

# STRUCTURAL OPTIMIZATION BY THE LEVEL SET METHOD

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1. Introduction
2. Setting of the problem
3. Shape differentiation
4. Front propagation by the level set method
5. Algorithm and numerical results

## **-I- INTRODUCTION**

Two main approaches in structural optimization:

### **1) Geometric optimization by boundary variations**

- ➡ Hadamard method of shape sensitivity: Murat-Simon, Pironneau, Zolésio...
- ➡ Ill-posed problem: many local minima, no convergence under mesh refinement.
- ➡ \* **Very costly because of remeshing.**
- ➡ \* **Very general: any model or objective function.**

### **2) Topology optimization (the homogenization method)**

- ➡ Developed by Murat-Tartar, Lurie-Cherkaev, Kohn-Strang, Bendsoe-Kikuchi...
- ➡ Well-posed problem ; topology changes.
- ➡ \* **Limited to linear models and simple objective functions.**
- ➡ \* **Very cheap because it captures shapes on a fixed mesh.**

## GOAL OF THIS WORK

### Combine the advantages of the two approaches:

- ➡ Fixed mesh: low computational cost.
- ➡ General method: based on shape differentiation.

### Main tool: the level set method of Osher and Sethian.

- ➡ Previous works: Sethian and Wiegmann (JCP 2000), Osher and Santosa (JCP 2001). Close but different approach of Bourdin and Chambolle (shape optimization with a phase field model).
- ➡ Some drawbacks remain: reduction of topology (rather than variation), many local minima.

## -II- SETTING OF THE PROBLEM

### Structural optimization in linearized elasticity.

Shape  $\Omega$  with boundary

$$\partial\Omega = \Gamma \cup \Gamma_N \cup \Gamma_D,$$

with Dirichlet condition on  $\Gamma_D$ , Neumann condition on  $\Gamma \cup \Gamma_N$ . **Only  $\Gamma$  is optimized.**

$$\left\{ \begin{array}{ll} -\operatorname{div} (A e(u)) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ (A e(u))n = g & \text{on } \Gamma_N \\ (A e(u))n = 0 & \text{on } \Gamma \end{array} \right.$$

with  $e(u) = \frac{1}{2} (\nabla u + \nabla^t u)$ , and  $A$  an homogeneous isotropic elasticity tensor.

## OBJECTIVE FUNCTIONS

### Two examples:

Compliance or work done by the load

$$J(\Omega) = \int_{\Gamma_N} g \cdot u \, ds = \int_{\Omega} A e(u) \cdot e(u) \, dx,$$

A least square criteria (useful for designing mechanisms)

$$J(\Omega) = \left( \int_{\Omega} k(x) |u - u_0|^\alpha \, dx \right)^{1/\alpha},$$

with a target displacement  $u_0$ ,  $\alpha \geq 2$  and  $k$  a given weighting factor.

## EXISTENCE THEORY

The “minimal” set of admissible shapes

$$\mathcal{U}_{ad} = \left\{ \Omega \subset D, \text{ vol}(\Omega) = V_0, \Gamma_D \cup \Gamma_N \subset \partial\Omega \right\}$$

with  $D$  a bounded open set  $\mathbb{R}^N$ . Usually, the minimization problem has no solution in  $\mathcal{U}_{ad}$ .

There exists an optimal shape if further conditions are required:

1. a uniform cone condition (D. Chenaus).
2. a perimeter constraint (L. Ambrosio, G. Buttazzo).
3. a bound on the number of connected components of  $D \setminus \Omega$  in two space dimensions (A. Chambolle).

## PROPOSED NUMERICAL METHOD

**First step:** we compute shape derivatives of the objective functions in a continuous framework.

**Second step:** we model a shape by a level-set function ; the shape is varied by advecting the level-set function following the flow of the shape gradient (the transport equation is of Hamilton-Jacobi type).

