_3 + \ldots + nx_n \geq n$$

$$x_j \geq 0, \text{ integer } \forall j$$

Subadditive Characterizations

- **Master Cyclic Group Problem**

  (π, π_r) is a facet of C(n,r) iff it is an extreme ray of the cone generated by: $\pi \geq 0$

  $\pi_i + \pi_j = \pi_r$ $\quad 1 \leq i \leq j \leq n\text{-}1,\ i+j = r \bmod n$

  $\pi_i + \pi_j \geq \pi_k$ $\quad 1 \leq i \leq j \leq n\text{-}1,\ i+j = k \bmod n,\ k \neq 0$

  Master Knapsack Partitioning Problem

  (ρ, ρ_n) is a facet of $K^=(n)$ iff it is an extreme ray of the cone generated by

  $\rho_i + \rho_{n-i} = \rho_n$ $\quad 1 \leq i \leq n\text{-}1$

  $\rho_i + \rho_j \geq \rho_k$ $\quad 1 \leq i \leq j \leq n\text{-}1,\ i+j = k,\ k \leq n$

  Master Knapsack Covering Problem

  (σ, σ_n) is a facet of $K^>(n)$ iff it is an extreme ray of the cone generated by $\sigma \geq 0$

  $\sigma_i + \sigma_{n-i} = \sigma_n$ $\quad 1 \leq i \leq n\text{-}1$

  $\sigma_i + \sigma_j \geq \sigma_k$ $\quad 1 \leq i \leq j \leq n\text{-}1,\ k = \min\{ n, i+j \}$

  or $K^=(n)$ plus $0 \leq \sigma_1 \leq \ldots \leq \sigma_n$

Knapsack facets

- $x_1 + x_3 - x_5 - x_7 \geq 0$ is a facet of $K^=(8)$

  Subtract $x_1+2x_2+3x_3+4x_4+5x_5+6x_6+7x_7+8x_8=8$

  Gives $-2x_2 -2x_3 -4x_4 -6x_5 -6x_6 -8x_7 -8x_8 \geq -8$
- $x_2 +x_3 +2x_4 +3x_5 +3x_6 +4x_7 +4x_8 \leq 4$ is a facet for the $\leq$ knapsack problem and all come from facets of $K^=(8)$ in this way
- Adding the equality constraint gives

  $2x_1+2x_2+4x_3+4x_4+4x_5+6x_6+6x_7+8x_8 \geq 8$

  or $x_1+x_2+2x_3+2x_4+2x_5+3x_6+3x_7+4x_8 \geq 4$

  which is a facet of $K^>(8)$ but not all facets of $K^>(8)$ come from this construction

Some Covering Facets

- Lemma: If $(\sigma_1,\ldots, \sigma_n)$ is complementary, subadditive, monotone, and has some k satisfying $\sigma_k = j$ for $j = 1, \ldots, J = \sigma_n$, then σ is facet-defining for $K^>(n)$
- Define $\kappa_j = \max \{ k \mid \sigma_k = j \}$
- Thm: σ is subadditive iff κ is superadditive
- Thm: σ is complementary iff $\kappa_{J-j} = n - (\kappa_{j-1} + 1)$
- Extends for even n and odd r to having one half integer interval around n/2

(1, 1, 1, 1, 1, 2) is always a facet for $n \geq 2$

Relationship between Problems

- The knapsack polytope $P(K^{=}_{n})$ is a facet for the polyhedra associated with four problems:
  - $C(n+1, n)$
  - $C(n,0)$
  - $K^{>}(n)$ – covering
  - The $\leq$ knapsack facets are a 1-1 mapping

- Every knapsack facet may be *tilted* to obtain a facet for *many* cyclic group (via automorphisms) and for covering polyhedra

3 Approaches to Facet Generation

- Seeds
  - Facets generated directly
- Mappings
  - Facets from other polyhedra directly mapped to current polyhedron
  - Tiltings (a particular type of mapping)
    - Mappings where facets for master knapsack polytopes are “tilted” to facets for master cyclic group polyhedron

