FIRST AND SECOND ORDER SHAPE SENSITIVITY AND LEVEL SET METHODS FOR OPTIMAL CONTROL OF PDEs AND IMAGE SEGMENTATION

Michael Hintermüller
joint work with Wolfgang Ring

SRC Optimization and Control & Institute of Mathematics
University of Graz, Austria

Institut für Mathematik
Karl-Franzens-Universität Graz

OUTLINE

- **State constrained optimal control of PDEs.**

  → First order shape sensitivity, LSM combined with techniques from numerical optimization.

- **Image segmentation via active contours and edge detectors.**

  → in addition to the above: shape Hessian, shape Newton method.

- **Image segmentation via Mumford-Shah functional.**

  → in addition to the above: inexact (shape) Newton combined with preconditioned CG for Newton equation.

OUTLINE

- **State constrained optimal control of PDEs.**
- Image segmentation via active contours and edge detectors.
- Image segmentation via Mumford-Shah functional.

- Control constrained optimal control problem

$$\min J(y,u) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)}$$

s.t.

$$\begin{cases} \Delta y + u = 0 \text{ in } \Omega \\ u \le b \text{ a.e. in } \Omega \end{cases}$$

and $(y,u) \in H^1_0(\Omega) \times L^2(\Omega)$,

$\Rightarrow$ Discretized optimality system corresponds to discretized control problem.

$\Rightarrow$ Standard optimization algorithms are mesh independent.

- State constrained optimal control problem

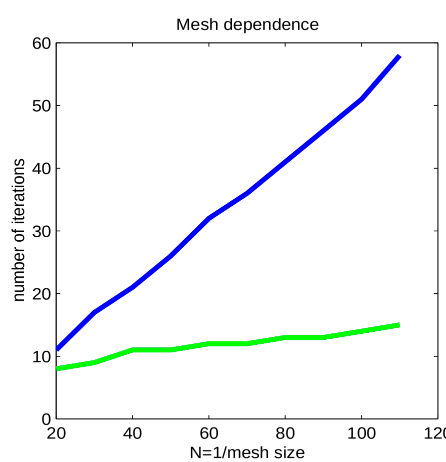
$$\min J(y,u) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)}$$

s.t.

$$\begin{cases} \Delta y + u = 0 \text{ in } \Omega \\ y \leq \psi \text{ a.e. in } \Omega \end{cases}$$

and $(y,u) \in H_0^1(\Omega) \times L^2(\Omega)$,

$\Rightarrow$ Standard optimization algorithms depend significantly on the mesh size.



* What causes problems?
* How to modify the algorithms?

Model problem.

Determine the (unique) solution (y^*, u^*) of the state constrained optimal control problem

$$\min J(y,u) = \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)}$$

s.t.

$$\begin{cases} \Delta y + u = 0 \text{ in } \Omega \\ y \leq \psi \text{ a.e. in } \Omega \end{cases}$$

and $(y,u) \in H^1_0(\Omega) \times L^2(\Omega)$,

where ψ is sufficiently smooth and $\alpha > 0$.

Optimality system known from the literature [e.g: Bergounioux, Casas, Kunisch, Tröltzsch]:

$$\Delta y^* + u^* = 0 \text{ in } \Omega,$$

$$y^* \leq \psi \text{ in } \Omega$$

$$-\alpha(u^*, \Delta y)_\Omega + \langle \lambda^*, y \rangle_{\mathcal{M},\mathcal{C}_0} = (y_d - y^*, y)_\Omega \; \forall y \in W$$

$$\langle \lambda^*, z - y^* \rangle_{\mathcal{M},\mathcal{C}_0} \leq 0 \; \forall z \in \mathcal{C}_0(\Omega), \; z \leq \psi,$$

where $W = H_0^1(\Omega) \cap H^2(\Omega)$ and $\mathcal{M}$ is the set of (regular) Borel measures.

Active and inactive sets.

$$\mathcal{A}^* = \{\mathbf{x} \in \Omega \,:\, y^*(\mathbf{x}) = \psi(\mathbf{x})\}, \quad \mathcal{I}^* = \Omega \setminus \mathcal{A}^*.$$

$$\Gamma^* = \partial \mathcal{A}^*.$$

$$\Sigma = \partial\Omega, \quad \Sigma \cap \Gamma^* = \emptyset.$$

Closer investigation of λ^*.

Theorem: Properties of the Lagrange multiplier λ^* are:

- λ^* is concentrated on $\mathcal{A}^*$, i.e. $\lambda^*|_{\mathcal{I}^*} = 0$.

- $\lambda^* = \lambda_s^* + \lambda_a^*$, with λ_s^* concentrated on Γ^*.

- $$\lambda_a^* = y_d - \psi - \alpha\Delta^2\psi \in L^2(\text{int}(\mathcal{A}^*))\,,$$
  $$\lambda_s^* = -\alpha\frac{\partial}{\partial n_{\mathcal{I}^*}}(u^* + \Delta\psi)|_{\Gamma^*} \in H^{-3/2}(\Gamma^*).$$

- $\lambda_a^* \geq 0$ a.e. in $\mathcal{A}^*$, $\lambda_s^* \geq 0$ as a measure on Γ^*.

$\Rightarrow$ Properties of λ^* on Γ^* must be taken care of!

Necessary and sufficient conditions

Proposition (Necessary conditions). Let $(y^*, u^*) \in H_0^1(\Omega) \times L^2(\Omega)$ be solution of (1), and let $\mathcal{A}^*$, $\mathcal{I}^*$ be sufficiently regular. Then $\Delta u^*|_{\mathcal{I}^*} \in L^2(\mathcal{I}^*)$, $y^* \in H^2(\Omega)$ and

$$\Delta y^* + u^* = 0 \text{ in } \mathcal{I}^* \tag{2a}$$

$$y^* - \alpha \Delta u^* = y_d \text{ in } \mathcal{I}^* \tag{2b}$$

$$y^* < \psi \text{ in } \mathcal{I}^* \tag{2c}$$

Boundary and interface conditions.

$$y^*|_\Sigma = 0, \quad u^*|_\Sigma = 0,$$

$$y^*|_{\Gamma^*} = \psi|_{\Gamma^*}, \quad u^*|_{\Gamma^*} = -\Delta\psi|_{\Gamma^*},$$

$$\frac{\partial y^*}{\partial n}\Big|_{\Gamma^*} = \frac{\partial \psi}{\partial n}|_{\Gamma^*} \tag{3a}$$

$$-\frac{\partial}{\partial n}(u^* + \Delta\psi)|_{\Gamma^*} \geq 0 \text{ as a measure on } \Gamma^* \tag{3b}$$

$$y_d - \psi - \alpha\Delta^2\psi \geq 0 \text{ a.e. in } \mathcal{A}^*. \tag{3c}$$

Conditions are also sufficient.

Change of paradigms

Philosophy of most iterative procedures for solving (1)

- So far:

> actual iterates (y_n, u_n) induce estimates $\mathcal{A}_n$, $\mathcal{I}_n$ and Γ_n.

- Our approach:

> actual interface estimate Γ_n induces (y_n, u_n) ($\Rightarrow$ Shape Optimization).

Level set methods

Level set methods $\Rightarrow$ Osher (UCLA), Sethian (UC Berkley)

Important properties:

- Very well suited for moving interface and free boundary problems.
- Interface as zero-level set of a signed distance function Φ defined.
- Robust and efficient (narrow band approach, fast marching,...), many applications.
- Core part: Level set equation

$$\Phi_t + F|\nabla\Phi| = 0 \quad \text{in } \Omega.$$

F denotes a speed function in Ω.

Approach: Define Γ^* as the zero-level set of Φ, with $\Phi < 0$ in int($\mathcal{A}^*$) and $\Phi > 0$ in $\mathcal{I}^*$.

Relaxed system and penalization

Assume that Γ_n is an estimate of Γ^*.

- Let $(\mathrm{O}_r(\Gamma_n))$ denote the following system induced by Γ_n:

$$\begin{aligned}
&\Delta y + u = 0 \text{ in } \mathcal{I}_n \\
&y - \alpha \Delta u = y_d \text{ in } \mathcal{I}_n \\
&y|_\Sigma = 0, \quad u|_\Sigma = 0, \\
&y|_{\Gamma_n} = \psi|_{\Gamma_n}, \quad u|_{\Gamma_n} = -\Delta\psi|_{\Gamma_n}.
\end{aligned}$$

- Violations of (2c), (3a), (3b) at Γ_n are taken care of by a penalty functional $K(\Gamma_n)$.

- (3c) is satisfied by the initial choice (feasibility).

Penalty functional

- The penalty functional is given by $K(\Gamma) = K(\Gamma, u(\Gamma), y(\Gamma))$:

$$K(\Gamma) = \frac{1}{2|\Gamma|} \int_\Gamma \left[\left| \frac{\partial}{\partial n}(y - \psi) \right|^2 + c_1 \max\left(0, \frac{\partial}{\partial n}(u + \Delta\psi)\right)^2 \right] d\Gamma + \frac{c_2}{2} \int_{\mathcal{I}} \big(\max(0, y - \psi) \big)^2 \, d\mathbf{x},$$

with penalty parameters $c_1, c_2 > 0$.

Level set based algorithm.

0. Choose a *feasible* Γ_0; set $n = 0$.

1. Compute (y_n, u_n) as solution to the relaxed system $(\mathrm{O}_r(\Gamma_n))$.

2. Evaluate $K(\Gamma_n)$, and compute its gradient $\nabla_\Gamma K(\Gamma_n)$. If $\|\nabla_\Gamma K(\Gamma_n)\| = 0$ then STOP; otherwise go to 3.

