Differential Optimization Problems: Analysis of the Hessian

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Issues in Differential Optimization

- Infinite dimensional
- Well-Posedness & Ill-Posedness
- Regularization
- Stability
- Fast Solvers

Motivating Applications

- Control Problems
- Shape Design Problems (Euler & NS)
- MDO Problems (structures-fluid coupling)
- Data Assimilation in Meteorology

Problem Types

- Unconstrained Optimization

- Optimization with Equality/Inequality Constraints

PDE Constraints: Elliptic systems, Hyperbolic systems, Mixed systems

Other Constraints: equality/inequality on design/state variables

Everything is in The (Reduced) Hessian

- Well-Posedness

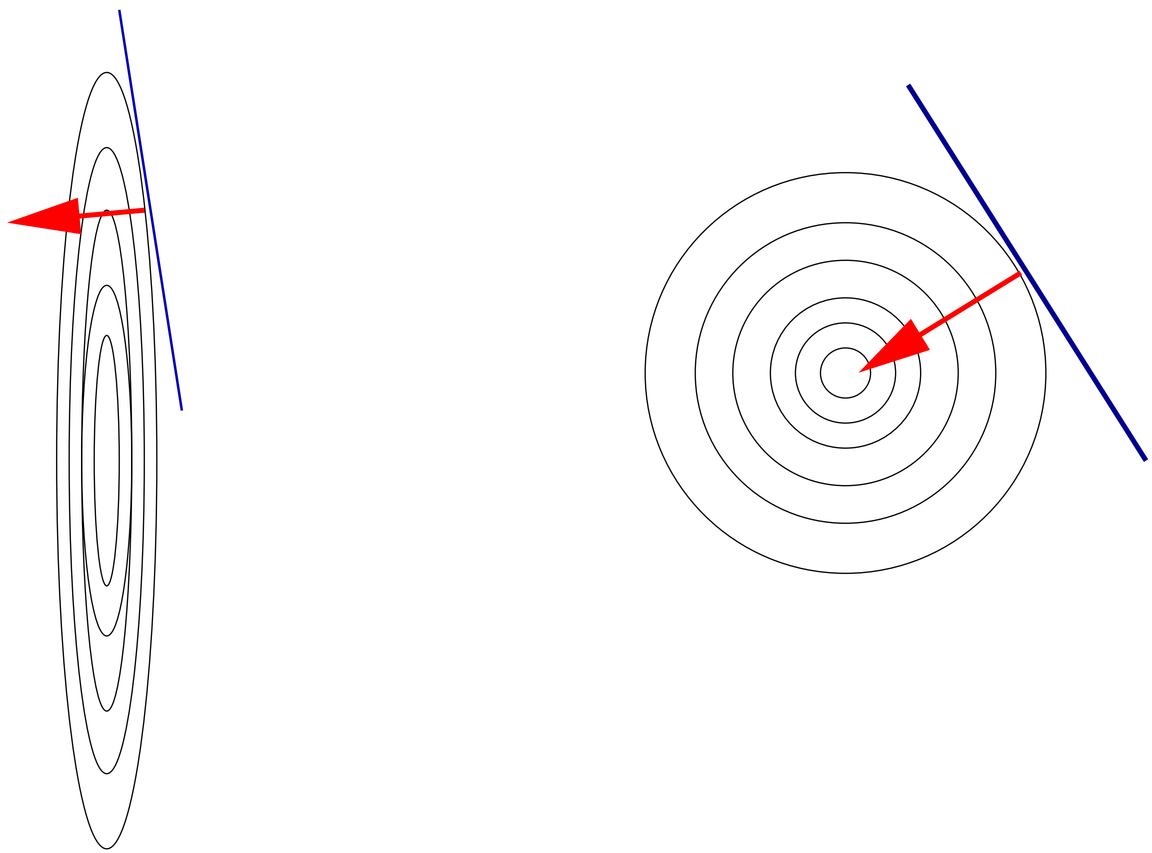
- Regularization choice

- Convergence to a minimum

- Stability

- Fast Solvers

The Basics in a Picture

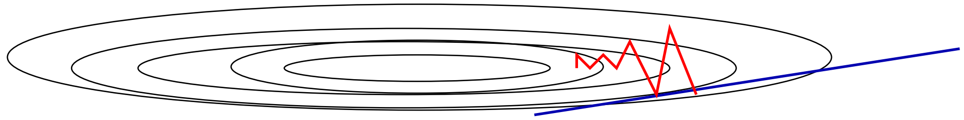


Unconstrained Optimization

$$\min_{\alpha} E(\alpha)$$

Gradient based minimization

$$\alpha \longleftarrow \alpha - \delta \nabla E \qquad \textit{with line search}$$



Error Analysis

If α^* is a minimum, then

$$E(\alpha, U(\alpha)) \approx E(\alpha^*, U(\alpha^*)) + \frac{1}{2}\tilde{\alpha}^T \mathcal{H} \tilde{\alpha}$$