Seeds

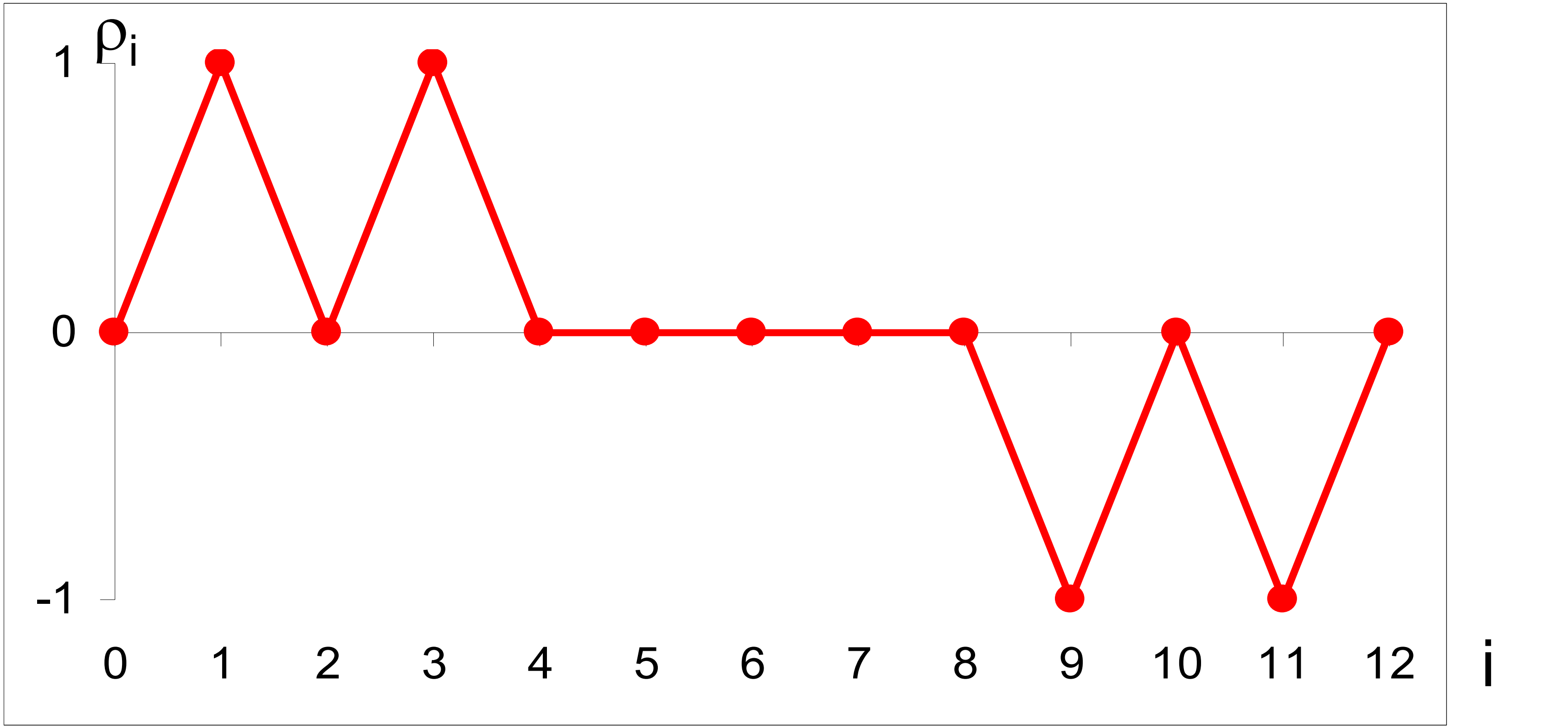
Class	For	Conditions
MIC	C(n,r)	none
2Slope	C(n,r)	none
3Slope	C(n,r)	$r \leq {}^{n-4}/_{3}$
1,0,-1	K(n)	n even or $n \geq 7$
2Lin	K(n)	$n \geq 8$
3Lin1Slp	K(n)	$n \geq 5$
Mod1Slp	K(n)	$n \geq 5$