3. Use an extension of $-\nabla_\Gamma K(\Gamma_n)$ as speed function in the level set equation. Update the level set function.

4. Put Γ_{n+1} equal to the zero-level set of the updated level set function. Further $n := n + 1$, and return to 1.

From the Eulerian derivative of K to $\nabla_\Gamma K$

- Eulerian derivative of $K(\Gamma)$ in direction of a smooth vector field V:

$$dK(\Gamma;V) = \lim_{t\downarrow 0}\frac{1}{t}(K(\Gamma_t) - K(\Gamma)), \tag{4}$$

where $\Gamma_t = T_t(\Gamma;V)$ with $T_t : \Omega \to \Omega$.

By Sokolowski, Zolesio under suitable assumptions there exists a distribution G, such that (in an appropriate sense)

$$dK(\Gamma;V) = \langle G, v_n\rangle_\Gamma, \quad v_n = V(0,\mathbf{x})\cdot\mathbf{n}(\mathbf{x}).$$

-G serves as speed function on Γ.

$\Rightarrow$Catalogue of Eulerian derivs.

Integrands do not depend on Ω or Γ.

- $J(\Omega) = \int_\Omega \phi \, d\mathbf{x}$

$$\boxed{dJ(\Omega; V) = \int_\Gamma \phi \langle V, \mathbf{n} \rangle dS.}$$

- $J(\Gamma) = \int_\Gamma \phi \, dS$

$$\boxed{dJ(\Gamma; V) = \int_\Gamma \Big(\frac{\partial \phi}{\partial n} + \phi\kappa\Big) \langle V, \mathbf{n} \rangle dS.}$$

κ denotes the mean curvature of Γ.

Integrands do depend on Ω or Γ.

- $J(\Omega) = \int_\Omega \phi(\Omega)\, d\mathbf{x}$

$$\boxed{dJ(\Omega; V) = \int_\Omega \phi'(\Omega; V) d\mathbf{x} + \int_\Gamma \phi \langle V, \mathbf{n} \rangle dS.}$$

ϕ' denotes the shape derivative of ϕ.

- $J(\Gamma) = \int_\Gamma \psi(\Gamma)\, dS$

$$\boxed{dJ(\Gamma; V) = \int_\Gamma \psi'(\Gamma; V) dS + \int_\Gamma \kappa \psi(\Gamma) \langle V, \mathbf{n} \rangle dS.}$$

If $\psi(\Gamma) = \phi(\Omega)|_\Gamma$. Then

$$\boxed{dJ(\Gamma; V) = \int_\Gamma \phi'(\Omega; V)|_\Gamma dS + \int_\Gamma (\frac{\partial \phi}{\partial n} + \kappa \phi)|_\Gamma \langle V, \mathbf{n} \rangle dS.}$$

The Eulerian derivative of $K(\Gamma)$.

$$
\begin{aligned}
dK(V;\Gamma) &= \frac{1}{|\Gamma|}\int_\Gamma \left(\frac{\partial}{\partial n}(y-\psi)\,\frac{\partial y'}{\partial n} + c_1\, m'\Big(\frac{\partial}{\partial n}(u+\Delta\psi)\Big)\,\frac{\partial u'}{\partial n}\right)\, d\Gamma \\
&+ \frac{c_1}{|\Gamma|}\int_\Gamma m'\Big(\frac{\partial}{\partial n}(u+\Delta\psi)\Big)\left(\frac{1}{\alpha}(y-y_d)+\Delta^2\psi\right) v_n\, d\Gamma \\
&- \frac{1}{|\Gamma|}\int_\Gamma \kappa\, \left(\frac{1}{2}\Big|\frac{\partial}{\partial n}(y-\psi)\Big|^2 + c_1 m\Big(\frac{\partial}{\partial n}(u+\Delta\psi)\Big)\right)\, v_n\, d\Gamma \\
&- \frac{1}{|\Gamma|^2}\int_\Gamma \kappa\, v_n\, d\Gamma\, \left(\int_\Gamma \left(\frac{1}{2}\Big|\frac{\partial}{\partial n}(y-\psi)\Big|^2 + c_1 m\Big(\frac{\partial}{\partial n}(u+\Delta\psi)\Big)\right)\, d\Gamma\right) \\
&+ c_2 \int_{\mathcal{I}} m'(y-\psi)\, y'\, d\mathbf{x}
\end{aligned}
$$

with $m(x) = \frac{1}{2}\max\{0,x\}^2$. Above $|\Gamma|$ denotes the length of Γ.

Adjoint equation

By means of the adjoint equation

$$\Delta\mu + \nu = c_2 m'(y-\psi) \text{ in } \mathcal{I},$$
$$\mu - \alpha\Delta\nu = 0 \text{ in } \mathcal{I}$$

with boundary conditions

$$\mu|_\Sigma = 0, \quad \nu|_\Sigma = 0,$$
$$\mu|_\Gamma = \frac{1}{|\Gamma|}\frac{\partial}{\partial n}(y-\psi)\Big|_\Gamma,$$
$$\nu|_\Gamma = -\frac{c_1}{\alpha|\Gamma|} m'\Big(\frac{\partial}{\partial n}(u+\Delta\psi)\Big)\Big|_\Gamma$$

the following characterization is obtained:

Speed function on Γ.

Theorem. The *gradient* of K with respect to Γ can be identified with

$$\nabla_\Gamma K = \Bigg(\alpha \frac{\partial \nu}{\partial n} \frac{\partial}{\partial n}(u + \Delta\psi) - \frac{\partial \mu}{\partial n} \frac{\partial}{\partial n}(y - \psi)$$

$$+ \frac{c_1}{|\Gamma|} m' \Big(\frac{\partial}{\partial n}(u + \Delta\psi) \Big) \Big(\frac{1}{\alpha}(y - y_d) + \Delta^2 \psi \Big)$$

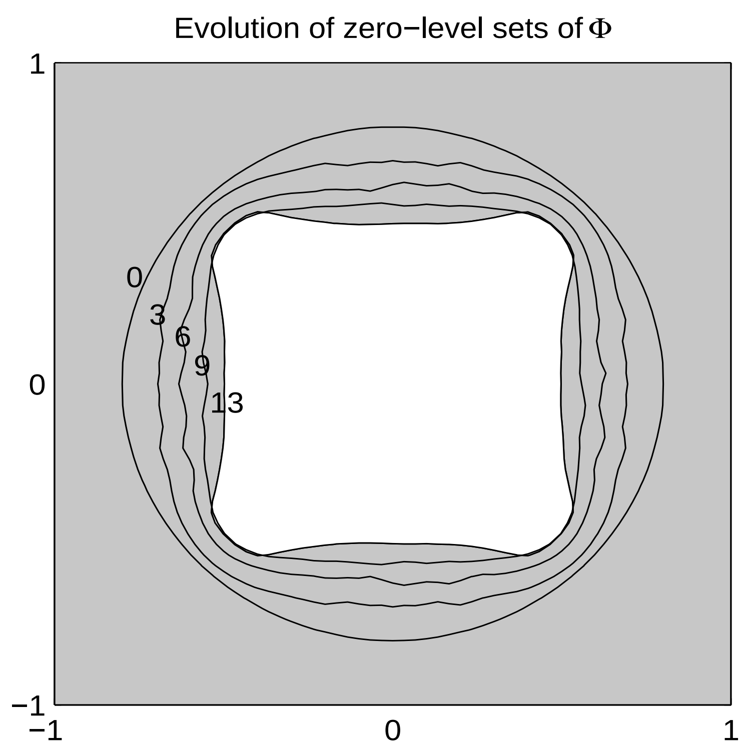
$$- \frac{1}{|\Gamma|} \kappa \left(\frac{1}{2} \Big| \frac{\partial}{\partial n}(y - \psi) \Big|^2 + c_1 m \Big(\frac{\partial}{\partial n}(u + \Delta\psi) \Big) \right)$$

$$- \frac{\kappa}{|\Gamma|^2} \int_\Gamma \left(\frac{1}{2} \Big| \frac{\partial}{\partial n}(y - \psi) \Big|^2 + c_1 m \Big(\frac{\partial}{\partial n}(u + \Delta\psi) \Big) \right) d\Gamma \Bigg) n$$

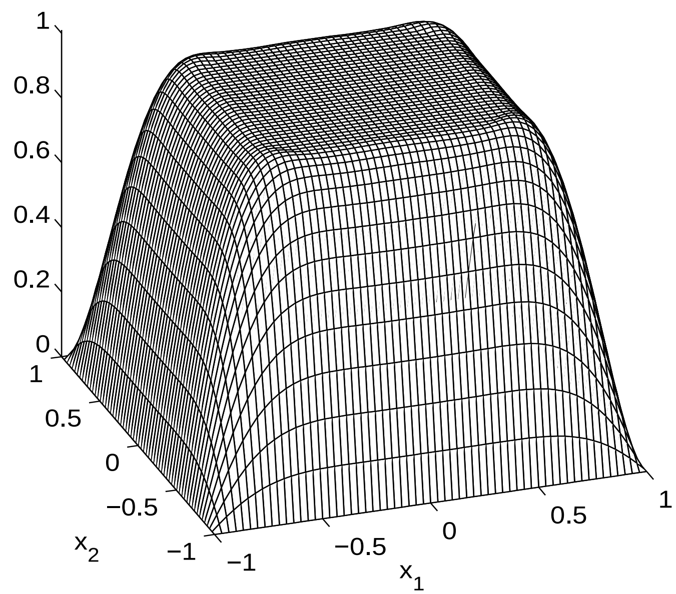
where (μ, ν) denotes the solution of the adjoint boundary value problem, κ is the (mean) curvature of Γ and $v_n = V \cdot \mathbf{n}$ on Γ.

