Subset algebra lifting

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Starting point

Balas, Pulleyblank, Barahona, others (pre 1990).

A polyhedron $P \subseteq R^n$ can be the **projection**

of a *simpler* polyhedron $Q \subseteq R^N$ $(N > n)$

More precisely:

There exist polyhedra $P \subseteq R^n$, such that

- P has exponentially (in n) many facets, and
- P is the projection of $Q \subseteq R^N$, where
- N is polynomial in n, and Q has polynomially many facets.

→ **Lovász and Schrijver (1989)**

Given $\mathcal{F} = \{\, x \in \{0,1\}^n \ : \ Ax \geq b \,\}$

Question: Given $x^* \in R^n_+$, is $x^* \in conv(\mathcal{F})$?

Idea: Let $N \gg n$.

Consider a function (a "lifting") that maps each

$v \in \{0,1\}^n$ into $\hat{z} = \hat{z}(v) \in \{0,1\}^N$

with $\hat{z}_i = v_i$, $1 \leq i \leq n$.

Let $\hat{\mathcal{F}}$ be the image of $\mathcal{F}$ under this operator.

Question: Can we find $y^* \in conv(\hat{\mathcal{F}})$, such that $y^*_i = x^*_i$, $1 \leq i \leq n$?

Concrete Idea

$v \in \{0,1\}^n$ mapped into $\hat{v} \in \{0,1\}^{2^n}$, where

(i) the entries of $\hat{v}$ are indexed by subsets of $\{1, 2, \cdots, n\}$, and

(ii) For $S \subseteq \{1, \ldots, n\}$, $\hat{v}_S = 1$ iff $v_j = 1$ for all $j \in S$.

Example: $v = (1,1,1,0)^T$ mapped to:

$\hat{v}_\emptyset = 1$, $\hat{v}_1 = 1$, $\hat{v}_2 = 1$, $\hat{v}_3 = 1$, $\hat{v}_4 = 0$,

$\hat{v}_{\{1,2\}} = \hat{v}_{\{1,3\}} = \hat{v}_{\{2,3\}} = 1$,

$\hat{v}_{\{1,4\}} = \hat{v}_{\{2,4\}} = \hat{v}_{\{3,4\}} = 0$,

$\hat{v}_{\{1,2,3\}} = 1$, $\hat{v}_{\{1,2,4\}} = \hat{v}_{\{1,3,4\}} = \hat{v}_{\{2,3,4\}} = 0$,

$\hat{v}_{\{1,2,3,4\}} = 0$.

Definitions: The subset *lattice* of $\{1, 2, \cdots, n\}$ is the set of subsets of $\{1, 2, \cdots, n\}$, ordered by inclusion.

Consider the $2^n \times 2^n$ matrix Z, with a column z^p for each $p \subseteq \{1, 2, \cdots, n\}$ such that

$$z^p_q = \begin{cases} 1 & \text{if } q \subseteq p \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Note:

- The lifting maps $v \in \{0, 1\}^n$ to $z^{supp(v)}$.
- Z is upper-triangular, with main diagonal $=$ $\emptyset$-row $= 1$,
- so Z is invertible. Its inverse is called the *Möbius* matrix of the lattice.

What can we say about

$$conv\{z^p \,:\, p \in \hat{\mathcal{F}}\}?$$

or, better, $cone\{z^p \,:\, p \in \hat{\mathcal{F}}\}$?

Let $M = Z^{-1}$

If

$$y = \Sigma_{p \in \hat{\mathcal{F}}} \alpha_p z^p,$$

then

$$m_p y = \alpha_p$$

$(m_p = p^{th}$ row of $M)$ and therefore

$\boldsymbol{cone(\hat{\mathcal{F}})}$ is defined by:

$$m_p y \geq 0, \ \ \forall y \in \hat{\mathcal{F}} \ \ (\forall p \in \hat{\mathcal{F}}) \ \ \ and$$

$$m_p y = 0, \ \ \forall y \in \hat{\mathcal{F}} \ \ (\forall p \notin \hat{\mathcal{F}})$$

$\rightarrow$ An exact formulation, but in 2^n variables.