## -III- SHAPE DIFFERENTIATION

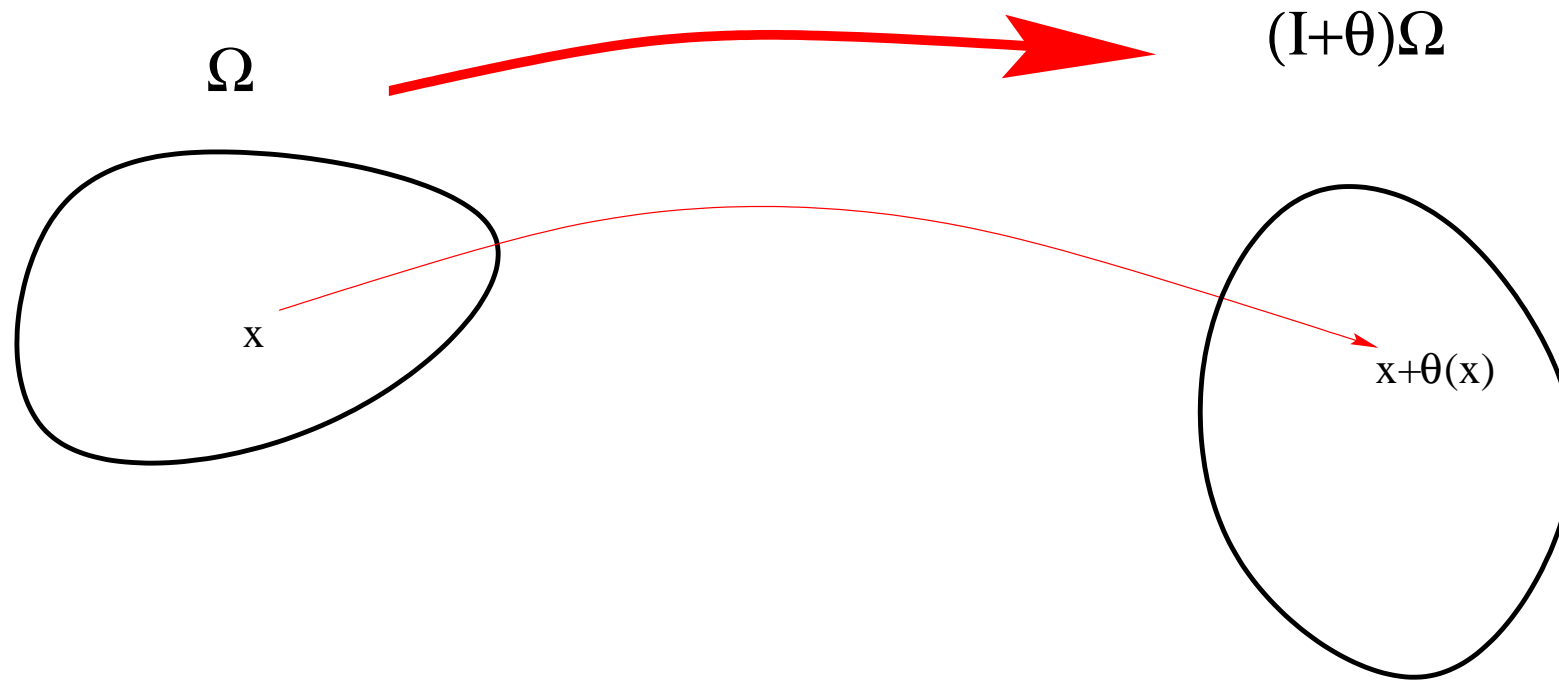
### Framework of Murat-Simon:

Let  $\Omega_0$  be a reference domain. Consider its variations

$$\Omega = (Id + \theta)\Omega_0 \quad \text{with} \quad \theta \in W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^N).$$

**Lemma.** For any  $\theta \in W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^N)$  such that  $\|\theta\|_{W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^N)} < 1$ ,  $(Id + \theta)$  is a diffeomorphism in  $\mathbb{R}^N$ .

**Definition:** the shape derivative of  $J(\Omega)$  at  $\Omega_0$  is the Fréchet differential of  $\theta \rightarrow J((Id + \theta)\Omega_0)$  at 0.