Numerical results

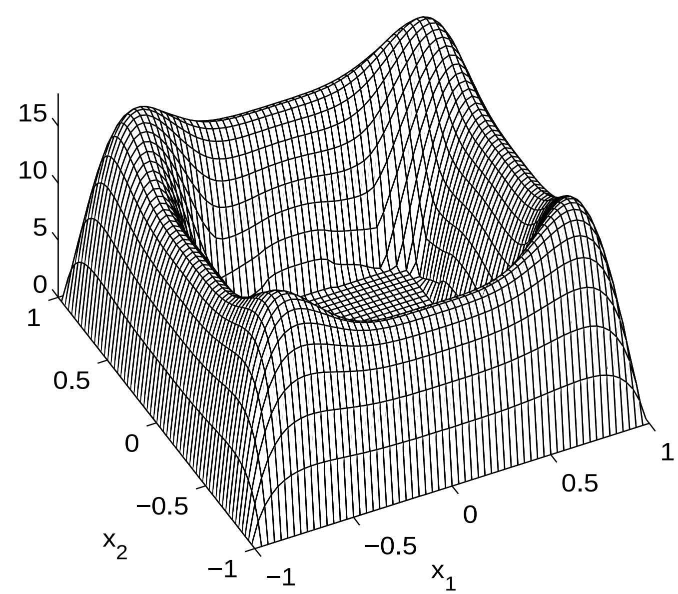
- Example 1: $y_d \equiv 1.2$, $\Psi \equiv 1$.



State y upon termination

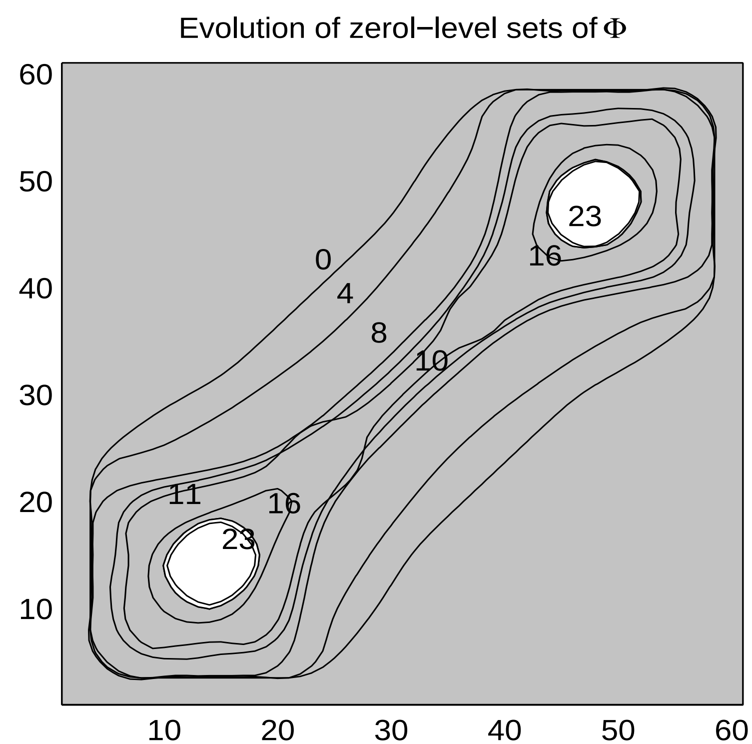


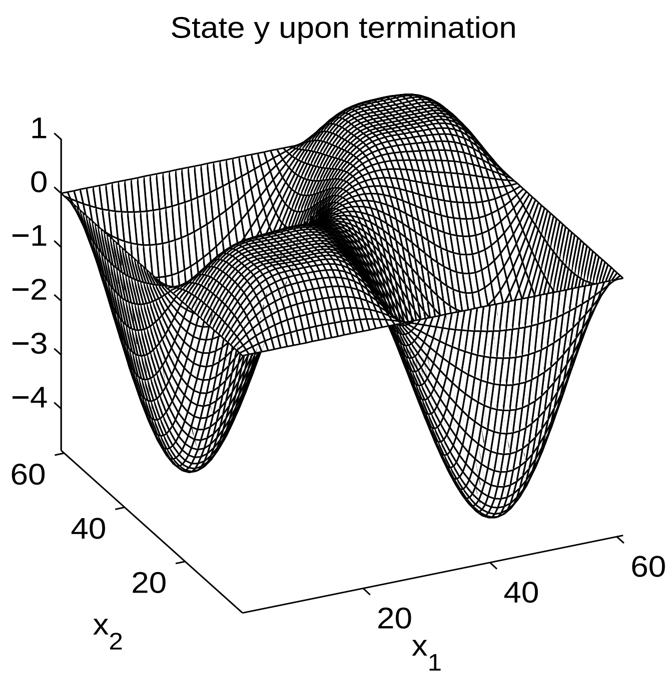
Control u upon termination

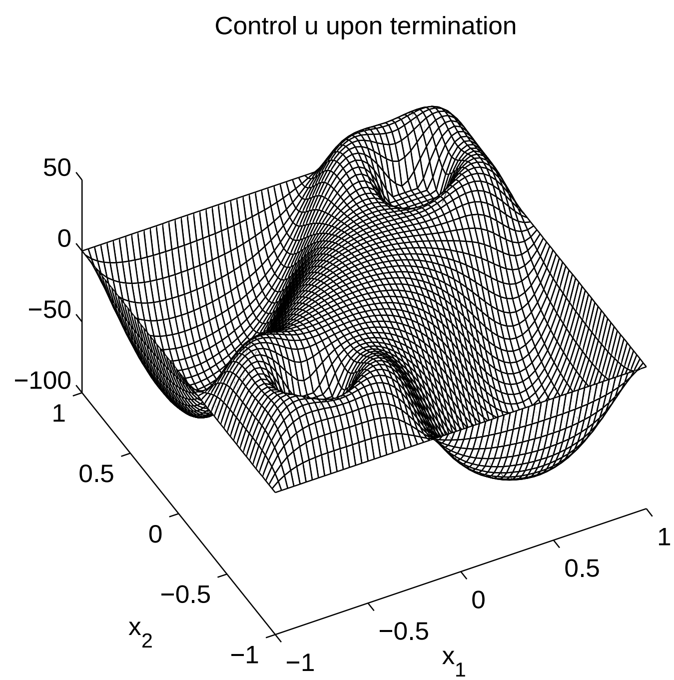


14 iterations on a 60×60 grid.

- Example 2: Topology change!