$\tilde{\alpha} = \alpha^* - \alpha$,

$$-(\nabla E)(\alpha) = \mathcal{H}\tilde{\alpha}.$$

Error equation

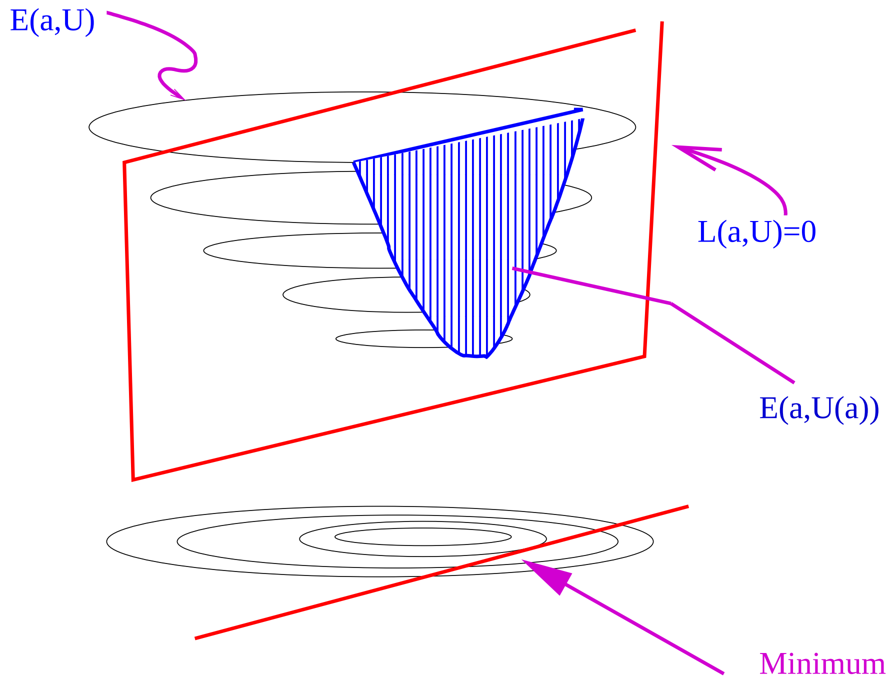
$$\tilde{\alpha} \leftarrow (I - \delta\mathcal{H})\tilde{\alpha}.$$

The Hessian Governs the Convergence Process!!

Constrained Optimization

$$\min_{\alpha} E(\alpha, U(\alpha))$$

$$L(U(\alpha), \alpha) = 0$$



The Adjoint Method: efficient calculation of gradients

Algorithm:

- Given α solve: $L(\alpha, U) = 0$ for U
- Given α, U solve: $L_U^* \lambda + E_U = 0$ for λ
- Update α: $\alpha \leftarrow \alpha - \delta[E_\alpha + L_\alpha^* \lambda]$, *with line search.*

Use Fourier Analysis to Understand the Hessian

High Frequencies Behavior

- Localized (elliptic equations)
- Can be Analyzed by Full/Half Space

Low Frequency Behavior

- Depends on the Whole Domain
- Cannot be Analyzed Using Fourier

The Symbol of an Operator

$$T\,\exp(i\mathbf{k}\cdot\mathbf{x}) = \widehat{T}(\mathbf{k})\,\exp(i\mathbf{k}\cdot\mathbf{x}).$$

Examples:

$$\frac{\partial}{\partial x} \quad\rightarrow\quad ik$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad\rightarrow\quad -k_1^2 - k_2^2$$

$$\begin{pmatrix} \beta^2\frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & -\frac{\partial}{\partial x} \end{pmatrix} \quad\rightarrow\quad \begin{pmatrix} i\beta^2 k_1 & ik_2 \\ ik_2 & -ik_1 \end{pmatrix}$$

Classification of DOP

Recall

$$\mathcal{H}\tilde{\alpha} = -g$$

Fourier analysis gives

$$\widehat{\mathcal{H}}(\mathbf{k})\widehat{\tilde{\alpha}}(\mathbf{k}) = -\widehat{g}(\mathbf{k})$$

High frequency behavior

`Well posedness:` unique solution stable to perturbations,

$$\widehat{\mathcal{H}}(\mathbf{k}) = O(|\mathbf{k}|^{\gamma}) \qquad \gamma \geq 0.$$

$$|\hat{\hat{\alpha}}(\mathbf{k})| \approx |\mathbf{k}|^{-\gamma}|\hat{g}(\mathbf{k})| \qquad \text{for large } |\mathbf{k}|$$

`Ill-Posedness:` Sensitivity to perturbations,

$$\hat{\mathcal{H}}(\mathbf{k}) = O(\frac{1}{|\mathbf{k}|^{\gamma}}) \qquad \gamma > 0.$$

$$|\hat{\hat{\alpha}}(\mathbf{k})| \approx |\mathbf{k}|^{\gamma}|\hat{g}(\mathbf{k})| \qquad \text{large } |\mathbf{k}|$$

Errors are amplified by small changes in data.

The Discretized Problem

$$\widehat{\mathcal{H}}^h(\mathbf{k}) = O(|\mathbf{k}|^\gamma)$$

For discrete level, $|\mathbf{k}| \le \pi/h$, so

$$\max_{|\mathbf{k}| \le \pi/h} |\widehat{\mathcal{H}}^h(\mathbf{k})| = O(\frac{1}{h^\gamma}).$$

For high frequencies,

$$\max_{\pi/(2h)\le|\mathbf{k}|\le\pi/h} |1-\delta\widehat{\mathcal{H}}^h(\mathbf{k})|$$

Smallest eigenvalue of $\mathcal{H}^h$ is $O(1)$

Convergence rate,

$$1-O(\frac{\mu_1}{\mu_q}) = 1-O(h^\gamma).$$

- *Easy problems:* $0 \le \gamma \ll 1$.
- *Difficult problems:* $\gamma \ge 1$

In Summary,

Symbol of the Hessian

$$\widehat{\mathcal{H}}(\mathbf{k}) = O(|\mathbf{k}|^{\gamma}) \qquad \text{large} \quad |\mathbf{k}|$$

`Well-Posed optimization problems:`	$\gamma \geq 0$
`Ill-Posed Optimization problems:`	$\gamma < 0$
`Easy optimization problems:`	$0 \leq \gamma \ll 1$
`Difficult optimization problems:`	$\gamma \geq 1$

Model Problems

Model I

State Eq:

$$\begin{aligned} \Delta\phi &= 0 \quad && \Omega \\ \frac{\partial\phi}{\partial n} &= \alpha_x \quad && \Gamma \\ \phi &= g \quad && \Gamma_0 \end{aligned}$$