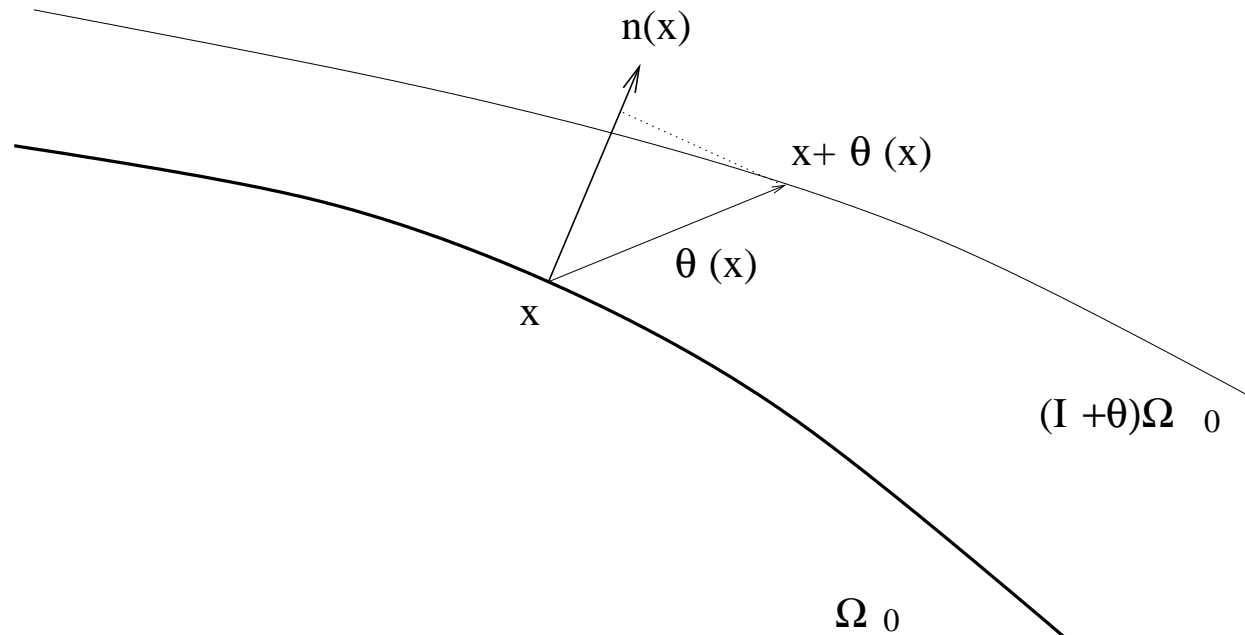


The set  $\Omega = (Id + \theta)(\Omega_0)$  is defined by

$$\Omega = \{x + \theta(x) \mid x \in \Omega_0\}.$$

The vector field  $\theta(x)$  is the displacement of  $\Omega_0$ .

The derivative  $J'(\Omega_0)(\theta)$  depends only on  $\theta \cdot n$  on the boundary  $\partial\Omega_0$ .



**Lemma.** Let  $\Omega_0$  be a smooth bounded open set and  $J(\Omega)$  a differentiable function at  $\Omega_0$ . Its derivative satisfies

$$J'(\Omega_0)(\theta_1) = J'(\Omega_0)(\theta_2)$$

if  $\theta_1, \theta_2 \in W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^N)$  are such that  $\theta_2 - \theta_1 \in C^1(\mathbb{R}^N; \mathbb{R}^N)$  and

$$\theta_1 \cdot n = \theta_2 \cdot n \quad \text{on } \partial\Omega_0.$$

**Example 1 of shape derivative**

Let  $\Omega_0$  be a smooth bounded open set and  $f(x) \in W^{1,1}(\mathbb{R}^N)$ . Define

$$J(\Omega) = \int_{\Omega} f(x) dx.$$

Then  $J$  is differentiable at  $\Omega_0$  and

$$J'(\Omega_0)(\theta) = \int_{\Omega_0} \operatorname{div} (\theta(x) f(x)) dx = \int_{\partial\Omega_0} \theta(x) \cdot n(x) f(x) ds$$

for any  $\theta \in W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^N)$ .