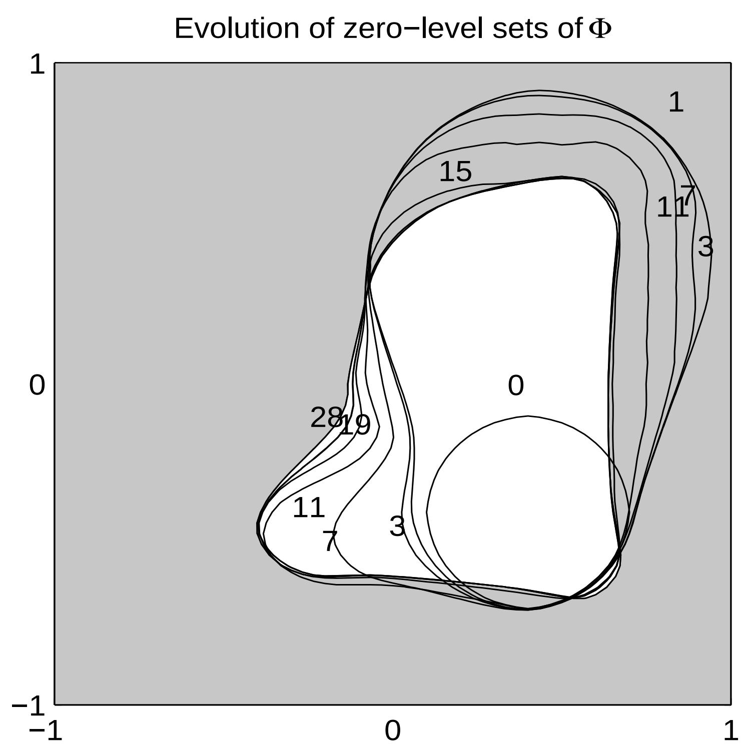


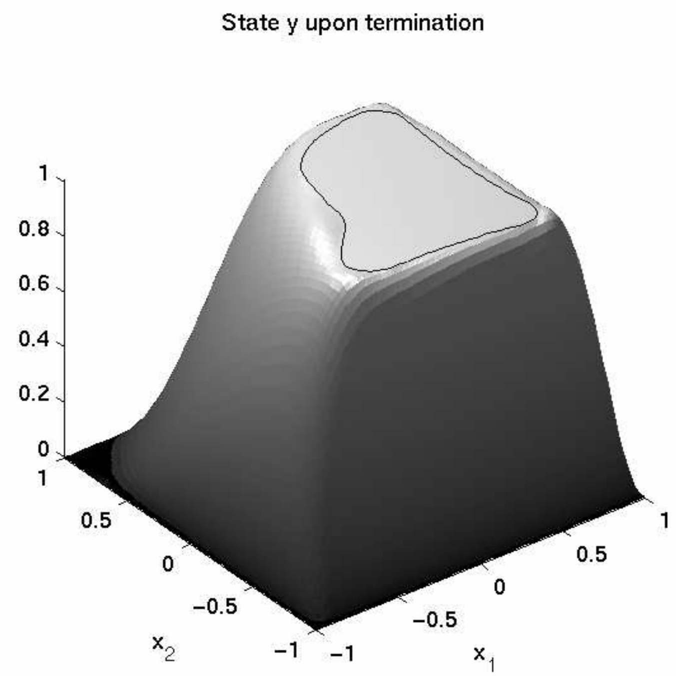


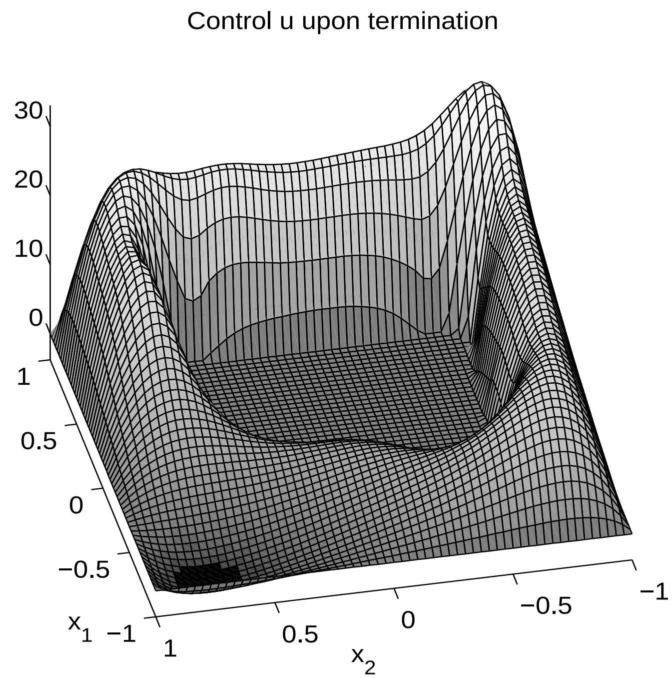


23 iterations on a 60×60 grid.

- Example 3: Large steps possible!







28 iterations on a 60×60 grid.

OUTLINE

- State constrained optimal control of PDEs.

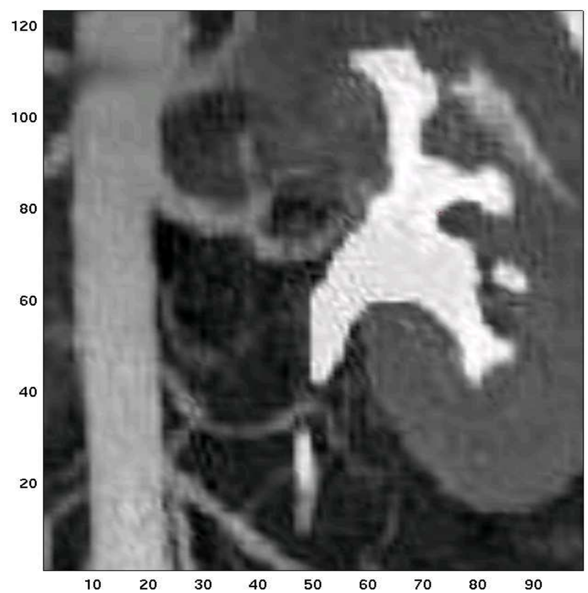
- **Image segmentation via active contours and edge detectors.**

- Image segmentation via Mumford-Shah functional.

Fundamental task of image segmentation.

⇒ Given a (possibly noise corrupted gray scale) image

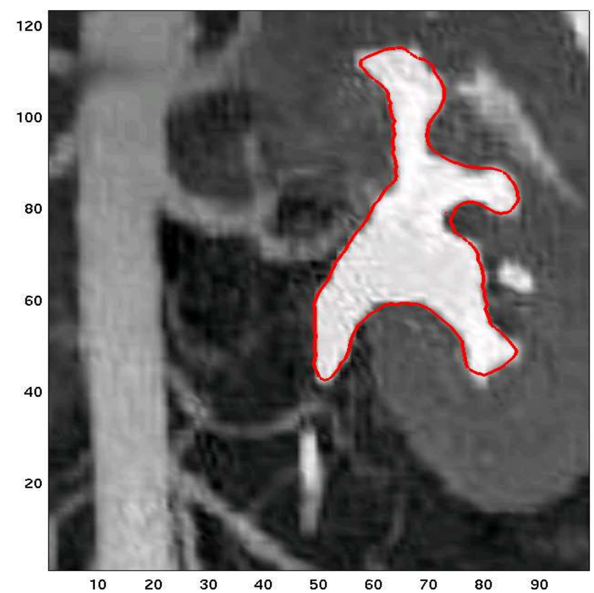
⇒ find boundary curves of regions of approximately constant gray levels (contours).



Fundamental task of image segmentation.

⇒ Given a (possibly noise corrupted gray scale) image

⇒ find boundary curves of regions of approximately constant gray levels (contours).



Some existing approaches.

⇒ Global energy principles satisfied by the optimal contour.

⇒ Snakes as deformable models based on energy minimization along the curve. Disadvantage: Depends on parametrization - non geometric (non intrinsic) model.

⇒ Geodesic active contours combining a geometric model with the energy minimization approach. Parametrization by Euclidean arclength of the curve. Curve evolution according to

$$\frac{d\mathcal{C}}{dt} = (g(\mathcal{C})\kappa - \langle \nabla g, \mathcal{N} \rangle)\mathcal{N}.$$

⇒ Deformable active contours with the contour as zero-level set of a time dependent function u in terms of a geometrically intrinsic formulation. E.g u as a signed distance function. Propagation according to

$$u_t + F|\nabla u| = 0.$$

Typical choices for F:

- $$F = g\left(\text{div}\left(\frac{\nabla u}{|\nabla u|}\right) + \nu\right),$$

- The choice

$$F = \text{div}\left(g\frac{\nabla u}{|\nabla u|}\right) = g\,\text{div}\left(\frac{\nabla u}{|\nabla u|}\right) + \frac{1}{|\nabla u|}\langle \nabla g, \nabla u\rangle,$$

can be interpreted as the gradient direction for the cost functional

$$J(\Gamma) = \int_\Gamma g\,dS \qquad (\Gamma \ldots \text{contour}).$$

g denotes an edge detector which is ideally zero at edges and positive elsewhere.

New approach.

⇒ Use shape sensitivity analysis in image segmentation.

⇒ Computed Newton-type flow F.

⇒ Use level set methodology combined with techniques from numerical optimization.

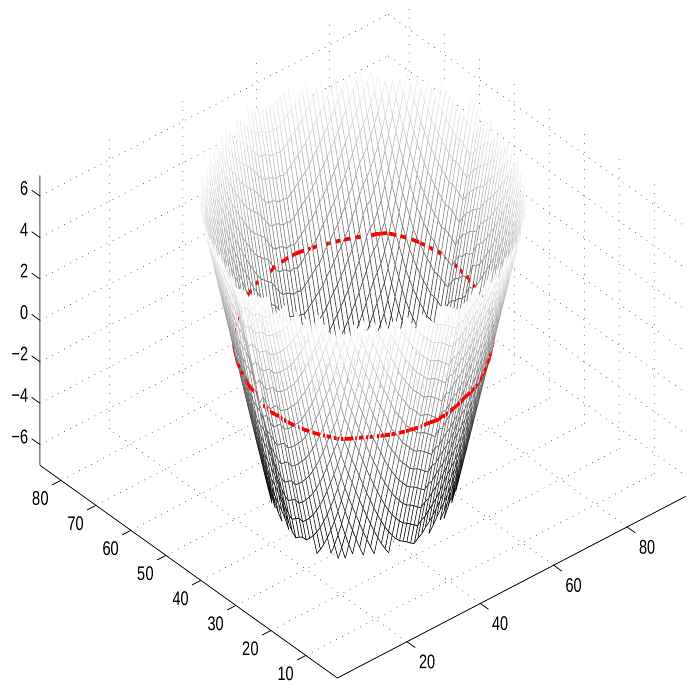
⇒ Benefits:

- Evolving contours are smoother (more regular).
- Global behaviour.
- Fast algorithm.
- Less parameter dependent.

Signed distance function

The signed distance function b_Ω of a bounded open set $\Omega \subset \mathbb{R}^2$ is defined as

$$b_\Gamma(\mathbf{x}) = d_\Omega(\mathbf{x}) - d_{\mathbb{R}^2 \setminus \Omega}(\mathbf{x}).$$



Properties:

- $|\nabla b_\Gamma|^2 = 1$ a.e. on $\mathbb{R}^2$ if meas(Γ)=0.
- $\nabla b_\Gamma\big|_\Gamma = \mathbf{n}$.
- $\Delta b_\Gamma\big|_\Gamma = \kappa$.
- $b'_\Gamma|_\Gamma = -v_n$ with $v_n = \langle V, \mathbf{n}\rangle|_\Gamma$.
- $\Delta b'_\Gamma|_\Gamma = -\Delta_\Gamma v_n$ with $\Delta_\Gamma w = \mathrm{div}_\Gamma(\nabla_\Gamma w)$ the Laplace-Beltrami operator.

Gradient and Newton-type level set flow.