C(n,r) : Master cyclic group polyhedron of size n with rhs r

K(n): Master knapsack polytope of size n

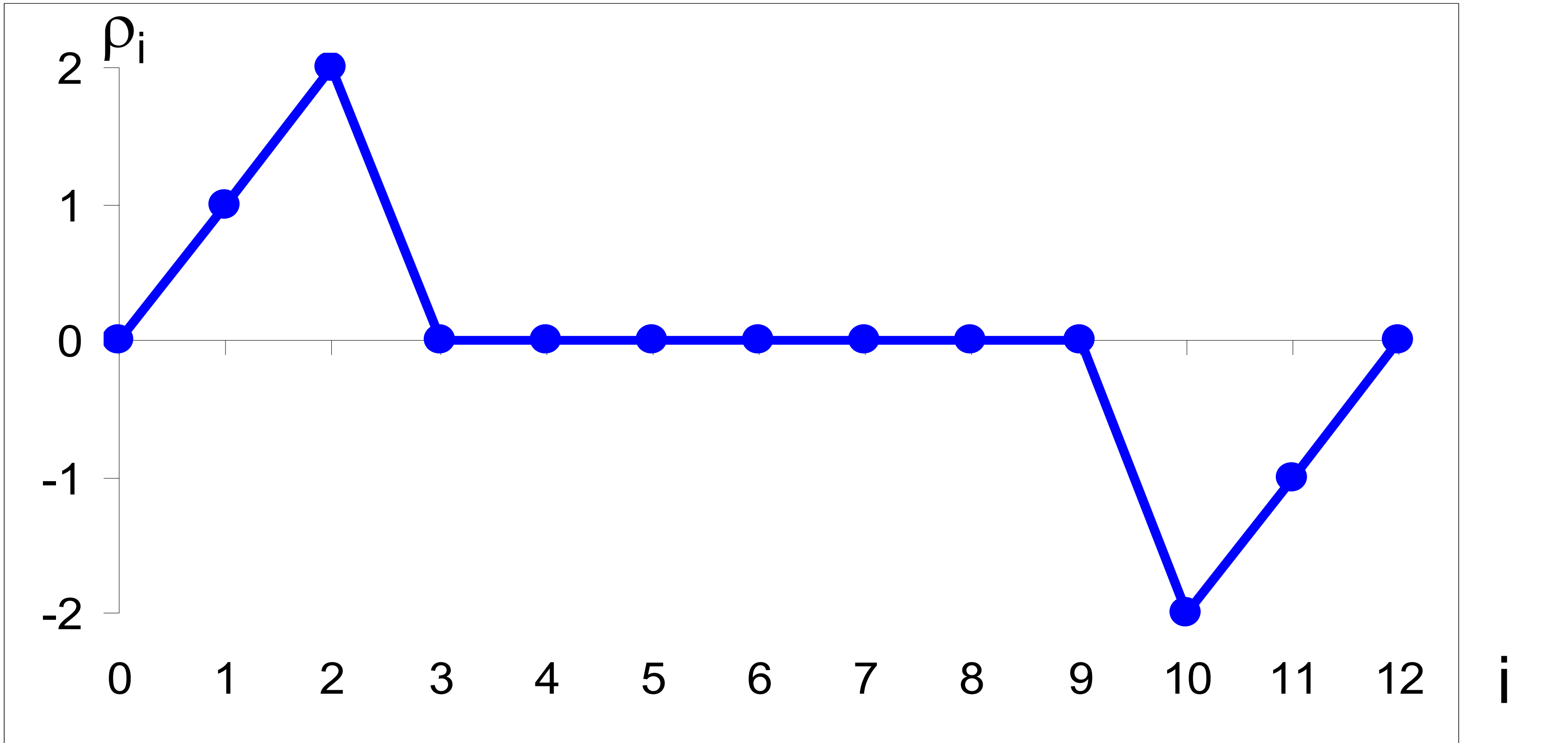
Example - (1,0,-1) Facet of $P(K_{12})$

	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8	ρ_9	ρ_{10}	ρ_{11}	ρ_{12}	ρ_{12}
1, 0, -1	1	0	1	0	0	0	0	0	-1	0	-1	0	0
2Lin	1	2	0	0	0	0	0	0	0	-2	-1	0	0
Lin1Slp	1	2	3	4	-1	0	1	-4	-3	-2	-1	0	0



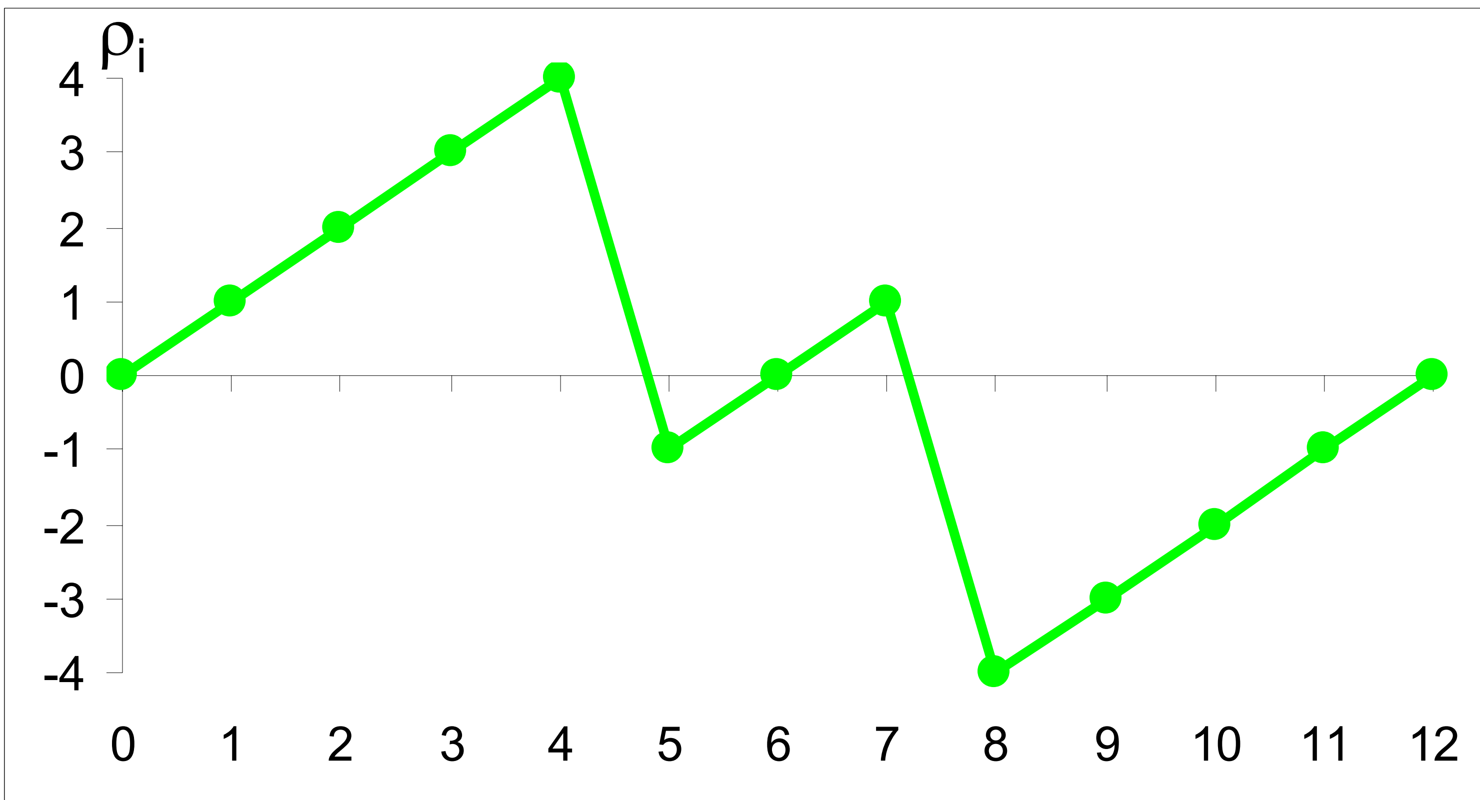
Example - 2Lin Facet of $P(K_{12})$

	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8	ρ_9	ρ_{10}	ρ_{11}	ρ_{12}	ρ_{12}
1, 0, -1	1	0	1	0	0	0	0	0	-1	0	-1	0	0
2Lin	1	2	0	0	0	0	0	0	0	-2	-1	0	0
Lin1Slp	1	2	3	4	-1	0	1	-4	-3	-2	-1	0	0



Example - 3Lin1Slp Facet of $P(K_{12})$

	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8	ρ_9	ρ_{10}	ρ_{11}	ρ_{12}	ρ_{12}
1, 0, -1	1	0	1	0	0	0	0	0	-1	0	-1	0	0
2Lin	1	2	0	0	0	0	0	0	0	-2	-1	0	0
Lin1Slp	1	2	3	4	-1	0	1	-4	-3	-2	-1	0	0