Cost Functional

$$E(\alpha) = \int_\Gamma (p - p^*)^2 dx \qquad p = \phi_x$$

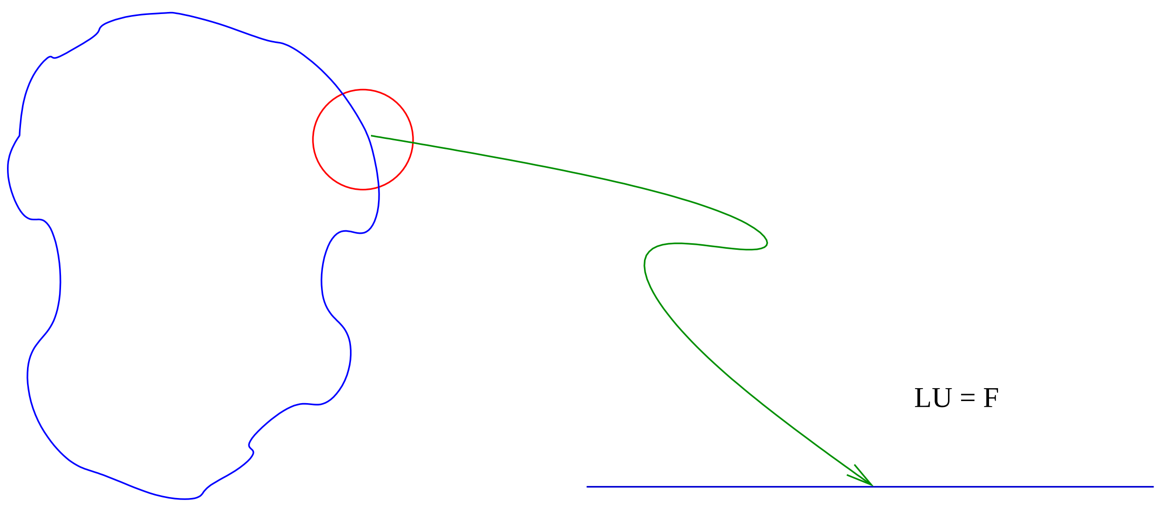
Adjoint Eq:

$$
\begin{aligned}
&\Delta\lambda = 0 && \Omega \\
&\frac{\partial\lambda}{\partial n} + 2(\phi_x - p^*)_x = 0 && \Gamma \\
&\lambda = 0 && \Gamma_0.
\end{aligned}
$$

$$\nabla E = -\lambda_x$$

Analysis

- Local analysis at a smooth pt of Γ
- Half Space


LU = F

In our model

$$\alpha \rightarrow \alpha + \epsilon\tilde{\alpha}$$
$$\phi \rightarrow \phi + \epsilon\tilde{\phi}$$

Now use Fourier

$$\tilde{\alpha} = \mathsf{exp}(ikx)$$

$$\tilde{\phi}|_{\Gamma} = \frac{ik}{|k|}\mathsf{exp}(ikx)$$

$$\tilde{\phi}_x = -|k|\tilde{\alpha}$$

Quadratic term in functional

$$\int |\phi_x|^2 ds \rightarrow \int |k|^2 |\widehat{\alpha}(k)|^2 dk$$

Hessian Symbol,

$$\widehat{\mathcal{H}}(k) = |k|^2$$

- Problem is well posed for h.f.
- Problem is not easy, condition number is $O(h^{-2})$

`Now lets change the functional`

Model II

State Eq:

$$\begin{aligned} \Delta\phi &= 0 && \Omega \\ \frac{\partial\phi}{\partial n} &= \alpha_x && \Gamma \\ \phi &= g && \Gamma_0 \end{aligned}$$

Cost Functional

$$E(\alpha) = \int_\Gamma (\phi - \phi^*)^2 dx$$

Adjoint Eq:

$$\begin{aligned} \Delta\lambda &= 0 && \Omega \\ \frac{\partial\lambda}{\partial n} + 2(\phi - \phi^*) &= 0 && \Gamma \\ \lambda &= 0 && \Gamma_0. \end{aligned}$$

Gradient

$$\nabla E = -\lambda_x$$

Hessian Symbol,

$$\widehat{\mathcal{H}}(k) = 1$$

- Problem is well posed for h.f.
- Problem is easy, condition number is $O(1)$

`Next lets change the BC`

Model III

State Eq:

$$\begin{aligned} \Delta\phi = 0 \quad & \Omega \\ \frac{\partial\phi}{\partial n} = \alpha \quad & \Gamma \\ \phi = g \quad & \Gamma_0 \end{aligned}$$

Cost Functional

$$E(\alpha) = \int_\Gamma (\phi - \phi^*)^2 dx$$

Adjoint Eq:

$$\begin{aligned} \Delta\lambda = 0 \quad & \Omega \\ \frac{\partial\lambda}{\partial n} + 2(\phi - \phi^*) = 0 \quad & \Gamma \\ \lambda = 0 \quad & \Gamma_0. \end{aligned}$$

$$\nabla E = \lambda$$

Fourier Analysis gives,

$$\widehat{\mathcal{H}}(k) = \frac{1}{|\mathbf{k}|^2}$$

- Problem is ill-posed for h.f.
- Problem is hard, condition number is $O(h^{-2})$