Take $v \in \{0,1\}^n$. The $2^n \times 2^n$ matrix $\hat{v}\hat{v}^t$

$\longrightarrow$ Is symmetric, and its main diagonal $=$ $\emptyset$-row.

Further, suppose $x^* \in R^n$ satisfies

$$x^* \;=\; \Sigma_i \lambda_i v_i,$$

where each $v_i \in \{0,1\}^n$, $0 \le \lambda$, and $\Sigma_i \lambda_i = 1$.

Let $W \;=\; W(x^*) \;=\; \Sigma_i \lambda_i \hat{v}_i \hat{v}_i^{\,t}$ and $y = \Sigma_i \lambda_i \hat{v}_i$.

- $y_{\{j\}} = x^*_j$, for $1 \le j \le n$.
- W is symmetric, $W_{\emptyset,\emptyset} = 1$, diagonal $=$ $\emptyset$-column $= y$.
- $W \succeq 0$.
- $\forall\, p, q \subseteq \{1, 2, \cdots, n\}$, $W_{p,q} = y_{p \cup q}$

So we can write $W = W^y$.

$x^* = \Sigma_i \lambda_i v_i, \quad 0 \le \lambda$ and $\Sigma_i \lambda_i = 1.$

$y = \Sigma_i \lambda_i \hat{v}_i, \qquad W^y \; = \; \Sigma_i \lambda_i \hat{v}_i \hat{v}_i{}^t,$ cont'd

Assume each $v_i \in \mathcal{F}$.

Theorem

Suppose $\Sigma_{j=1}^n \alpha_j x_j \ge \alpha_0 \;\; \forall x \in \mathcal{F}$.

Let $p \subseteq \{1, 2, \cdots, n\}$.

Then:

$$\Sigma_{j=1}^n \alpha_j W^y_{\{j\},p} \; - \; \alpha_0 W^y_{\emptyset,p} \; \ge \; 0$$

e.g. the p-column of W satisfies **every** constraint valid for $\mathcal{F}$, homogenized.

$\rightarrow$ just show that for every i,

$$\Sigma_{j=1}^n \alpha_j [\hat{v}_i \hat{v}_i{}^t]_{\{j\},p} \; - \; \alpha_0 [\hat{v}_i \hat{v}_i{}^t]_{\emptyset,p} \; \ge \; 0$$

Also holds for the $\emptyset$-column minus the u^{th}-column.

Lovász-Schrijver Operator

Given $\mathcal{F} = \{\, x \in \{0,1\}^n \;:\; Ax \geq b \,\}$

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1. Form an $(n+1) \times (n+1)$-matrix W of **variables**
2. **Constraint:** $W_{0,0} = 1$, W symmetric, $W \succeq 0$.
3. **Constraint:** $0 \leq W_{i,j} \leq W_{0,j}$, for all i, j.
4. **Constraint:** The main diagonal of W equals its 0-row.
5. **Constraint:** For every column u of W,

   $$\Sigma_{h=1}^{n} a_{i,h} u_h \;-\; b_i u_0 \;\geq\; 0, \quad \forall \text{ row } i \text{ of } A$$

   and

   $$\Sigma_{h=1}^{n} a_{i,h}(W_{h,0} - u_h) \;-\; b_i(1 - u_0) \;\geq\; 0, \quad \forall \text{ row } i \text{ of } A$$

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Let $C = \{\, x \in R^n : \; 0 \leq x \leq 1, \; Ax \geq b \,\}$
and $N_+(C)$ = set of $x \in R^n$, such that
there exists W satisfying 1-5, with $W_{j,0} = x_j$, $1 \leq j \leq n$.

Lemma. $C \supseteq N_+(C) \supseteq N_+^2(C) \supseteq \cdots$.

Theorem. $N_+^n(C) = conv(\mathcal{F})$.

$x^* = \Sigma_i \lambda_i v_i, \quad 0 \leq \lambda$, each $v_i \in \mathcal{F}$.