**Example 2 of shape derivative**

Let  $\Omega_0$  be a smooth bounded open set and  $f(x) \in W^{2,1}(\mathbb{R}^N)$ . Define

$$J(\Omega) = \int_{\partial\Omega} f(x) ds.$$

Then  $J$  is differentiable at  $\Omega_0$  and

$$J'(\Omega_0)(\theta) = \int_{\partial\Omega_0} (\nabla f \cdot \theta + f(\operatorname{div} \theta - \nabla \theta n \cdot n)) ds$$

for any  $\theta \in W^{1,\infty}(\mathbb{R}^N; \mathbb{R}^N)$ .

An integration by parts on the manifold  $\partial\Omega_0$  yields

$$J'(\Omega_0)(\theta) = \int_{\partial\Omega_0} \theta \cdot n \left( \frac{\partial f}{\partial n} + Hf \right) ds,$$

where  $H$  is the mean curvature of  $\partial\Omega_0$  defined by  $H = \operatorname{div} n$ .

**SHAPE DERIVATIVE OF THE COMPLIANCE**

$$J(\Omega) = \int_{\Gamma_N} g \cdot u_\Omega \, ds = \int_{\Omega} A e(u_\Omega) \cdot e(u_\Omega) \, dx,$$

$$J'(\Omega_0)(\theta) = - \int_{\Gamma} A e(u) \cdot e(u) \theta \cdot n \, ds,$$

where  $u$  is the state variable in  $\Omega_0$ .

**Remark:** self-adjoint problem (no adjoint state is required).

## SHAPE DERIVATIVE OF THE LEAST-SQUARE CRITERIA

$$J(\Omega) = \left( \int_{\Omega} k(x) |u_{\Omega} - u_0|^{\alpha} dx \right)^{1/\alpha},$$

$$J'(\Omega_0)(\theta) = \int_{\Gamma} \left( -Ae(p) \cdot e(u) + \frac{C_0}{\alpha} k |u - u_0|^{\alpha} \right) \theta \cdot n ds,$$

with the state  $u$  and the adjoint state  $p$  defined by

$$\begin{cases} -\operatorname{div} (Ae(p)) = C_0 k(x) |u - u_0|^{\alpha-2} (u - u_0) & \text{in } \Omega_0 \\ p = 0 & \text{on } \Gamma_D \\ (Ae(p))n = 0 & \text{on } \Gamma_N \cup \Gamma, \end{cases}$$

and  $C_0 = \left( \int_{\Omega_0} k(x) |u(x) - u_0(x)|^{\alpha} dx \right)^{1/\alpha-1}$ .

Idea of the proof.

The proof is classical.

Rigorous but lengthy proof:

- ⇒ Change of variables  $y = x + \theta(x)$ : rewrite all integrals in the fixed reference domain  $\Omega_0$ .
- ⇒ Write a variational formulation of the p.d.e. in  $\Omega_0$ .
- ⇒ Differentiate with respect to  $\theta$ .

Formal but simpler proof:

⇒ Write a Lagrangian for  $(v, q) \in \left(H^1(\mathbb{R}^d; \mathbb{R}^d)\right)^2$

$$\begin{aligned} \mathcal{L}(\Omega, v, q) = & \int_{\Omega} j(x, v) dx + \int_{\Omega} Ae(v) \cdot e(q) dx - \int_{\Omega} q \cdot f dx \\ & - \int_{\Gamma_N} q \cdot g ds - \int_{\Gamma_D} \left( q \cdot Ae(v)n + v \cdot Ae(q)n \right) ds. \end{aligned}$$

⇒ Stationarity of  $\mathcal{L}$  gives the state and adjoint equations.

⇒ Remark that  $J(\Omega) = \mathcal{L}(\Omega, u_{\Omega}, p_{\Omega})$ , and thus

$$J'(\Omega)(\theta) = \frac{\partial \mathcal{L}}{\partial \Omega}(\Omega, u, p)$$

## -IV- FRONT PROPAGATION BY LEVEL SET

Shape capturing method on a fixed mesh of a “large” box  $D$ .