Conclusion: `Be Careful with formulation of DOP`

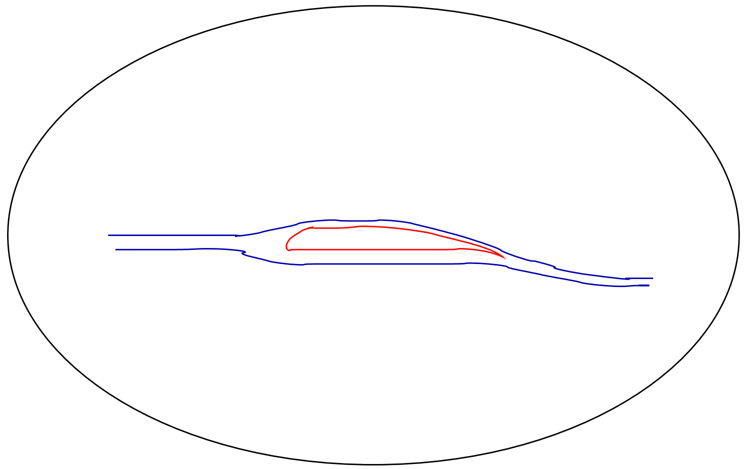
Shape Design: Full Potential Equation

$$
\begin{aligned}
&\nabla\rho\nabla\phi = 0 && \Omega \\
&\phi = \phi_\infty && \partial\Omega - \Gamma \\
&\nabla\phi\cdot\mathbf{n} = 0 && \Gamma
\end{aligned}
$$

where $\rho = f(q)$ $\qquad q = \frac{1}{2}|\nabla\phi|^2$

Cost Functional:

$$\min_{\Gamma}\frac{1}{2}\int_{\Gamma}(p-p^*)^2 ds \qquad p = g(q)$$



Gradient,

$$\nabla J = -\rho\lambda\frac{\partial^2\phi}{\partial n^2} - \frac{(p-p^*)^2}{2R} + \sum_{j=1}^{d-1}\frac{\partial}{\partial t_j}(\lambda\rho\frac{\partial\phi}{\partial t_j}) \qquad \Gamma.$$

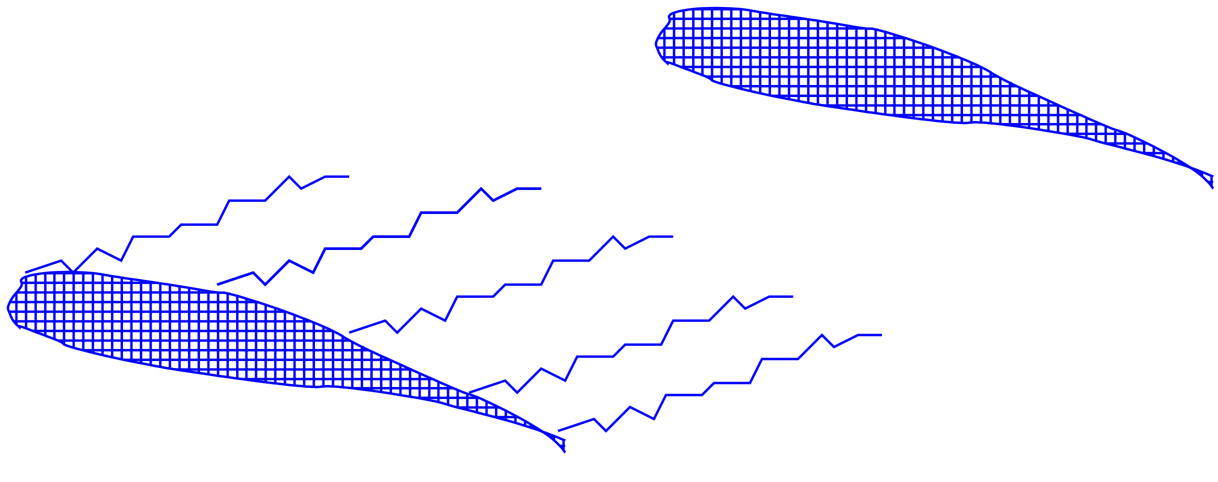
λ: adjoint variable.

Hessian symbol, (after some work)

$$\widehat{\mathcal{H}}(\mathbf{k}) = \rho_0^2 u_0^4 \frac{k_1^4}{|(1-M^2)k_1^2 + k_2^2|}$$

k_1 correspond to direction of flow k_2 spanwise direction

- H is not a differential operator
- The functional is almost flat in some directions (Inf-Dim)



Shape Design: Euler Equations

Euler equations

$$f_x + g_y + h_z = 0$$

$$f = AU \qquad g = BU \qquad h = CU,$$

$U = (\rho, \rho\mathbf{u}, E)$ and $\mathbf{u} = (u, v, w)$.

Characteristic BC for $\tilde{U}$:

Cost Functional

$$\min_{\Gamma} \frac{1}{2} \int_{\Gamma} (p - p^*)^2 ds \qquad p = p(U)$$

Gradient,

$$\nabla_{\Gamma} J = -\frac{(p - p^*)^2}{2R} + (p - p^*) \frac{\partial p}{\partial n} - div(p\lambda).$$

$\Lambda = (\lambda_1, \lambda, \lambda_5), \lambda = (\lambda_2, \lambda_3, \lambda_4)$ adjoint variable

Hessian symbol, (after more work)

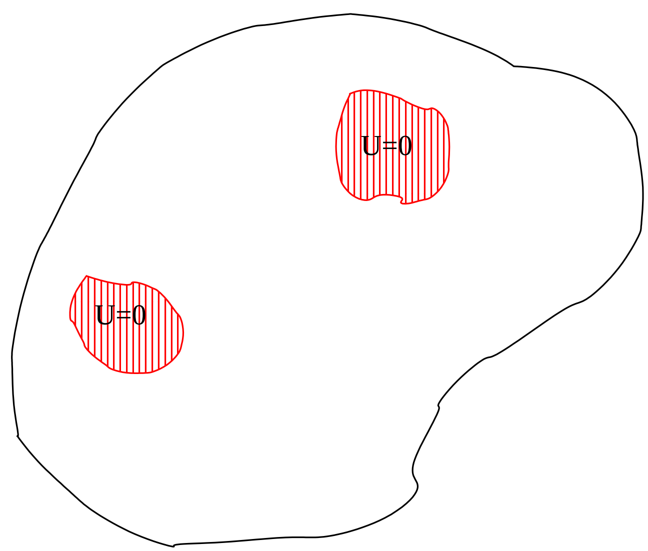
$$\widehat{\mathcal{H}}(\mathbf{k}) = \rho_0^2 u_0^4 \frac{k_1^4}{|(1 - M^2) k_1^2 + k_2^2|}$$

The Same as the Full Potential Case!