$y = \Sigma_i \lambda_i \hat{v}_i, \qquad W^y \;=\; \Sigma_i \lambda_i \hat{v}_i \hat{v}_i{}^t, \quad$ again

Suppose $\alpha^T x \geq \alpha_0$ for all $x \in \mathcal{F}$.

Then

$$V^{(\alpha,\alpha_0)} = \Sigma_i \gamma_i \hat{v}_i \hat{v}_i^T \succeq 0,$$

where

$$\gamma_i = \lambda_i (\alpha^T v_i - \alpha_0) \geq 0.$$

The $\emptyset -$ column of $V^{(\alpha,\alpha_0)}$ equals $W^y \hat{\alpha}$, where

$$\hat{\alpha}_\emptyset \;=\; -\alpha_0$$

$$\hat{\alpha}_{\{j\}} \;=\; \alpha_j \;\; j = 1, 2, \cdots, n$$

$$\hat{\alpha}_p \;=\; 0, \;\; \text{all other} \;\; p$$

So: $V^{(\alpha,\alpha_0)} = W^z$, where $z = W^y \hat{\alpha}$.

$\mathcal{F} = \{\, x \in \{0,1\}^n \;:\; Ax \geq b \,\}$

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Laurent (2001):

Sherali-Adams level-k Operator ($k = 1, 2, \cdots, n$) **(1990)**

$\triangleright$ For each $p \subseteq \{1, 2, \cdots, n\}$ with $|p| = \min\{k+1, n\}$:

Form $W[p]$, the minor of W induced by p and its subsets.

$\longrightarrow$ Require $W[p] \succeq 0$.

$\triangleright$ For each row $a_i x \geq b_i$, and for each $q \subseteq \{1, 2, \cdots, n\}$ with $|q| = k$:

Form $V[q, i]$, the minor of $V^{(a_i, b_i)}$ induced by q and its subsets.

$\longrightarrow$ Require $V[q, i] \succeq 0$.

Note: we obtain the $V[q]$ from $W[p](a_i^T, -b_i)$.

$\mathcal{F} = \{ x \in \{0,1\}^n \ : \ Ax \geq b \}$

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Laurent (2001):

Lasserre level-k Operator ($k = 1, 2, \cdots, n$) (2001)

▷ Let $W\{k\}$ = minor of W induced by

$$\{p \subseteq \{1, 2, \cdots, n\} \ : \ |p| \leq \min\{k + 1, n\}\}$$

→ Require $W\{k\} \succeq 0$.

▷ For each row $a_i x \geq b_i$:

Form $V\{i\}$, the minor of $V^{(a_i, b_i)}$ induced by

$$\{p \subseteq \{1, 2, \cdots, n\} \ : \ |p| \leq k\}$$

→ Require $V\{i\} \succeq 0$.

Operator Comparison

(a) k-iterate convexification
(Balas, Ceria, Cornuejols) C^k

(b) k-iterate Lovász-Schrijver N_+^k, N^k

(c) level-k Sherali-Adams S^k

(d) level-k Lasserre L^k

S^k stronger than N^k stronger than C^k

L^k stronger than both N_+^k and S^k

Bound on max. rank $= n$ for N_+ is tight:
Cook-Dash (2001), Laurent (2001).

Lovász-Schrijver revisited

$v \in \{0,1\}^n$ lifted to $\hat{v} \in \{0,1\}^{2^n}$, where

(i) the entries of $\hat{v}$ are indexed by subsets of $\{1, 2, \cdots, n\}$, and

(ii) For $S \subseteq \{1, \ldots, n\}$, $\hat{v}_S = 1$ iff $v_j = 1$ for all $j \in S$.

$\rightarrow$ this approach makes statements about *sets* of variables that simultaneously equal 1

How about more complex logical statements?

Subset algebra lifting

For $1 \leq j \leq n$, let

$$Y_j = \{z \in \{0,1\}^n : z_j = 1\}$$

$$N_j = \{z \in \{0,1\}^n : z_j = 0\}$$

Let $\mathcal{A}$ denote the set of all set-theoretic expressions

involving the Y_j, the N_j, and $\emptyset$.