Mappings

Mapping	From	To	Conditions
Auto (Gomory)	C(n,r)	C(n,s)	Automorph maps r -> s
Homomorphism (Gomory)	C(d,r)	C(n,s)	d | n and $s \equiv r \pmod d$
0-Homom.	C(d,0), $C(^n/_d, {}^r/_d)$	C(n,r)	d|n, d|r, and condition
Cyc	C(d,r)	K(n)	$d \leq {}^{n+1}/_2$, d † n, $r \equiv n \pmod d$
Cyc0	C(d,r)	K(n)	$d \leq {}^n/_4$ and d | n
Tilting	K(r)	C(n,r)	$n \geq r+1$

C(n,r) : Master cyclic group polyhedron of size n with rhs r

K(n): Master knapsack polytope of size n

Mappings for $P(C_{n,r})$: Homomorphisms

Given a homomorphism Φ: $\mathbf{C_n} \rightarrow \mathbf{C_d}$ such that $\Phi(\mathbf{s}) = \mathbf{r}$ and facet (π,π_r) of $P(C_{d,r})$, where d divides n,

$$(\pi',\pi'_s) = ((\pi_{\Phi(1)}, \pi_{\Phi(2)}, \ldots ,\pi_{\Phi(n-1)}), \pi_{\Phi(s)})$$

is a facet of $P(C_{n,s})$.

Example: $P(C_{5,2}) \rightarrow P(C_{10,7})$

	n	r	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	$\pi_{r,}$
MIC:	5	2	3	**6**	4	2						6
homo $\Phi(i) = i \pmod 5$	10	7	3	6	4	2	0	3	**6**	4	2	6

Mappings for $P(K_n)$ from cyclic groups (I)

If $\mathbf{d} \leq {}^{\mathbf{n+1}}/_{\mathbf{2}}$, **d does not divide n**, and $\mathbf{r} \equiv \mathbf{n}$ **(mod d)**, then a facet (π,π_r) of $P(C_{d,r})$ give the facet (ρ,ρ_n) of $P(K_n)$ as follows:

$$\rho_i = \pi_{i \,(\mathrm{mod}\, d)} \quad \text{for } i=1,\ldots,n$$

Example: $P(C_{5,2}) \rightarrow P(K_{10})$

$$(\pi,\pi_2) = ((1,2,1),2)$$

$$\Downarrow$$

$$(\rho,\rho_{10})=((1,2,1,0,1,2,1,0,1,2),2)$$

Tilting Knapsack Facets

Given a facet (ρ, ρ_r) of master knapsack polytope $P(K_r)$, we may "tilt" ρ by adding any multiple α times the mixed integer cut for $C(r+1,r)$. The resulting inequality is given by:

$$\pi_i = \rho_i + \alpha({}^{i}/_{r})$$

For (π,π_r) to be a facet for $C(r+1,r)$, α must be chosen so that all subadditive relations are satisfied, and at least one new subadditive relation is satisfied at equality.

Tilting Knapsack Facets-Example

$\rho = [2\ 1\ 0\ 2\ 1\ 0\ 2]$ is a facet $\rho x \geq \rho_7$ for K(7).

A facet $\pi x \geq \pi_7$ for $C_{8,7}$ must satisfy the subadditive relation

$$\pi_3 + \pi_6 \geq \pi_1.$$

If $\alpha = {}^7/_4$, then

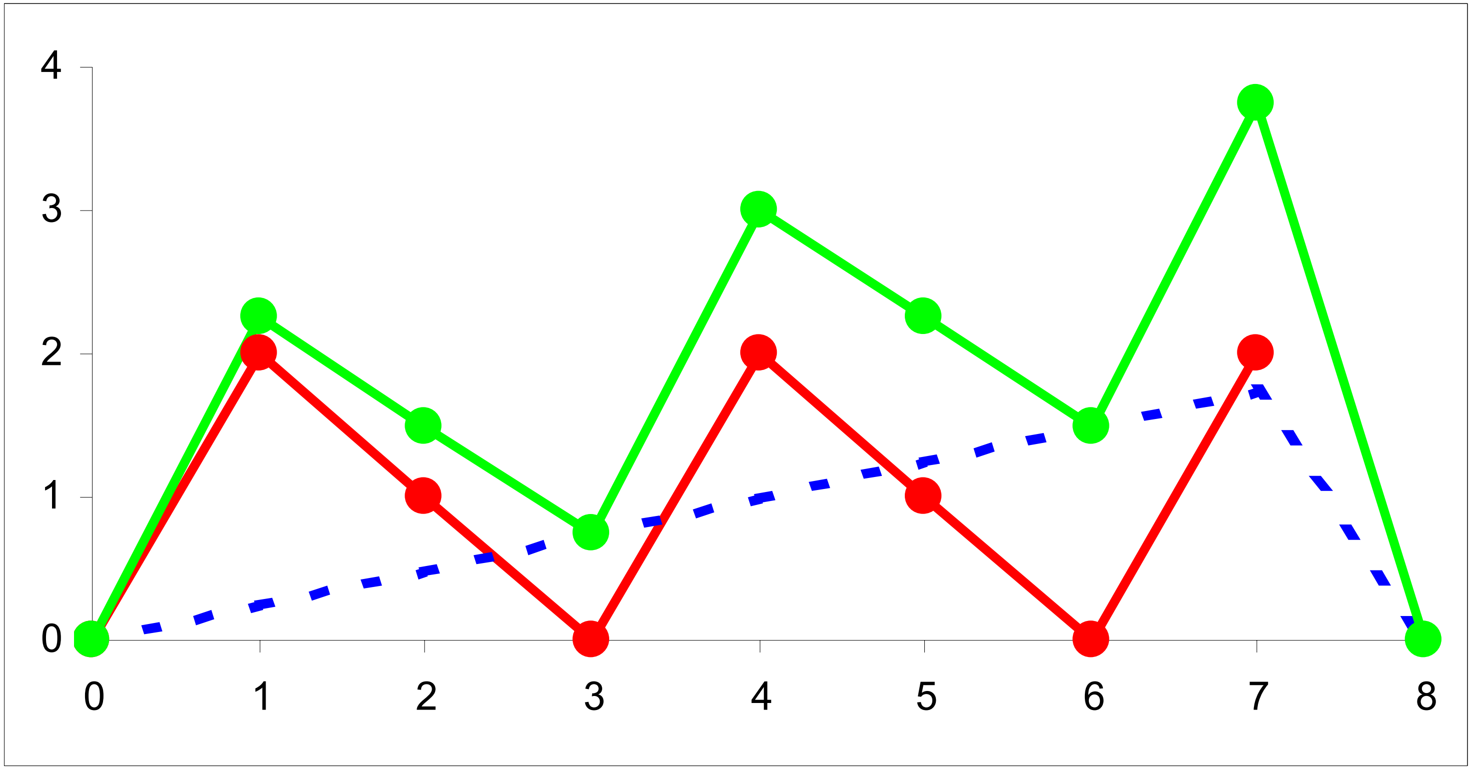
$$\begin{aligned} \pi_3 + \pi_6 &= {}^7/_4 * {}^3/_7 + {}^7/_4 * {}^6/_7 \\ &= {}^{63}/_{28} \\ &= 2 + {}^7/_4 * {}^1/_7 \\ &= \pi_1. \end{aligned}$$