Inequality Constraints: Inf Dim Example

$$\min_{u \geq 0, u|_{\partial\Omega}=0} \int_{\Omega} (|\nabla u|^2 - 2fu)dx$$

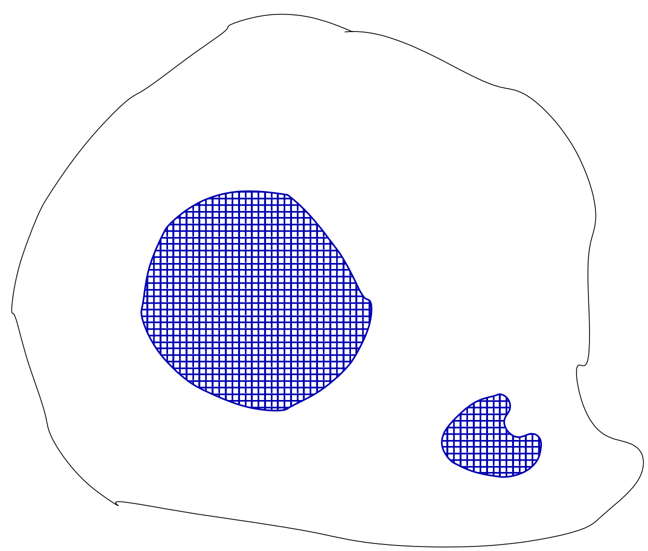
Active set: $I = \{x \in \Omega |\ u(x) = 0\}$, $\Gamma = \partial I$



Inactive Constraints $\Longrightarrow -\triangle u = f$.

Active Constraints $\Longrightarrow$ Shape design problem

$$\min_{(u,\Gamma)} \int_{\Omega - I} (|\nabla u|^2 - 2fu)dx$$

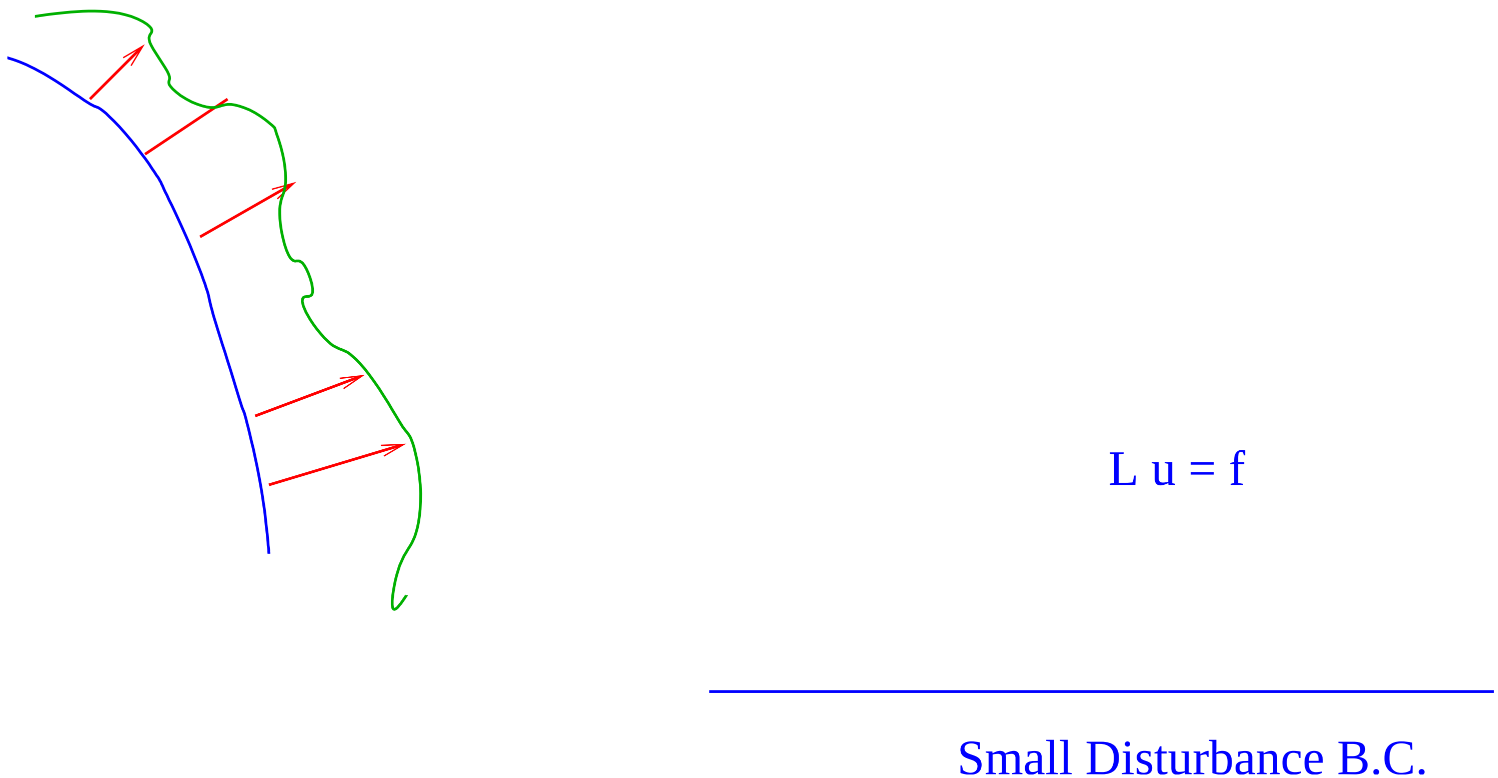


Active Constraints $\Longrightarrow$

Small Disturbance Analysis $\Longrightarrow$

Half Space Fourier $\Longrightarrow$

Hessian Symbol



$$\Gamma \to \Gamma + \epsilon\alpha\mathbf{n}$$

$$E(u,\Gamma) \to E(u,\Gamma) - \epsilon \int_\Gamma \alpha(\frac{\partial u}{\partial n})^2 ds$$