A shape  $\Omega$  is parametrized by a **level set** function

$$\begin{cases} \psi(x) = 0 & \Leftrightarrow x \in \partial\Omega \cap D \\ \psi(x) < 0 & \Leftrightarrow x \in \Omega \\ \psi(x) > 0 & \Leftrightarrow x \in (D \setminus \Omega) \end{cases}$$

The normal  $n$  to  $\Omega$  is given by  $\nabla\psi/|\nabla\psi|$  and the curvature  $H$  is the divergence of  $n$ . **These formulas make sense everywhere in  $D$**  on not only on the boundary  $\partial\Omega$ .

### Hamilton Jacobi equation

Assume that the shape  $\Omega(t)$  evolves in time  $t$  with a normal velocity  $V(t, x)$ .

Then

$$\psi(t, x(t)) = 0 \quad \text{for any } x(t) \in \partial\Omega(t).$$

Deriving in  $t$  yields

$$\frac{\partial\psi}{\partial t} + \dot{x}(t) \cdot \nabla_x \psi = \frac{\partial\psi}{\partial t} + Vn \cdot \nabla_x \psi = 0.$$

Since  $n = \nabla_x \psi / |\nabla_x \psi|$  we obtain

$$\frac{\partial\psi}{\partial t} + V|\nabla_x \psi| = 0.$$

This Hamilton Jacobi equation is posed in the whole box  $D$ , and not only on the boundary  $\partial\Omega$ , if the velocity  $V$  is known everywhere.

Idea of the method

Shape derivative

$$J'(\Omega_0)(\theta) = \int_{\Gamma} j(u, p, n) \theta \cdot n \, ds.$$

Gradient algorithm for the shape:

$$\Omega_{k+1} = \left( Id - j(u_k, p_k, n_k) n_k \right) \Omega_k$$

since the normal  $n_k$  is “automatically” defined everywhere in  $D$ . In other words, the normal advection velocity of the shape is  $-j$ . Introducing a “pseudo-time” (a descent parameter), we solve [the Hamilton-Jacobi equation](#)

$$\frac{\partial \psi}{\partial t} - j |\nabla_x \psi| = 0 \quad \text{in } D$$

Choice of the advection velocity

Simplest choice:

$$J'(\Omega_0)(\theta) = \int_{\Gamma} j \theta \cdot n \, ds \quad \Rightarrow \quad \theta = -j n.$$

However,  $j$  may be not smooth enough (typically  $j \in L^1(\Omega_0)$  if there are “corners”).

Classical trick: **one can smooth the velocity.** For example:

$$\begin{cases} -\Delta \theta = 0 & \text{in } \Omega_0 \\ \theta = 0 & \text{on } \Gamma_D \cup \Gamma_N \\ \frac{\partial \theta}{\partial n} = -j n & \text{on } \Gamma \end{cases}$$

It increases of one order the regularity of  $\theta$  and

$$\int_{\Omega_0} |\nabla \theta|^2 \, dx = - \int_{\Gamma} j \theta \cdot n \, ds$$

which guarantees the decrease of  $J$ .

## -V- NUMERICAL ALGORITHM

1. Initialization of the level set function  $\psi_0$  (including holes).
2. Iteration until convergence for  $k \geq 1$ :
  - (a) Computation of  $u_k$  and  $p_k$  by solving linearized elasticity problem with the shape  $\psi_k$ . Evaluation of the shape gradient = normal velocity  $V_k$
  - (b) Transport of the shape by  $V_k$  (Hamilton Jacobi equation) to obtain a new shape  $\psi_{k+1}$ .
  - (c) (Occasionally, re-initialization of the level set function  $\psi_{k+1}$  as the signed distance to the interface).

Algorithmic issues

- ✗ Quadrangular mesh.
- ✗ Finite difference scheme, upwind of order 1 or 2, for the Hamilton Jacobi equation ( $\psi$  is discretized at the mesh nodes).
- ✗ Q1 finite elements for the elasticity problems in the box  $D$

$$\left\{ \begin{array}{ll} -\operatorname{div} (A^* e(u)) = 0 & \text{in } D \\ u = 0 & \text{on } \Gamma_D \\ (A^* e(u))n = g & \text{on } \Gamma_N \\ (A^* e(u))n = 0 & \text{on } \partial D \setminus (\Gamma_N \cup \Gamma_D). \end{array} \right.$$

- ✗ Elasticity tensor  $A^*$  defined as a “mixture” of  $A$  and a weak material mimicking holes

$$A^* = \theta A \quad \text{with} \quad 10^{-3} \leq \theta \leq 1$$

and  $\theta = \text{volume of the shape } \psi < 0 \text{ in each cell (piecewise constant proportion)}$ .