$\Rightarrow$ The level set idea and descent flow. Suppose $F : \Gamma \to \mathbb{R}$ is a descent direction, e.g. the negative shape gradient (Hadamard-Zolesio).

Propagating front formulation for $\Gamma(t)$:

$$\dot{\mathbf{x}}(t) = F\big(\mathbf{x}(t), \Gamma(t)\big)\,\mathbf{n}(\mathbf{x}(t)) \text{ for } \mathbf{x}(t) \in \Gamma(t).$$

An equivalent formulation is given by the level set equation

$$u_t + \tilde{F}|\nabla u| = 0 \text{ on } \mathbb{R}^2 \times (0, T)$$

where the propagating front is the zero level set of the function u, i.e.

$$\Gamma(t) = \{\mathbf{x} \in \mathbb{R}^2 : u(\mathbf{x}, t) = 0\}.$$

The scalar function $\tilde{F} : \mathbb{R}^2 \times [0, T) \to \mathbb{R}$ is chosen such that $\tilde{F}\big|_{\Gamma(t)} = F(\Gamma(t))$.

$\Rightarrow$ Extension velocity. There is some freedom in extending $F : \Gamma \to \mathbf{R}$ to $\tilde{F} : \mathbb{R}^2 \times [0, T) \to \mathbb{R}$. In our context it turns out that constructing $\tilde{F}$ as solution to the transport equation

$$\langle \nabla \tilde{F}, \nabla b_\Gamma \rangle = 0 \text{ on } \mathbb{R}^2; \quad \tilde{F}\big|_\Gamma = F$$

is most appropriate.

Define

$$V_F = \tilde{F}\, \nabla b_\Gamma.$$

Then $\langle V_F, \mathbf{n} \rangle = F$.

$\Rightarrow$ Newton-type speed function. The Newton-type speed function is the solution $F : \Gamma \to \mathbb{R}$ to

$$d^2 J(\Gamma; V_F; V_G) = -dJ(\Gamma; V_G) \text{ for all } G : \Gamma \to \mathbb{R}.$$

$\Rightarrow$ shape Hessian. Utilizing the speed method and the particular choice of extension velocities the second Eulerian derivatives of the functionals

$$J_1(\Gamma) = \int_\Gamma \phi \, dS$$

and

$$J_2(\Omega) = \int_\Omega \phi \, d\mathbf{x}$$

are given by

$$d^2 J_1(\Gamma; V_F; V_G) = \int_\Gamma \Big[\Big(\frac{\partial^2 \phi}{\partial n^2} + 2 \frac{\partial \phi}{\partial n} \, \kappa \Big) \, F \, G \, + \phi \langle \nabla_\Gamma F, \nabla_\Gamma G \rangle \Big] \, dS.$$

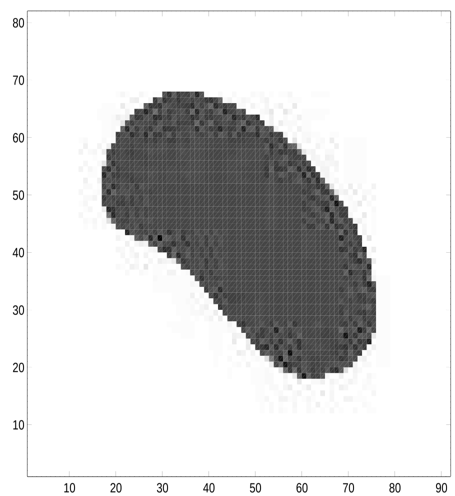
and

$$d^2 J_2(\Omega; V_F; V_G) = \int_\Gamma \Big(\frac{\partial \phi}{\partial n} + \kappa \phi \Big) \, F \, G \, dS.$$

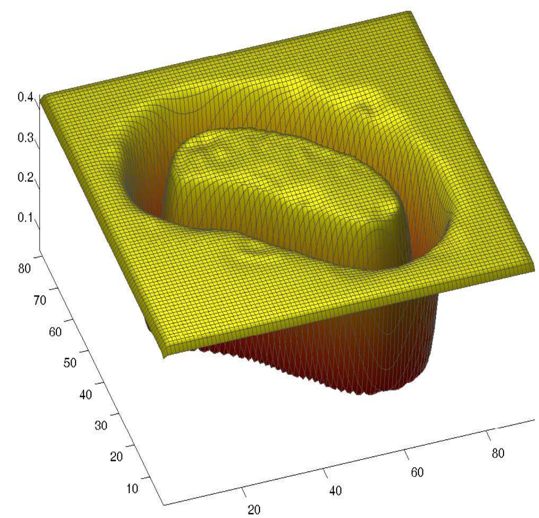
For image segementation one seeks to locally minimize the functional

$$J(\Gamma) = \int_\Gamma g_I \, dS + \nu \int_\Omega g_I \, d\mathbf{x}.$$

Here g_I is an edge detector



Image!



Edge detector!

Utilizing the results on the first and second Eulerian derivative, we obtain

$$dJ(\Gamma; V) = \langle D_\Gamma J, V\rangle = \int_\Gamma \Big\langle \Big(\frac{\partial g_I}{\partial n} + g_I\,(\kappa + \nu)\Big)\,\mathbf{n}, V \Big\rangle\, dS.$$

The Newton-type speed function F solves

$$\int_\Gamma \Big[\Big(\frac{\partial^2 g_I}{\partial n^2} + (2\kappa + \nu)\,\frac{\partial g_I}{\partial n} + \nu\,\kappa\, g_I\Big)\, F\, G + g_I \langle \nabla_\Gamma F, \nabla_\Gamma G\rangle\Big]\, dS = - \int_\Gamma \Big(\frac{\partial g_I}{\partial n} + (\kappa + \nu)\, g_I\Big)\, G\, dS$$

$\Rightarrow$ Coercivity.

$$\int_\Gamma \Big[\Big(\frac{\partial^2 g_I}{\partial n^2} + (2\kappa + \nu)\,\frac{\partial g_I}{\partial n} + \nu\,\kappa\, g_I\Big)_+ F\, G + g_I \langle \nabla_\Gamma F, \nabla_\Gamma G\rangle\Big]\, dS = - \int_\Gamma \Big(\frac{\partial g_I}{\partial n} + (\kappa + \nu)\, g_I\Big)\, G\, dS$$

which we refer to as (*).

The algorithm.

Shape Newton-Algorithm with narrow band.

1. **Initialization.** Choose Γ_0. Initialize the level set function u^0 such that Γ_0 is the zero level set of u^0; set $k = 0$. Choose a bandwidth $w \in \mathbb{N}$ and $\nu \in \mathbb{R}$.

2. **Newton direction.** Find the zero level set Γ_k of the actual level set function u^k. Solve (*) to obtain the Newton-type direction F^k.

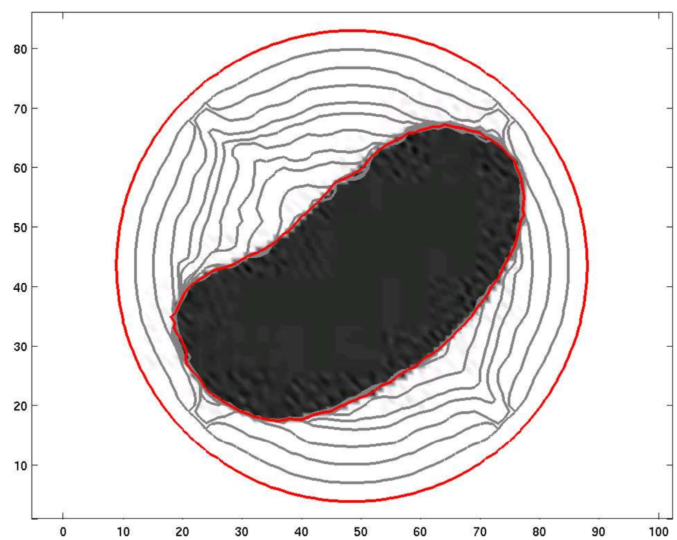
3. **Extension.** Extend F^k to a band around the actual zero level set Γ_k with bandwidth w yielding F^k_{ext}.

4. **Update.** Perform a time step in the level set equation with speed function F^k_{ext} to update u^k on the band. Let $\hat{u}^{k+1}$ denote this update.

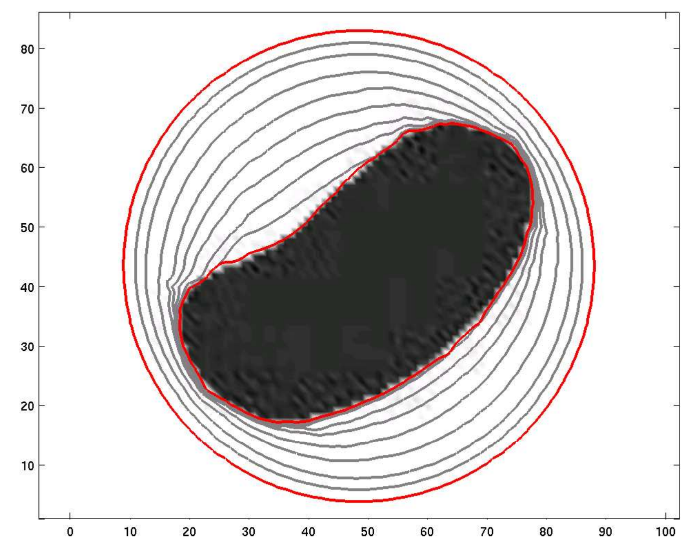
5. **Reinitialization.** Reinitialize $\widehat{u}^{k+1}$ in order to obtain a signed distance function u^{k+1} with zero level set given by the zero level set of $\widehat{u}^{k+1}$. Set $k = k + 1$ and go to (2).