$$\nabla_\Gamma E = -(\frac{\partial u}{\partial n})^2 \qquad \Gamma$$

$$\nabla_u E = -\triangle u - f \qquad \Omega - I$$

$\hat{H}(\mathbf{k}) \sim |k| \qquad \Gamma$ dependence

$\hat{H}(\mathbf{k}) \sim |k|^2 \qquad u$ dependence

Fast Solvers: One Shot Methods

- Infinite Dimensional Preconditioners
- Multigrid Methods

Infinite Dimensional Preconditioners

Iteration Matrix In Fourier space

$$I - \delta \widehat{\mathcal{H}}^h(\mathbf{k})$$

Convergence Difficulties are related to

$$\max_{0 \leq |\mathbf{k}| \leq \pi/h} |\widehat{\mathcal{H}}^h(\mathbf{k})| = O(\frac{1}{h^\gamma})$$

Construct Preconditioners from

$$\widehat{\mathcal{H}}(\mathbf{k}) \qquad \text{large } \mathbf{k}$$

Let $\hat{\mathcal{R}}(\mathbf{k}) \approx \mathcal{H}^{-1}(\mathbf{k})$ for large $|\mathbf{k}|$

Optimization procedure is now

$$\alpha \leftarrow \alpha - \delta \mathcal{R} g$$

Its iteration matrix

$$I - \delta \mathcal{R} \mathcal{H}$$

In Fourier Space,

$$I - \delta \widehat{\mathcal{R}}(\mathbf{k}) \widehat{\mathcal{H}}(\mathbf{k})$$

$\delta = 1$ efficiently reduce high frequencies.

High Frequencies Behavior

- Localized (elliptic equations)

- Can be Analyzed by Half Space

- Inf-Dim Preconditioner is Accurate

Low Frequency Behavior

- Depends on the Whole Domain
- Cannot be Analyzed By Half Space
- Inf-Dim Preconditioner is not Accurate

Hence, $\mathcal{R}(\mathbf{k})$ should satisfy,

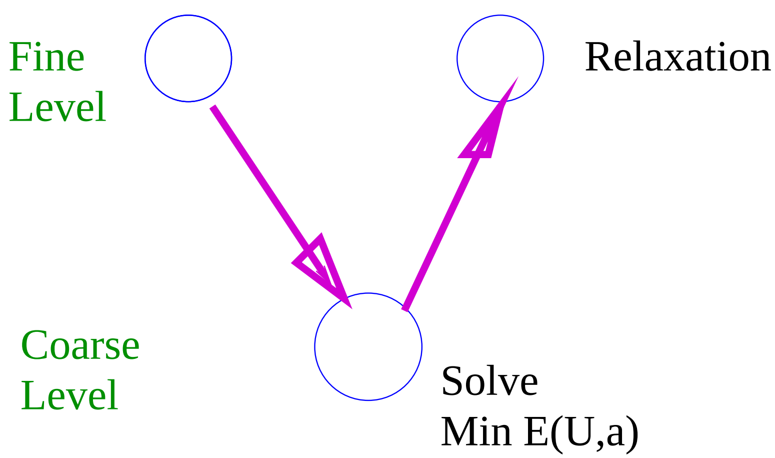
$$\hat{\mathcal{R}}(\mathbf{k}) \to 1 \qquad \text{for } |\mathbf{k}| \to 0.$$

Now For Multigrid

Multigrid for the Full Optimization Problem

Algorithm: Two Level

1. Relax (smooth) the state, adjoint and design variables
2. Accelerate convergence using a coarse grid optimization problem



Issues

Discretization

Coarsening

Smoother

Discretization Issues

`h-ellipticity:` $\widehat{\mathcal{H}}^h(\theta) \geq C \sum_j |sin(\theta_j/2)|^m$

`quasi-ellipticity:` $\widehat{\mathcal{H}}^h(\theta) \geq C \sum_j |sin(\theta_j)|^m$

Example I: A control problem

Cost functional

$$\min_{\alpha} \frac{1}{2} \int_{\partial\Omega} (\frac{\partial \phi}{\partial n} - d)^2 ds$$

State equation

$$\begin{aligned} \Delta\phi = 0 \quad & \Omega \\ \phi = \alpha \quad & \partial\Omega \end{aligned}$$

$$\Omega = \{(x,y) | y > 0\}.$$

Optimality conditions,

$$\begin{aligned} &\Delta\phi = 0 && \Omega \\ &\Delta\lambda = 0 && \Omega \\ &\phi = \alpha && \partial\Omega \\ &\lambda + \frac{\partial\phi}{\partial n} = d && \partial\Omega \\ &\frac{\partial\lambda}{\partial n} = 0 && \partial\Omega. \end{aligned}$$

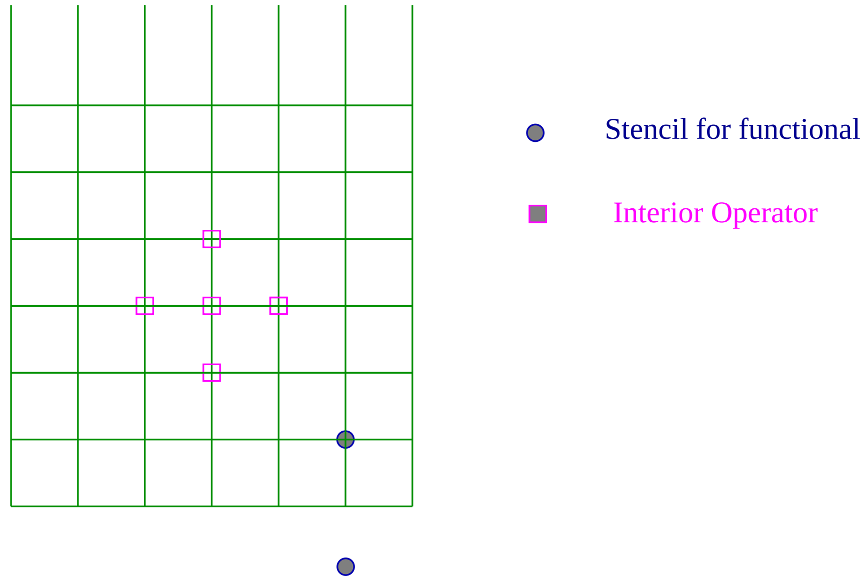
Symbol of Hessian

$$\widehat{\mathcal{H}}(\mathbf{k}) = |\mathbf{k}|^2.$$

Discretization I: h-elliptic Hessian.