Tilting Knapsack Facets

$$[\,2 \quad 1 \quad 0 \quad 2 \quad 1 \quad 0 \quad 2\,]$$

$$+{}^{7}/_{4}*\ [{}^{1}/_{7} \quad {}^{2}/_{7} \quad {}^{3}/_{7} \quad {}^{4}/_{7} \quad {}^{5}/_{7} \quad {}^{6}/_{7} \quad 1\,]$$

$$[\,{}^{9}/_{4} \quad {}^{6}/_{4} \quad {}^{3}/_{4} \quad {}^{12}/_{4} \quad {}^{9}/_{4} \quad {}^{6}/_{4} \quad {}^{15}/_{4}\,]$$



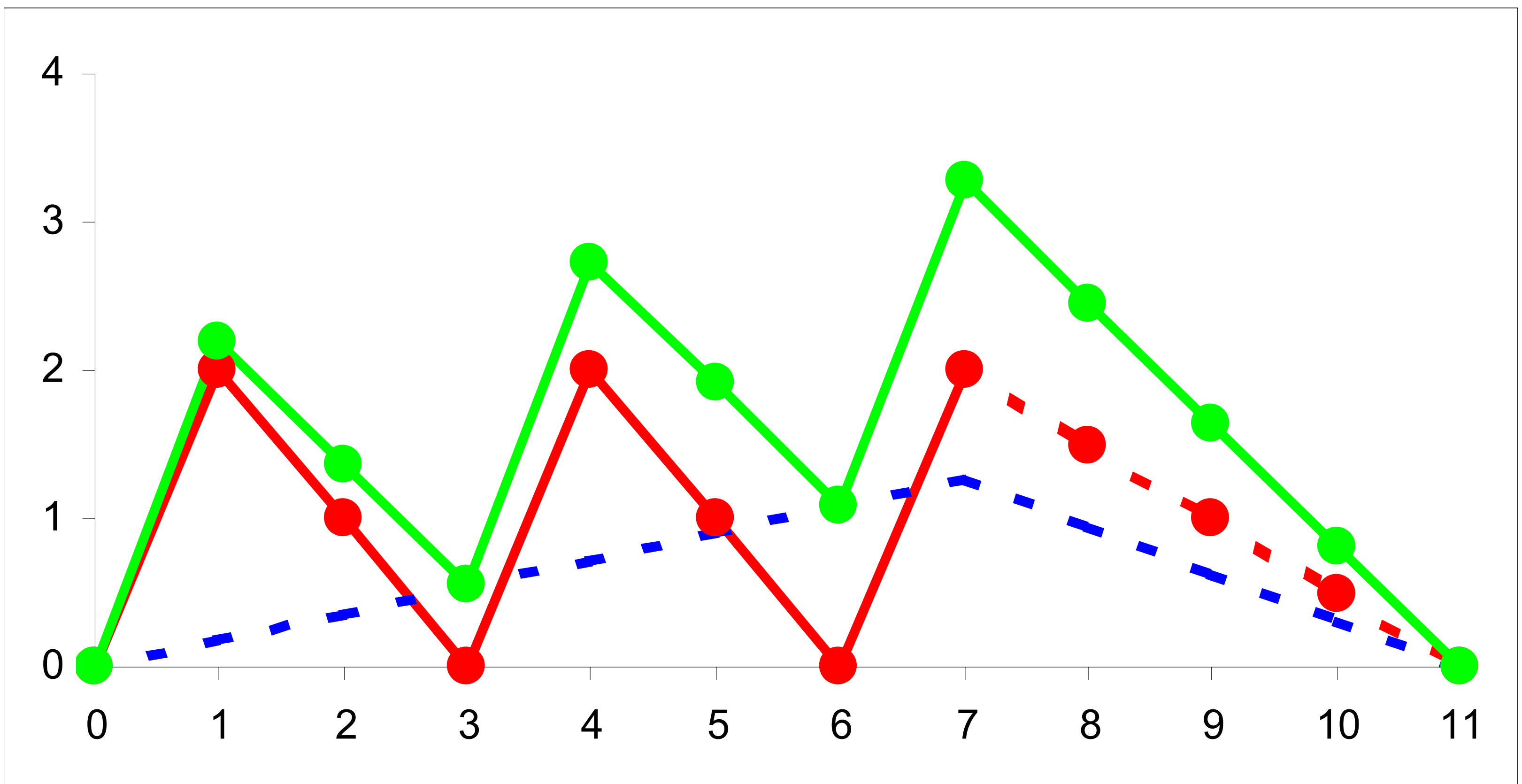
Tilting Knapsack Facets-Extension

We can also find facets for larger cyclic group problems by adding multiples of the MIC for C(n,r) for any n > r.