Note:

(i) $\mathcal{A}$ is isomorphic to the set of subsets of $\{0,1\}^n$.

(ii) $|\mathcal{A}| = 2^{2^n}$

(iii) $\mathcal{A}$ is a lattice under $\supseteq$

(iv) $\mathcal{A}$ contains an isomorphic copy of the *lattice* of subsets of $\{1, 2, \cdots, n\}$.

Lift $v \in \{0,1\}^n$ to $\breve{v} \in \{0,1\}^{\mathcal{A}}$

where for each $S \subseteq \{0,1\}^n$, $\breve{v}_S = 1$ iff $v \in S$.

Example

$v = (1,1,1,0,0) \in \{0,1\}^5$ is lifted to

$\breve{v} \in \{0,1\}^{2^{64}}$ which satisfies

$\breve{v}[(Y_1 \cap Y_2) \cup Y_5] = 1$

$\breve{v}[Y_3 \cap Y_4] = 0$

$\breve{v}[Y_3 \cap (Y_4 \cup N_5)] = 1$

$\ldots$

$\breve{v}[S] = 1$ iff $(1,1,1,0,0) \in S$

Note: if $v \in \mathcal{F}$ then $\breve{v}[\mathcal{F}] = 1$.

$\rightarrow$ Family of algorithms that generalize Lovász-Schrijver, Sherali-Adams, Lasserre

Generic Algorithm

1. Form a family of set-theoretic indices $\mathcal{V}$.
Among them, for $1 \leq j \leq n$,

$$\begin{array}{l} \mathrm{Y}_j, \text{ to represent } \{x \in \{0,1\}^n \,:\, x_j = 1\} \\ \mathrm{N}_j, \text{ to represent } \{x \in \{0,1\}^n \,:\, x_j = 0\} \end{array}$$

Also $\emptyset$, $\mathcal{F}$ (and others).

2. Impose all constraints known to be valid for $\mathcal{F}$:
e.g.

$$x_1 + 4x_2 \geq 3 \text{ valid } \rightarrow X[\mathrm{Y}_1] + 4X[\mathrm{Y}_2] - 3 \geq 0$$

Also set theoretic constraints, e.g.

$$X[\mathrm{N}_5] \geq X[\mathrm{Y}_2 \cap \mathrm{N}_5]$$

3. Form a matrix $U \in R^{\mathcal{V} \times \mathcal{V}}$ of variables

- U symmetric, main diagonal = $\mathcal{F}$-row = X
- For $p, q \in \mathcal{V}$,

$$\begin{array}{rcl} U_{p,q} & = & X[p \cap q] \text{ if } p \cap q \in \mathcal{V} \\ U_{p,q} & = & \text{a new variable, otherwise} \end{array}$$

- Each column of U satisfies all constraints,
- $U \succeq 0$ (optional)

How do we algorithmically choose small $\mathcal{V}$?

Obstructions

→ An expression $\omega \in \mathcal{A}$ is an **obstruction** if

$$v \in \mathcal{F} \text{ implies } \breve{v}[\omega] = 0$$

(simply means: v does not satisfy ω)

Example: given

$$x_1 + 5x_2 + x_3 + x_4 + x_5 - 2x_6 \geq 2$$

then

$$\mathrm{N}_1 \cap \mathrm{N}_2 \cap \mathrm{Y}_6$$

is an obstruction

- it is *minimal*
- it is $3 - small$
  - ◇ at most 2 or at least 4 Y_j
  - ◇ at most 2 or at least 4 N_j

→ Set covering, set partitioning, set packing: all minimal obstructions are k-small, $k \leq 2$.