Numerical results.

⇒ Example 1



Steepest descent!



Newton-type direction!

Iteration history for Newton-type direction.

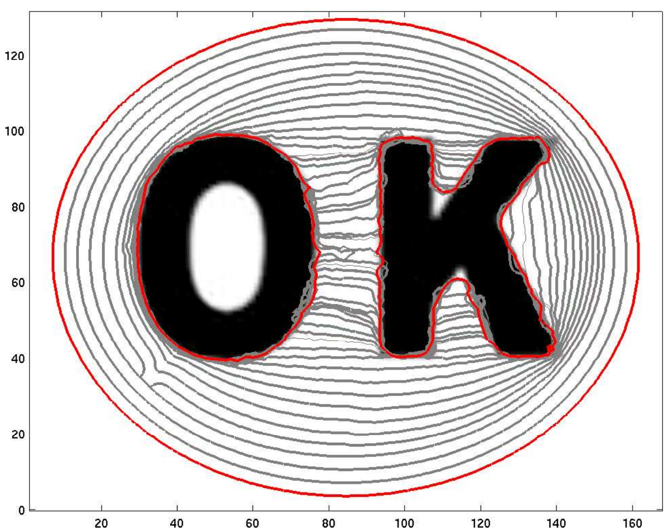
k	Δt^k	Δt^k_{CFL}	J^k_h	$J^k_{h,r}$
1	0.00027	0.00014	67.71894	67.73983
2	0.00916	0.00458	63.62859	63.58714
3	0.05119	0.01462	55.69355	55.30486
4	0.07655	0.02187	45.59301	45.34222
5	0.11608	0.03317	37.06772	36.81020
6	0.16018	0.04577	28.19008	27.54977
7	0.20494	0.05856	16.41064	15.95286
8	0.31020	0.08862	9.73240	9.92598
9	0.34469	0.09848	4.01012	3.83231

Comparison of Algorithms.

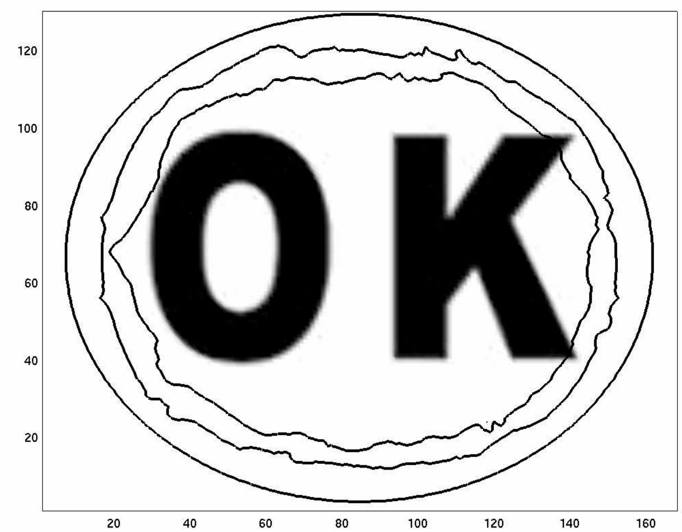
	Newton $\nu = 0$ LS	gradient $\nu = 1$ LS	gradient $\nu = 1$ no LS	gradient $\nu = 0$ LS
# it.	9	13	31	327

LS ... line search

$\Rightarrow$ Example 2.

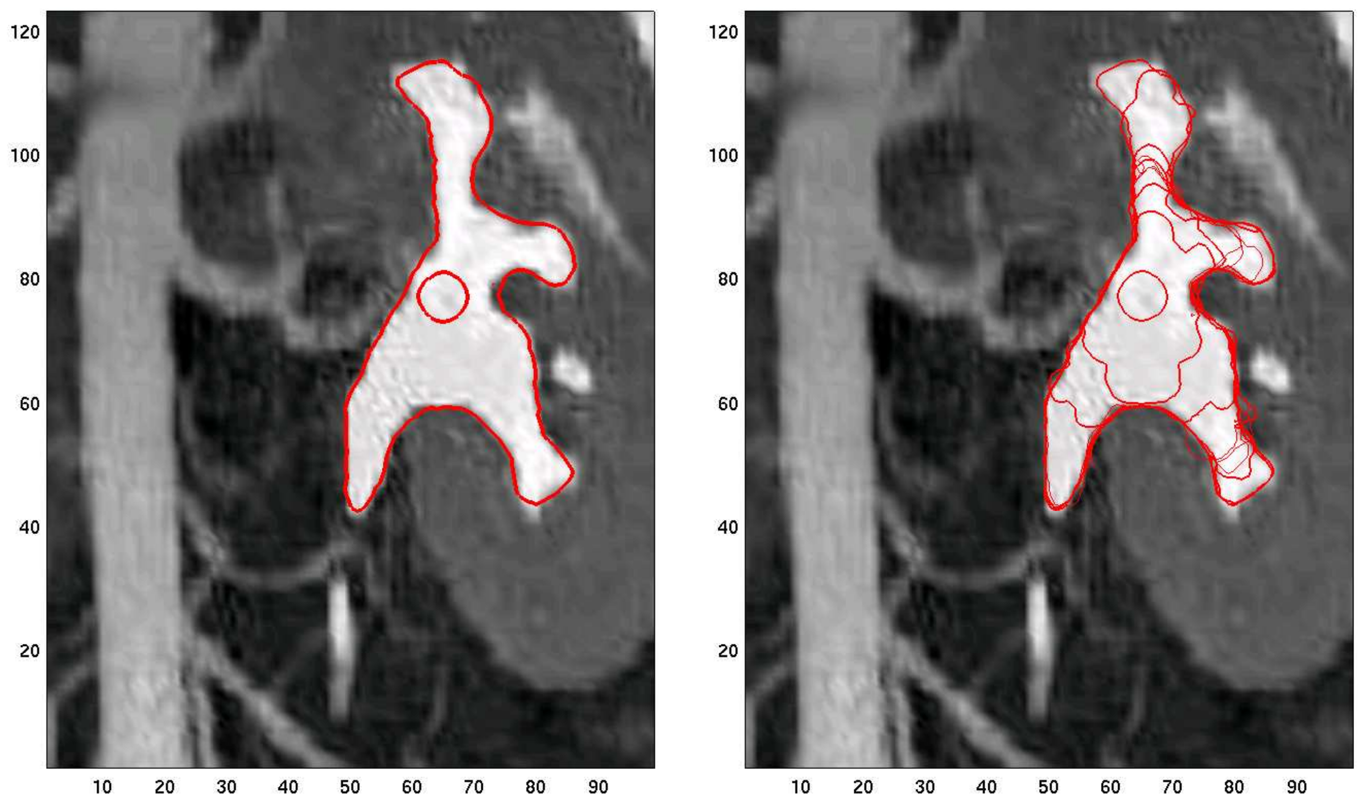


Newton-type direction!



Steepest descent!

⇒ Example 3.



Newton-type direction!

OUTLINE

- State constrained optimal control of PDEs.
- Image segmentation via active contours and edge detectors.
- **Image segmentation via Mumford-Shah functional.**

Problem formulation

- Given: Gray value image $u_d : D \to \mathbb{R}$ with $D = (0,1) \times (0,1)$ (noisy and/or blurred).

- Aim: Find denoised and deblurred approximation u to the given data u_d and a set $\Gamma \subset \Omega$ — the *edge-set* of the given image u_d — as the minimizers of the Mumford-Shah functional

$$J(u,\Gamma) = \frac{1}{2}\int_D |u - u_d|^2 \, d\mathbf{x} + \frac{\mu}{2}\int_{D\setminus\Gamma} |\nabla u|^2 \, d\mathbf{x} + \nu \int_\Gamma 1 \, d\mathcal{H}_1. \qquad \text{(MS)}$$

Here $\mu, \nu \geq 0$ and $\mathcal{H}_1$ denotes the 1-dimensional Hausdorff measure.

We consider

$$\Gamma = \partial\Omega_1 = \{\mathbf{x} \in D : \phi(\mathbf{x}) = 0\}, \quad \Omega_1 = \{\mathbf{x} \in D : \phi(\mathbf{x}) < 0\}$$

with $\Omega_1 \subset D$ open.

$$\Omega_2 = D \setminus \overline{\Omega_1} = \{\mathbf{x} \in D : \phi(\mathbf{x}) > 0\}.$$

Under suitable assumptions we have

$$\boxed{\inf_{(u,\Gamma)\in H^1(D\setminus\Gamma)\times\mathcal{E}} J(u,\Gamma) = \inf_{\Gamma\in\mathcal{E}} \min_{u\in H^1(D\setminus\Gamma)} J(u,\Gamma).}$$

$\mathcal{E}$ denotes the set of admissible edges.

- Set $u_k = u|_{\Omega_k}$ for $k = 1, 2$.

- Note: $u \in H^1(D \setminus \Gamma) \Leftrightarrow u_k \in H^1(\Omega_k)$ for $k = 1, 2$.

- The solution $u(\Gamma) = u_1(\Gamma)\chi_{\Omega_1} + u_2(\Gamma)\chi_{\Omega_2}$ to the inner minimization is then given as the solution to the optimality system

$$\int_{\Omega_k} \big(u_k(\Gamma)\,\varphi + \mu\langle \nabla u_k(\Gamma), \nabla\varphi\rangle\big)\,d\mathbf{x} = \int_{\Omega_k} u_d\,\varphi\,d\mathbf{x}$$

for all $\varphi \in H^1(\Omega_k)$ and for $k = 1, 2$.