$$\pi_i = \begin{cases} \rho_i + \alpha\,({}^{i}/_{r}) & \text{for } i = 1,\ldots,r \\ {}^{n-i}/_{n-r} + \alpha\,({}^{n-i}/_{n-r}) & \text{for } i = r+1,\ldots,n-1 \end{cases}$$

Tilting Knapsack Facets-Extension

	[2	1	0	2	1	0	2	$^3/_2$	1	$^1/_2$]
$+ ^{14}/_{11} *$	[$^1/_7$	$^2/_7$	$^3/_7$	$^4/_7$	$^5/_7$	$^6/_7$	1	$^3/_4$	$^2/_4$	$^1/_4$]
	[$^{24}/_{11}$	$^{15}/_{11}$	$^6/_{11}$	$^{30}/_{11}$	$^{21}/_{11}$	$^{12}/_{11}$	$^{36}/_{11}$	$^{27}/_{11}$	$^{18}/_{11}$	$^9/_{11}$]



Generating Facets-Summary

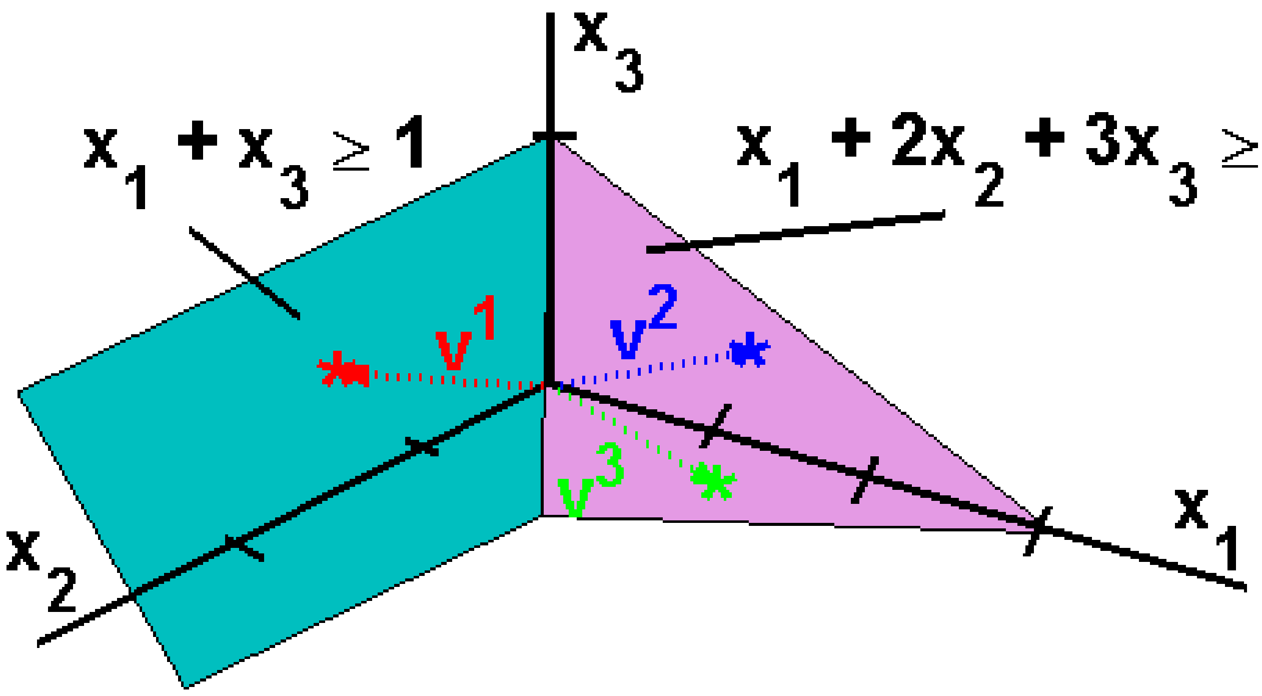
- Finding one seed for any master problem gives **many** facets for larger problems of both types through mappings

- A subset of the facets for any master problem can be found using seeds and mappings

Gomory's Shooting Experiment

Which facets are "important"?

Idea: Shoot random vectors from the origin and see which facets they hit.



Shooting Theorem

Theorem: The facet $\pi^*(v)$ hit by a random vector v is the basic optimal solution to the LP:

$$\min v\pi$$

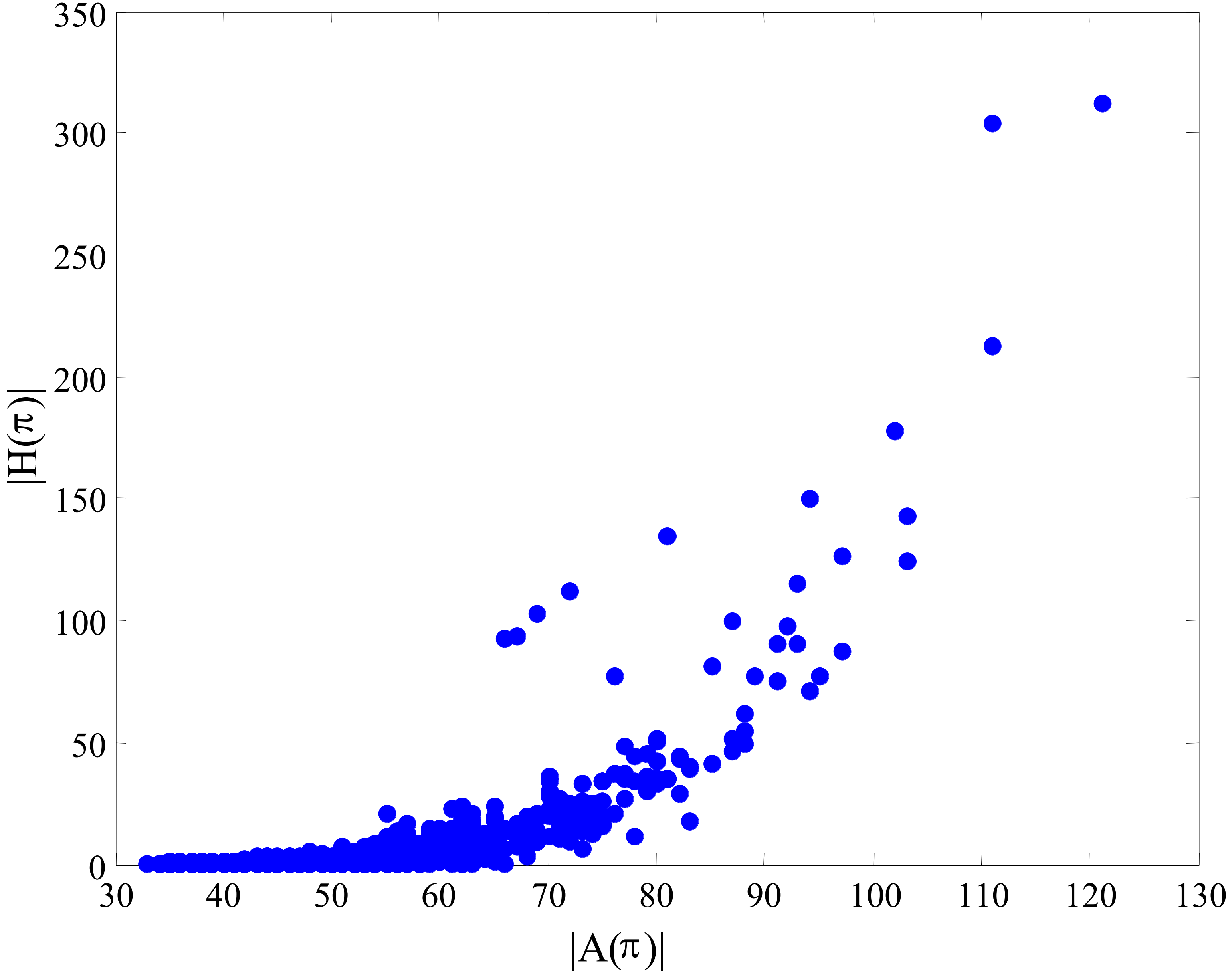
$$\pi_i + \pi_j = \pi_r \qquad \text{if } i+j = r \bmod n, \quad 1 \le i \le j \le n-1$$

$$\pi_i + \pi_j \ge \pi_k \qquad \text{if } i+j = k \bmod n,\ k \ne 0, \quad 1 \le i \le j \le n-1$$