Walls

$\rightarrow$ A wall is an intersection (conjunction) of obstructions

Example:

$$x_1 + 5x_2 + x_3 + x_4 + x_5 - 2x_6 \geq 2 \rightarrow \mathrm{N}_1 \cap \mathrm{N}_2 \cap \mathrm{Y}_6$$

$$-x_2 + 2x_3 + x_4 + x_6 \leq 3 \rightarrow \mathrm{N}_2 \cap \mathrm{Y}_3 \cap \mathrm{Y}_4 \cap \mathrm{Y}_6$$

$$x_1 + x_2 + x_3 - x_4 \geq 1 \rightarrow \mathrm{N}_1 \cap \mathrm{N}_2 \cap \mathrm{N}_3 \cap \mathrm{Y}_4$$

$\rightarrow$ $\omega = \mathrm{N}_1 \cap \mathrm{N}_2 \cap \mathrm{Y}_4 \cap \mathrm{Y}_6$ is the derived **wall**

$\rightarrow$ $\mathrm{Y}_1 \cap \mathrm{Y}_2 \cap \mathrm{Y}_4 \cap \mathrm{Y}_6$ is a **negation** of ω of **order 2**

so is $\mathrm{N}_1 \cap \mathrm{N}_2 \cap \mathrm{N}_4 \cap \mathrm{N}_6$

also:

$$\omega^{>2} = \bigcup_{t>2} \{\omega' : \omega' \text{ is a negation of order } t \text{ of } \omega\}$$

In general, the $w^{>r}$ expressions are unions (disjuntions) of exponentially many intersections.

Constraints

"Box" constraints:

$$0 \leq X, \quad X[\mathcal{F}] = 1, \quad X[p] - X[\mathcal{F}] \leq 0$$

Also, say:

$$\omega = \mathrm{N}_1 \cap \mathrm{N}_2 \cap \mathrm{Y}_4 \cap \mathrm{Y}_6.$$

Then e.g.

$$X[\mathrm{N}_1] - X[\omega] \geq 0.$$

Also,

$$X[\mathrm{Y}_1] + X[\mathrm{Y}_2] + X[\mathrm{N}_4] + X[\mathrm{N}_6] - 2X[\omega^{>1}] \geq 0.$$

Finally,

$$X[\omega] + X[\mathrm{Y}_1 \cap \mathrm{N}_2 \cap \mathrm{Y}_4 \cap \mathrm{Y}_6] + X[\mathrm{N}_1 \cap \mathrm{Y}_2 \cap \mathrm{Y}_4 \cap \mathrm{Y}_6] +$$

$$+ X[\mathrm{N}_1 \cap \mathrm{N}_2 \cap \mathrm{N}_4 \cap \mathrm{Y}_6] + X[\mathrm{N}_1 \cap \mathrm{N}_2 \cap \mathrm{Y}_4 \cap \mathrm{N}_6] +$$

$$+ X[\omega^{>1}] \ = \ 1$$

$\rightarrow$ Implications for "matrix of variables"

Critical part of algorithm - excerpt

$\mathcal{V}$ = family of set-theoretic variables
Step 3. Form a matrix $U \in R^{\mathcal{V} \times \mathcal{V}}$ of variables

- U symmetric, main diagonal = $\mathcal{F}$-row = X
- For $p, q \in \mathcal{V}$,

$$\begin{aligned} U_{p,q} &= X[p \cap q] \text{ if } p \cap q \in \mathcal{V} \\ U_{p,q} &= \text{a new variable, otherwise} \end{aligned}$$

- Each column of U satisfies all constraints known to be valid.

⋆ Say $\omega = \mathrm{N}_1 \cap \mathrm{N}_2 \cap \mathrm{Y}_4 \cap \mathrm{Y}_6$ is a wall.

Then we impose

$$X[\omega] + X[\mathrm{Y}_1 \cap \mathrm{N}_2 \cap \mathrm{Y}_4 \cap \mathrm{Y}_6] + X[\mathrm{N}_1 \cap \mathrm{Y}_2 \cap \mathrm{Y}_4 \cap \mathrm{Y}_6] +$$

$$+ X[\mathrm{N}_1 \cap \mathrm{N}_2 \cap \mathrm{N}_4 \cap \mathrm{Y}_6] + X[\mathrm{N}_1 \cap \mathrm{N}_2 \cap \mathrm{Y}_4 \cap \mathrm{N}_6] +$$

$$+ X[\omega^{>1}] - X[\mathcal{F}] = 0$$

→ **The columns of U satisfy this equation**

⋆ Say $p = \mathrm{N}_1 \cap \mathrm{N}_2$, $q = \mathrm{N}_3 \cap \mathrm{N}_4$ and $\mathrm{N}_1 \cap \mathrm{N}_2 \cap \mathrm{N}_3 \cap \mathrm{N}_4$ is an obstruction. Then

$$U_{p,q} = 0$$

can be enforced.