- Weak form of the Neumann problem for the Helmholtz

$$\begin{cases} -\mu\Delta u_k(\Gamma) + u(\Gamma) = u_d \text{ on } \Omega_k \\ \frac{\partial u_k(\Gamma)}{\partial n_k} = 0 \text{ on } \partial\Omega_k \end{cases}$$

for $k = 1, 2$.

Shape optimization problem.

- Remaining shape optimization problem.

$$\text{minimize } \hat{J}(\Gamma) = \sum_{k=1}^{2} \int_{\Omega_k} \left(\frac{1}{2}|u_k(\Gamma) - u_d|^2 + \frac{\mu}{2}|\nabla u_k(\Gamma)|^2 \right) d\mathbf{x} + \nu \int_{\Gamma} 1 \, d\mathcal{H}_1$$

over $\Gamma \in \mathcal{E}$

- Let $V_F = F \nabla b_\Gamma$ with a scalar function F. Eulerian derivative.

$$d\hat{J}(\Gamma; V_F) = \int_{\Gamma} \left(\frac{1}{2} [\![|u - u_d|^2]\!] + \frac{\mu}{2} [\![|\nabla_\Gamma u(\Gamma)|^2]\!] + \nu\kappa \right) F \, d\mathcal{H}_1.$$

where $[\![|u - u_d|^2]\!] = |u_1 - u_d|^2 - |u_2 - u_d|^2$ and $[\![|\nabla u(\Gamma)|^2]\!] = |\nabla u_1(\Gamma)|^2 - |\nabla u_2(\Gamma)|^2$ denote the jumps of $|u - u_d|^2$ and $|\nabla u|^2$, respectively, across Γ.

Shape Hessian.

$$d^2\hat{J}(\Gamma;V_F;V_G) = \int_\Gamma \Bigg[\frac{1}{2}\Big(\kappa\,(\llbracket|u-u_d|^2\rrbracket - \mu\llbracket|\nabla_\Gamma u|^2\rrbracket) + \frac{\partial}{\partial n}\llbracket|u-u_d|^2\rrbracket\Big)\,G + \llbracket(u-u_d)\,u'_G\rrbracket + \mu\llbracket\langle\nabla u, \nabla u'_G\rangle\rrbracket - \nu\Delta_\Gamma G\Bigg]\,F\,d\mathcal{H}_1.$$

The shape derivative u'_G solves the Helmholtz problem

$$\begin{cases} -\mu\Delta u'_{k,G} + u'_{k,G} = 0 \text{ on } \Omega_k \\ \dfrac{\partial u'_{k,G}}{\partial n_1} = \operatorname{div}_\Gamma(G\,\nabla_\Gamma u_k) + \dfrac{1}{\mu}(u_d - u_k)\,G \text{ on } \Gamma, \end{cases}$$

for $k = 1, 2$.

$\Rightarrow$ Shape Hessian evaluation too expensive!

Descent direction and PCG.

- Let $B(\Gamma; V_F; V_G)$ denote the shape Hessian or a positive semidefinite approximation. A descent direction G_N for $\hat{J}$ is obtained from

$$\boxed{\mathrm{B}(\Gamma_k; V_F, V_{G_N}) = -d\hat{J}(\Gamma_k; V_F) \quad \forall F.}$$

  by means of the preconditioned conjugate gradient method.

- **Preconditioned shape CG-method.**

  1. Choose a preconditioner $C(\Gamma; \cdot, \cdot)$, an initial scalar function G^0 and a stopping tolerance $\epsilon > 0$. Compute $r^0 = r^0(\cdot) = B(\Gamma; \cdot, V_{G^0}) + d\hat{J}(\Gamma; \cdot)$. Obtain d^0 as the solution to

  $$C(\Gamma; V_F, V_{d^0}) = -r^0(V_F) \quad \text{for all } F.$$

     Set $k = 0$.

  2. If $\|r^k\| \leq \epsilon$, then stop; otherwise continue with step 3.

3. Compute the solution q^k to

$$C(\Gamma; V_F, V_{q^k}) = r^k(V_F) \quad \text{for all } F.$$

Put

$$t_k := \frac{\langle r^k, q^k \rangle}{B(\Gamma; V_{d^0}, V_{d^0})}.$$

4. Compute

$$\begin{aligned} G^{k+1} &:= G^k + t_k B(\Gamma; \cdot, V_{G^k}), \\ r^{k+1} &:= r^k + t_k B(\Gamma; \cdot, V_{G^k}). \end{aligned}$$

Obtain q^{k+1} as the solution to

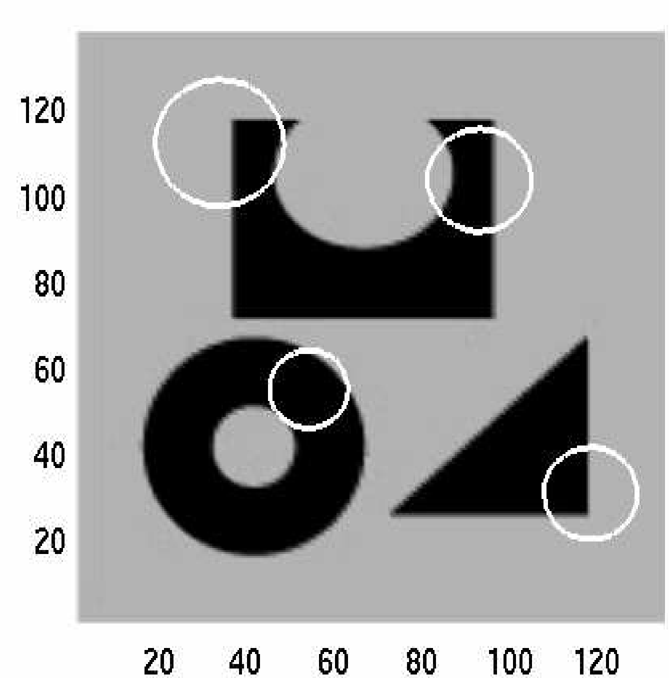
$$C(\Gamma; V_F, V_{q^{k+1}}) = r^{k+1}(V_F) \quad \text{for all } F.$$

and compute

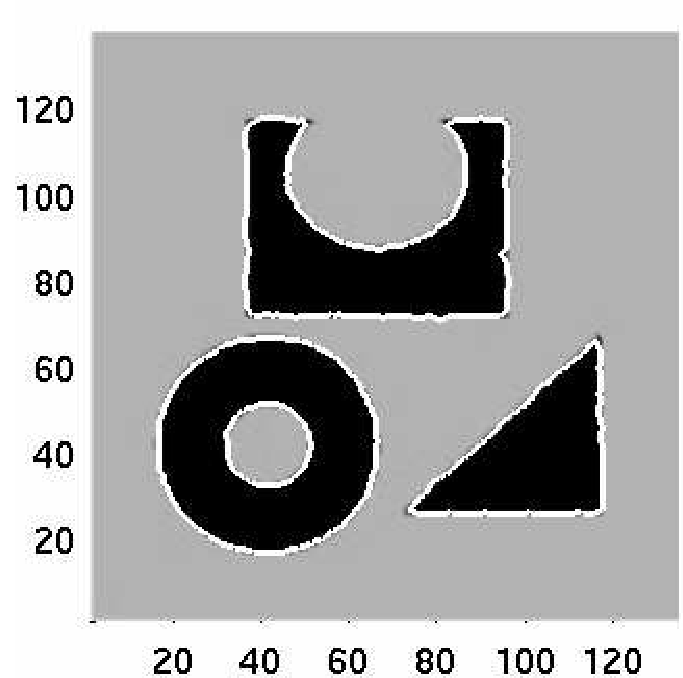
$$\begin{aligned} \beta_k &:= \frac{\langle r^{k+1}, q^{k+1} \rangle}{\langle r^k, q^k \rangle}, \\ d^{k+1} &:= -q^{k+1} + \beta_k d^k, \\ k &:= k + 1 \text{ and go to step 2.} \end{aligned}$$

Numerical results.