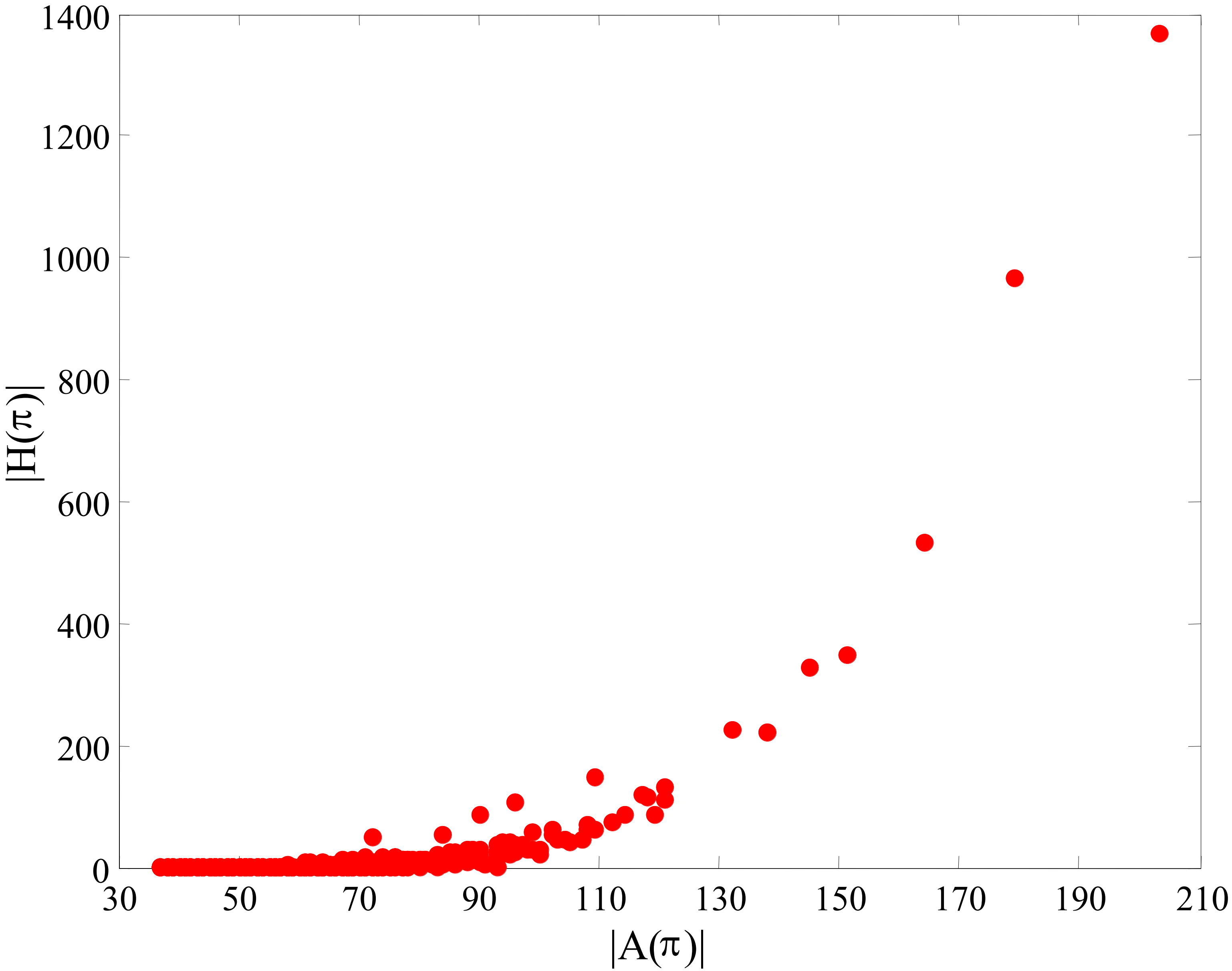
$$\pi_r = 1$$

$$\pi \ge 0$$

Additive Relations vs Hits-C(23,22)



Additive Relations vs Hits-C(24,23)



Shooting Results for Facet Classes

						Tilted	Tilted	Tilted	
n	r	Homo	MIC	2slope	3slope	1,0,-1	3lin1slp	mod1slp	Total
2	1		100%						100%
3	2		100%						100%
4	2		100%						100%
4	3	49%	100%						100%
5	4		100%						100%
6	2	54%	100%						100%
6	3	43%	100%						100%
6	5	76%	100%						100%
7	6		83%		17%				100%
8	2	37%	100%						100%
8	4		49%	51%					100%
8	7	53%	80%	16%	4%				100%
9	3		64%	36%					100%
9	8	31%	77%	8%	9%	6%			100%
10	2	38%	82%	14%	4%				100%
10	5	27%	74%	24%					98%
10	9	57%	89%		5%	1%	5%		100%
11	10		79%		13%	5%	3%		100%
12	2	59%	87%		5%	4%	4%		100%
12	3	57%	73%	15%		2%	1%	5%	96%
12	4	37%	74%	20%		6%			100%
12	6	24%	52%	38%		10%			100%
12	11	63%	87%	3%	4%	1%	1%		96%
13	12		66%	4%	6%	14%	5%	1%	96%
14	2	33%	62%	11%	8%	6%	8%	1%	96%
14	7	23%	69%	18%			3%		90%

Generating and Shooting Results

n	r	Total Facets	# in Shot(n,r)	# in Gen(n,r)	% Shots in Gen(n,r)
15	3	112	112	55	87%
15	5	78	78	45	87%
15	14	68	68	39	90%
16	2	58	58	40	93%
16	4	94	94	56	83%
16	8	104	104	50	82%
16	15	173	173	63	84%
17	16	251	250	108	85%
18	2	154	151	51	85%
18	3	566	479	89	73%
18	6	208	207	63	78%
18	9	574	505	79	73%
18	17	369	309	64	80%
19	18	726	605	168	79%
20	2	377	341	69	79%
20	4	753	587	94	66%
20	5	1,749	929	140	69%
20	10	401	371	77	67%
20	19	1,393	781	108	76%
21	3	2,634	1,355	163	60%
21	7	1,400	872	121	61%
21	20	1,661	840	148	71%
22	2	1,608	896	144	67%
22	11	3,939	1,303	126	58%
22	21	4,640	896	187	67%
23	22	7,188	1,875	347	63%

n	r	Total Facets	# in Shot(n,r)	# in Gen(n,r)	% Shots in Gen(n,r)
24	2	3,509	1,096	116	70%
24	3		2,046	234	59%
24	4	6,737	1,486	177	62%
24	6	5,381	1,605	137	56%
24	8	5,443	1,462	137	58%
24	12	8,184	2,058	148	51%
24	23	11,891	1,527	176	67%
25	5		3,024	408	45%
25	24		2,469	410	57%
26	2		2,213	279	55%
26	13		2,622	235	51%
26	25		2,564	385	57%
27	3		3,734	548	45%
27	9		3,677	364	39%
27	26		2,850	465	55%
28	2		2,532	281	55%
28	4		3,389	350	45%
28	7		3,169	467	49%
28	14		2,749	199	39%
28	27		3,061	476	53%
29	28		4,324	954	45%
30	2		2,975	309	53%
30	3		3,680	532	49%
30	5		3,509	559	50%
30	6		3,922	312	40%
30	10		3,381	266	43%
30	15		4,234	302	43%

Generating Cutting Planes

- Given:
  - Updated row from optimal LP tableau:

$$x_k + \sum_{j \in N} \overline{a}_{ij} x_j = \overline{b}_j$$

  - *Subadditive* function $f : [0,1] \rightarrow [0,1]$

- Let $\hat{u} = \overline{u} \pmod 1$

- Cutting plane: $\sum_{j \in N} f(\hat{a}_{ij}) x_j \geq f(\hat{b}_j)$

Subadditive Functions $f:[0,1]\rightarrow[0,1]$

1. Non – negativity : $f(u)\geq 0$ for $u\in[0,1]$

2. Subadditivity : $f(u)+f(v)\geq f((u+v)\bmod 1)$
for $u,v\in[0,1]$

3. Complementarity : $f(u)+f(r-u)=f(r)$ for $u\in[0,r]$
$f(u)+f(1+r-u)=f(r)$ for $u\in(r,1]$

4. $f(r)=1$

5. Extremality