vertex discretization with the 5-point Laplacian

$$L^h = \frac{1}{h^2}\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix}.$$



Normal derivative,

$$\frac{\partial \phi}{\partial n} \approx \frac{1}{2h}(\phi_{l,1} - \phi_{l,-1})$$

Fourier analysis

$$\phi_{l,j} = \exp(i\theta l) d(\theta)^j$$
$$\lambda_{l,j} = A \exp(i\theta l) d(\theta)^j$$

where $d(\theta) \leq 1$ is the solution of

$$d(\theta) + 1/d(\theta) - 4 + 2cos(\theta) = 0.$$

Define

$$\mu(\theta) = \frac{1}{2h}[1/d(\theta) - d(\theta)]$$

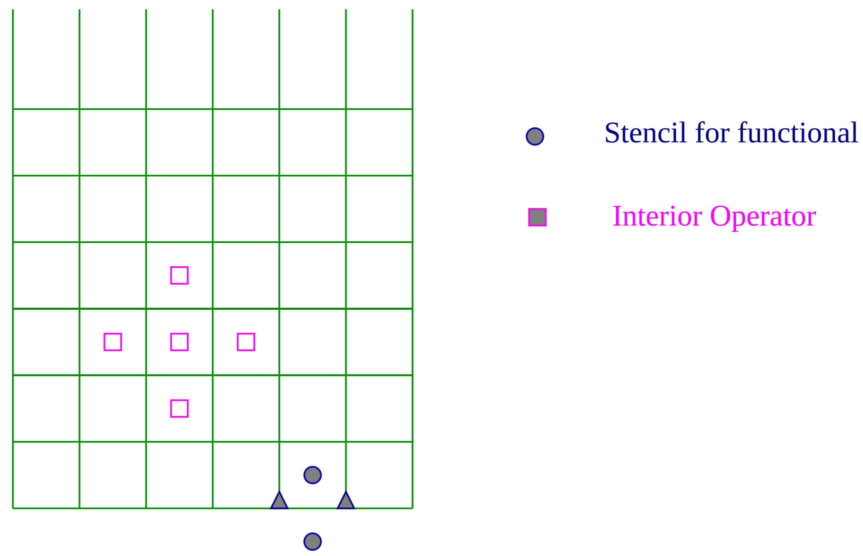
Then of discrete Hessian

$$\widehat{\mathcal{H}}^h(\theta) = \mu^2(\theta)$$

Discretization II: Quasi-elliptic Hessian.

cell-centered scheme with the 5-point Laplacian

$$L^h = \frac{1}{h^2}\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix}.$$



Boundary condition,

$$\frac{1}{2}(\phi_{l+\frac{1}{2},\frac{1}{2}} + \phi_{l+\frac{1}{2},-\frac{1}{2}}) = \frac{1}{2}(\alpha_l + \alpha_{l+1})$$

Normal derivative (for cost functional)

$$(\frac{\partial \phi}{\partial n})_{l+1/2,0} \approx \frac{1}{h}(\phi_{l+1/2,1/2} - \phi_{l+1/2,-1/2})$$

Define

$$\mu_c(\theta) = \frac{1}{h}(\frac{1}{\sqrt{d(\theta)}} - \sqrt{d(\theta)}).$$

And

$$C(\theta) = \frac{1}{\sqrt{d(\theta)}} + \sqrt{d(\theta)}$$

Fourier analysis,

$$\phi_{l,j} = \frac{\cos(\theta/2)}{C(\theta)} \exp(i\theta l) d(\theta)^j$$

$$\lambda_{i,j} = \mu_c(\theta) \frac{\cos(\theta/2)}{C(\theta)} \exp(i\theta l) d(\theta)^j.$$

Symbol of the Hessian

$$\widehat{\mathcal{H}}(\theta) = \mu_c^2(\theta)\frac{\cos(\theta/2)}{C(\theta)}.$$

$cos(\pi/2) = 0$: the symbol is not h-elliptic.

Example II:

Cost functional

$$\min_{\alpha} \frac{1}{2} \int_{\partial\Omega} (\phi - d)^2 ds$$

State equation,

$$\begin{array}{ll} \Delta\phi = 0 & \Omega \\ \frac{\partial\phi}{\partial n} = \alpha_x & \partial\Omega \end{array}$$

$\Omega = \{(x, y) | y > 0\}.$

Optimality conditions,

$$
\begin{aligned}
\Delta\phi = 0 & \quad \Omega \\
\Delta\lambda = 0 & \quad \Omega \\
\frac{\partial\phi}{\partial n} = \alpha_x & \quad \partial\Omega \\
\frac{\partial\lambda}{\partial n} = \phi - d & \quad \partial\Omega \\
-\lambda_x = 0 & \quad \partial\Omega
\end{aligned}
$$

Symbol of the Hessian

$$\widehat{\mathcal{H}}(\mathbf{k}) = 1.$$

Discretization I: h-elliptic Hessian.

Cell centered scheme with the usual 5-point Laplacian.