Set covering problems

$$\begin{aligned} \min \quad & c^T x \\ s.t. \ & x \in \mathcal{F} \end{aligned}$$

$\mathcal{F} = \{\, x \in \{0,1\}^n \ : \ Ax \geq 1 \,\}$, $\quad A$ a $0-1$-matrix.

$\longrightarrow$ All faces $\alpha^T x \geq \alpha_0$ satisfy $\alpha \geq 0$
$\longrightarrow$ can assume all coefficients are integral

Balas and Ng (1989):
all facets with $\alpha_j \in \{0,1,2\}$, $j = 0,1,\cdots,n$.

$\longrightarrow$ There exist Gomory rank-2 valid inequalities

$$\alpha^T x \geq \alpha_0$$

where some $\alpha_j = 3$.

Circulant matrices

$$\min \quad c^T x$$

$$s.t. \quad \Sigma_{j \neq h} x_j \geq 1, \quad \text{for each } h$$

$$x_j = 0 \text{ or } 1, \quad \text{all } j$$

$$\rightarrow \Sigma_j x_j \geq 2 \ \text{ is valid.} \qquad (2)$$

Theorem

The combination of the Lovász-Schrijver N_+ operator and the Sherali-Adams operator requires at least rank $n-2$ to guarantee (2). ■

$\Rightarrow$ exponential work

$\mathcal{F} = \{ x \in \{0,1\}^n \ : \ Ax \geq 1 \}$, A a 0-1 matrix.

Definition:
The **pitch** of $\alpha^T x \geq \beta$ is the smallest integer π, such that the sum of the π smallest positive α_j is at least β.

e.g. $2x_1 + 3x_2 + 4x_3 + 4x_4 \geq 6$ has pitch 3.

Theorem

Let $k \geq 1$, and

$P_k = \{x \in R^n \ : \ 0 \leq x \leq 1,$ and
x satisfies all valid inequalities with pitch $\leq k \ \}$.
(includes $Ax \geq 1$)

Then: there is a polyhedron

(a) given by a formulation with polynomially many rows and columns,

(b) whose **projection** W_k to R^n satisfies:

$$conv(\mathcal{F}) \subseteq W_k \subseteq P_k. \ \blacksquare$$

$\rightarrow$ P_k satisfies all inequalities with coefficients in $\{0, 1, \cdots, k\}$.

$\rightarrow$ There are examples of set-covering problems with **exponentially** many facets with all coefficients in $\{0, 1, 2\}$.

Additional Results

Cook and Dash, 2001 (Goemans and Tuncel, 2000):

$$\mathcal{F} = \left\{ x \in \{0,1\}^n \,:\, x_1 + x_2 + \cdots x_n \geq \frac{1}{2} \right\}$$

the N_+-rank of

$$x_1 + x_2 + \cdots x_n \geq 1$$

is n.

→ the subset algebra lifting algorithm proves
it in polynomial-time (rank 2).

Cook, Chvátal, Hartmann (1985):

$$\mathcal{F} = \left\{ x \in \{0,1\}^n \,:\, \sum_{j \in J} x_j + \sum_{j \notin J} (1 - x_j) \geq \frac{1}{2}, \forall J \right\}$$

Hence, $\mathcal{F} = \emptyset$.

The Chvátal rank is n (Cornuéjols and Li, 2001: the mixed-integer rank).

→ the subset algebra lifting algorithm proves $\mathcal{F} = \emptyset$ in polynomial-time (rank 1).