⇒ Example 1



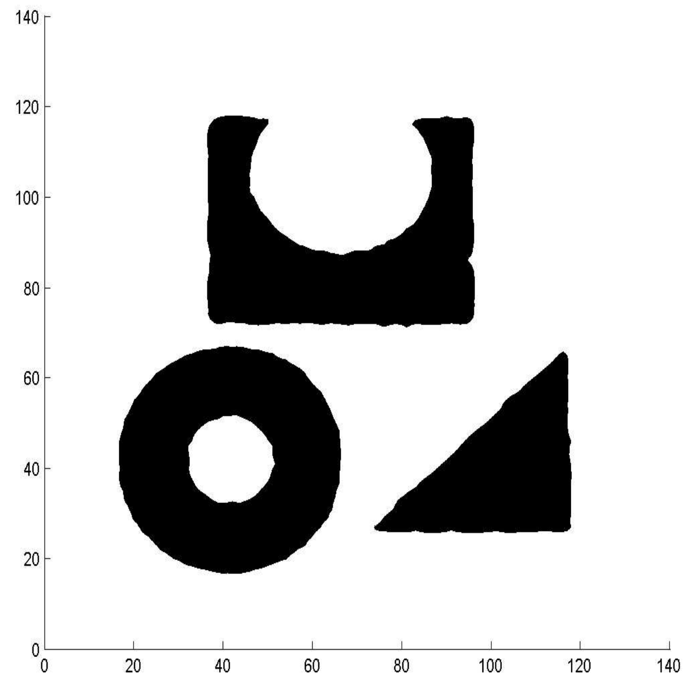
Initialization



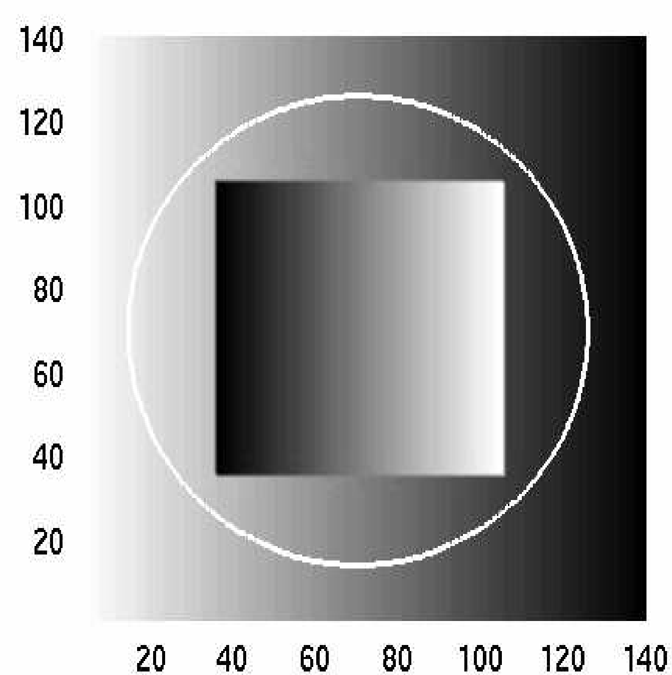
Segmented image

15 iterations.

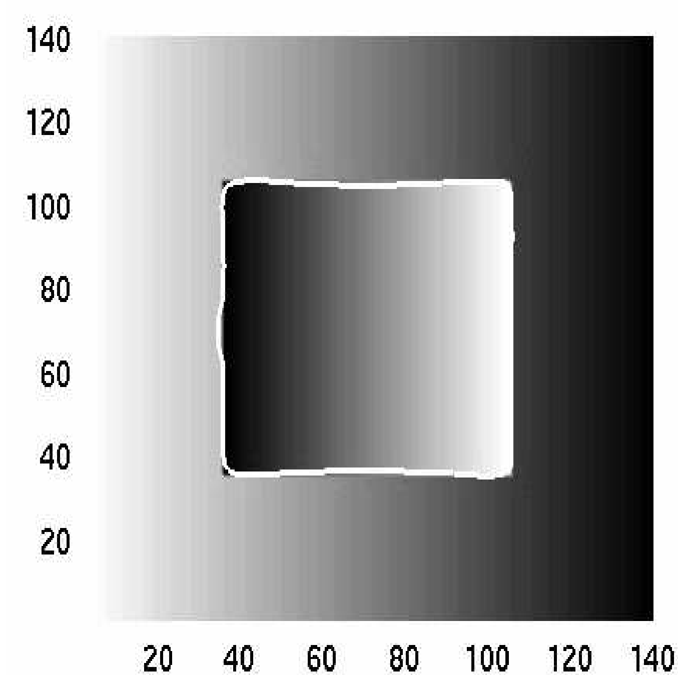
Solution of the Helmholtz equation for u.



⇒ Example 2



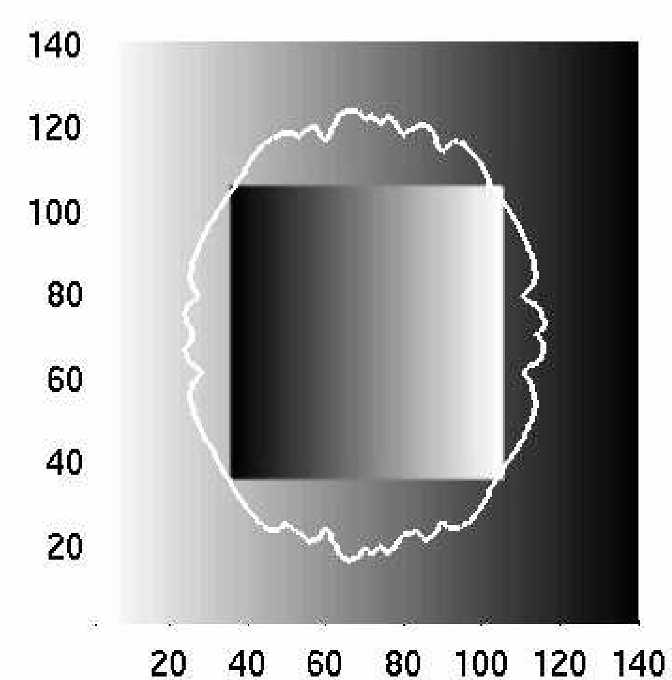
Initialization



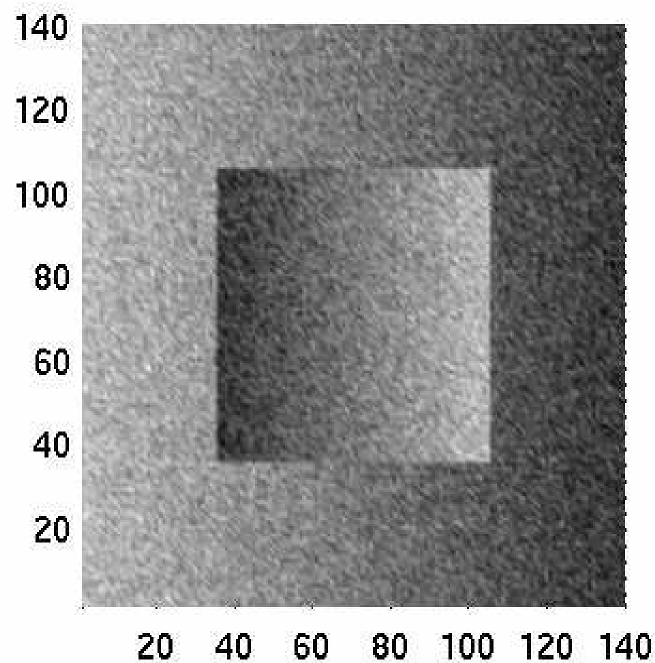
Segmented image

5 iterations!

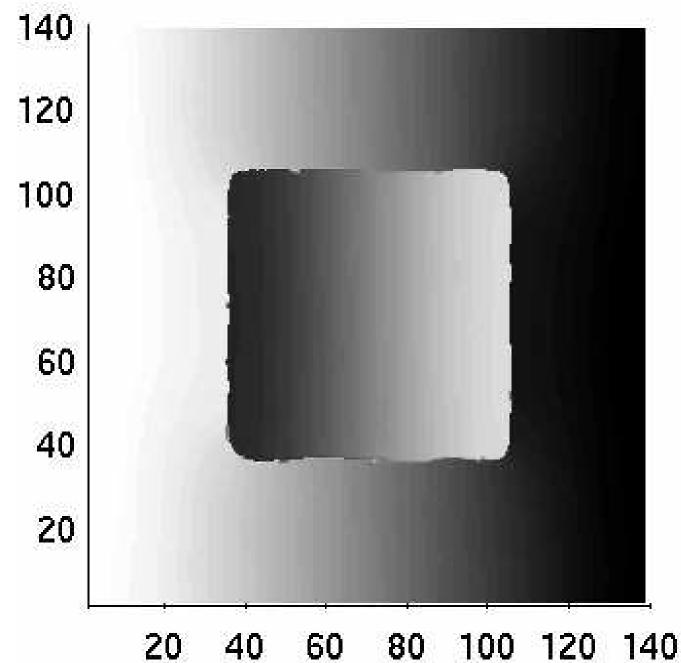
Steepest descent method at iteration 8.



Denoising (and simultaneous segmentation).



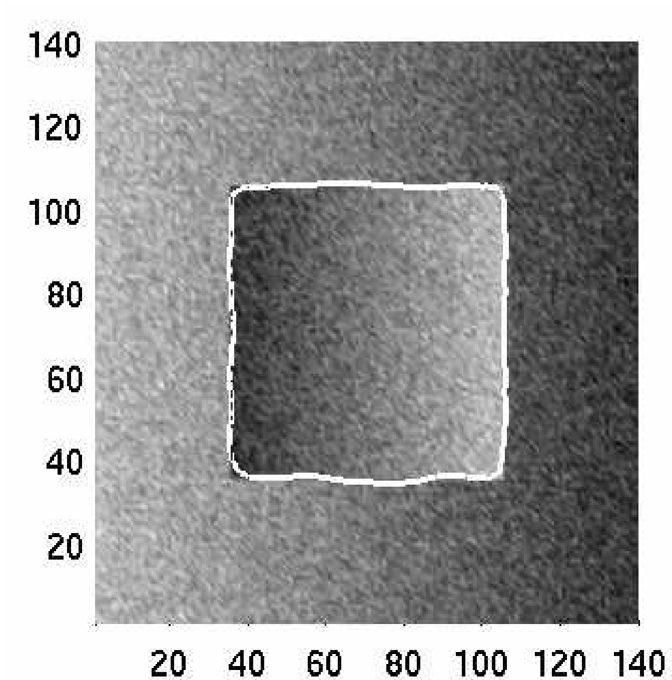
Initialization



Denoised image

5 iterations!

Segmentation.



Give me F and I'll make it work!