Normal derivative,

$$(\frac{\partial\alpha}{\partial x})_{l+1/2} \approx \frac{1}{h}(\alpha_{l+1} - \alpha_l)$$

Its symbol

$$s(\theta) = \frac{2i}{h}\sin(\theta/2).$$

Fourier analysis,

$$\phi_{l,k} = \frac{s(\theta)}{\mu_c(\theta)}\exp(i\theta l)d(\theta)^j$$
$$\lambda_{l,k} = \frac{s(\theta)}{\mu_c^2(\theta)}\exp(i\theta l)d(\theta)^j$$

Hessian's symbol

$$\widehat{\mathcal{H}}^h(\theta) = \frac{4\sin^2(\theta/2)}{\mu_c^2(\theta)}.$$

Discretization II: Quasi-elliptic Hessian.

Cell-Vertex using 5-point Laplacian

$$\widehat{\mathcal{H}}^h(\theta) = \frac{\sin^2(\theta)}{\mu^2(\theta)}$$

A quasi-elliptic symbol

Coarse Grid Optimization Problem

Consider,

$$\min_{\alpha^h} E^h(u^h, \alpha^h)$$
$$L^h(u^h, \alpha^h) = f^h.$$

Optimality conditions,

$$L^h(u^h, \alpha^h) = f^h$$
$$L_u^{h*}(u^h, \alpha^h)\lambda + E_u^h = 0$$
$$L_\alpha^h \ (u^h, \alpha^h)\lambda^h + E_\alpha^h = 0.$$

Wrong coarsening:

$$\min_{\alpha^H} E^H(u^H, \alpha^H)$$
$$L^H(u^H, \alpha^H) = f^H.$$

The optimality condition (coarse grid)

$$L^H(u^H, \alpha^H) = f^H$$

$$L_u^{H*}(u^H, \alpha^H)\lambda^H + E_u^H(u^H, \alpha^H) = 0$$

$$L_\alpha^H(u^H, \alpha^H)\lambda^H + E_\alpha^H(u^H, \alpha^H) = 0.$$

Does not keep the fine grid minimum unchanged. NO Convergence!

Correct Coarsening: Use adjoint variables.

FAS equations for the necessary conditions.

$$
\begin{gathered}
L^H(u^H, \alpha^H) = f^H \\
L_u^{H*}(u^H, \alpha^H)\lambda^H + E_u^H(u^H, \alpha^H) = g_1^H \\
L_\alpha^H(u^H, \alpha^H)\lambda^H + E_\alpha^H(u^H, \alpha^H) = g_2^H
\end{gathered}
$$

Proper coarse grid optimization problem

$$
\text{min}_{\alpha^H} E^H(u^H, \alpha^H) - < g_1^H, u^H > - < g_2^H, \alpha^H >
$$

$$
L^H(u^H, \alpha^H) = f^H.
$$

FAS transfers for g_1^H, g_2^H.

Relaxations for Optimization Problems

Three Step Relaxation

1. Relax the state variable
2. Relax the adjoint variable
3. Relax the design variables

- **Design Space of a Small Dimension**

- **Design Space of Moderate Size**

- **Infinite Dimensional Design Space**

Design Space of a Small Dimension

Design variable,

$$\alpha(x) = \sum \alpha_j f_j(x),$$

```
RELAXATION: Level l

    (1) Relax State Equation
    (2) Relax Adjoint Equation
    (3) If level = 1 Use BFGS
```

Design Space of Moderate Size

Redefine Design variables

$$\alpha(x) = \sum_{m=1}^{m_l} \sum_{l=1}^{n_d} \beta_{m,l} g_{m,l}(x)$$

Relaxation: Level l

(1) Relax State Equation

(2) Relax Adjoint Equation

(3) Update design variables $\beta_{m,l}, m = 1, \ldots, m_l$

Infinite Dimensional Design Space: Smoothers

Convergence rate, (Fourier analysis)

$$1 - \delta\widehat{\mathcal{H}}^h(\theta)$$

Good Smoothing for

$$\delta = 2./[\widehat{\mathcal{H}}^h(\pi/2) + \widehat{\mathcal{H}}^h(\pi)]$$

Above relaxation may update smooth components of design variables.

In a previous example,

$$\widehat{\mathcal{H}}^h(\theta) = \frac{\sin^2(\theta/2)}{\mu_c^2(\theta)}$$

and

$$0 < c_1 \leq \widehat{\mathcal{H}}^h(\theta) \leq c_2 \qquad\qquad |\theta| \leq \pi$$

Can choose

$$\delta = 2/(c_1 + c_2)$$

To have good convergence for all components.

The Problem Need to update state and adjoint also by smooth components.

EXPENSIVE !!!

Efficient Multigrid: Update ONLY high frequencies in design variables

Preconditioners for Multigrid Relaxation.

Need to update ONLY high frequencies in the design variables.

Use a preconditioner $\mathcal{R}^h$ with

$$\widehat{\mathcal{R}}^h(\theta)\widehat{\mathcal{H}}^h(\theta) \geq C \sum_j \sin^\gamma(\theta_j/2)$$

with $\gamma > 0$.

Do construction on differential level,

$$\widehat{\mathcal{R}}(\mathbf{k})\widehat{\mathcal{H}}(\mathbf{k}) \geq |\mathbf{k}|^\gamma$$

with $\gamma > 0$.

Smoother

$$\alpha^h \leftarrow \alpha^h - \delta \mathcal{R}^h g$$

Fourier analysis of iteration matrix

$$1 - \delta \widehat{\mathcal{R}}^h(\theta) \widehat{\mathcal{H}}^h(\theta).$$

Step size (for good smoothing)

$$\delta = 2/[\mathcal{R}^h(\pi/2)\widehat{\mathcal{H}}^h(\pi/2) + \mathcal{R}^h(\pi)\widehat{\mathcal{H}}^h(\pi)]$$

Example: For a previous case,

$$\mathcal{R}^h(\theta) = \frac{h^2}{4\sin^2(\theta/2)},$$

$\mathcal{R}^h$ is the inverse of T^h given by

$$(T^h g)_k = \frac{1}{h^2}(g_{k+1} - 2g_k + g_{k-1})$$