Upwind scheme

$$\frac{\partial \psi}{\partial t} - j |\nabla_x \psi| = 0 \quad \text{in } D$$

solved by an **explicit 1st order upwind scheme**

$$\frac{\psi_i^{n+1} - \psi_i^n}{\Delta t} - \max(j_i^n, 0) g^+(D_x^+ \psi_i^n, D_x^- \psi_i^n) - \min(j_i^n, 0) g^-(D_x^+ \psi_i^n, D_x^- \psi_i^n) = 0$$

with  $D_x^+ \psi_i^n = \frac{\psi_{i+1}^n - \psi_i^n}{\Delta x}$ ,  $D_x^- \psi_i^n = \frac{\psi_i^n - \psi_{i-1}^n}{\Delta x}$ , and

$$g^-(d^+, d^-) = \sqrt{\min(d^+, 0)^2 + \max(d^-, 0)^2},$$

$$g^+(d^+, d^-) = \sqrt{\max(d^+, 0)^2 + \min(d^-, 0)^2}.$$

☞ 2nd order extension.

☞ Easy computation of the curvature (for perimeter penalization).

## Re-initialization

In order to regularize the level set function (which may become too flat or too steep), we reinitialize it periodically by solving

$$\begin{cases} \frac{\partial \psi}{\partial t} + \text{sign}(\psi_0) \left( |\nabla_x \psi| - 1 \right) = 0 & \text{in } D \\ \psi(t = 0, x) = \psi_0(x) \end{cases}$$

which admits as a stationary solution the **signed distance** to the initial interface  $\{\psi_0(x) = 0\}$ .

- ➡ Classical idea in fluid mechanics.
- ➡ A few iterations are enough.
- ➡ Improve the convergence of the optimization process.

## Choice of the descent step

Two different strategies:

At each elasticity analysis, we perform a single time step of transport:

- ➡ The descent step is controlled by the CFL of the transport equation.
- ➡ Smooth descent.
- ➡ Lengthy computations (sub-optimal descent step).

At each elasticity analysis, we perform many time steps of transport:

- ➡ The descent step is controlled by the decrease of the objective function.
- ➡ Fast descent but requires a good heuristic for monitoring the number of time steps.

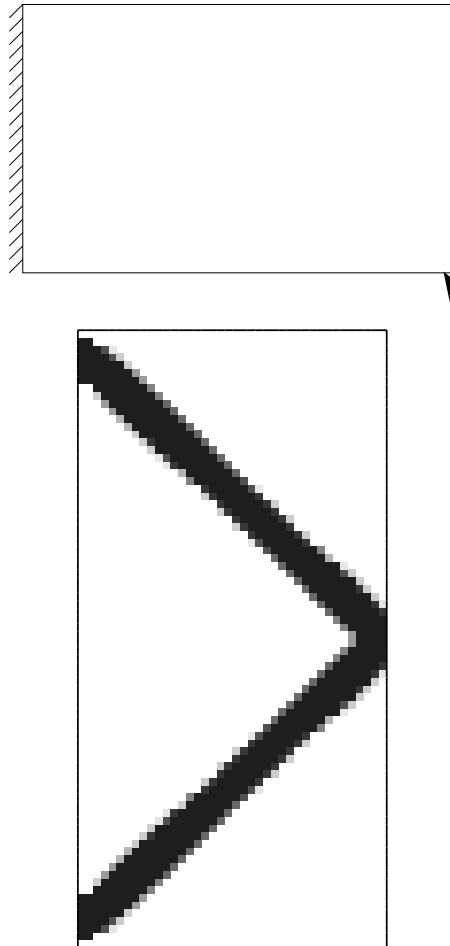
## NUMERICAL EXAMPLES

Definition of the test cases.

Results obtained with the homogenization method.

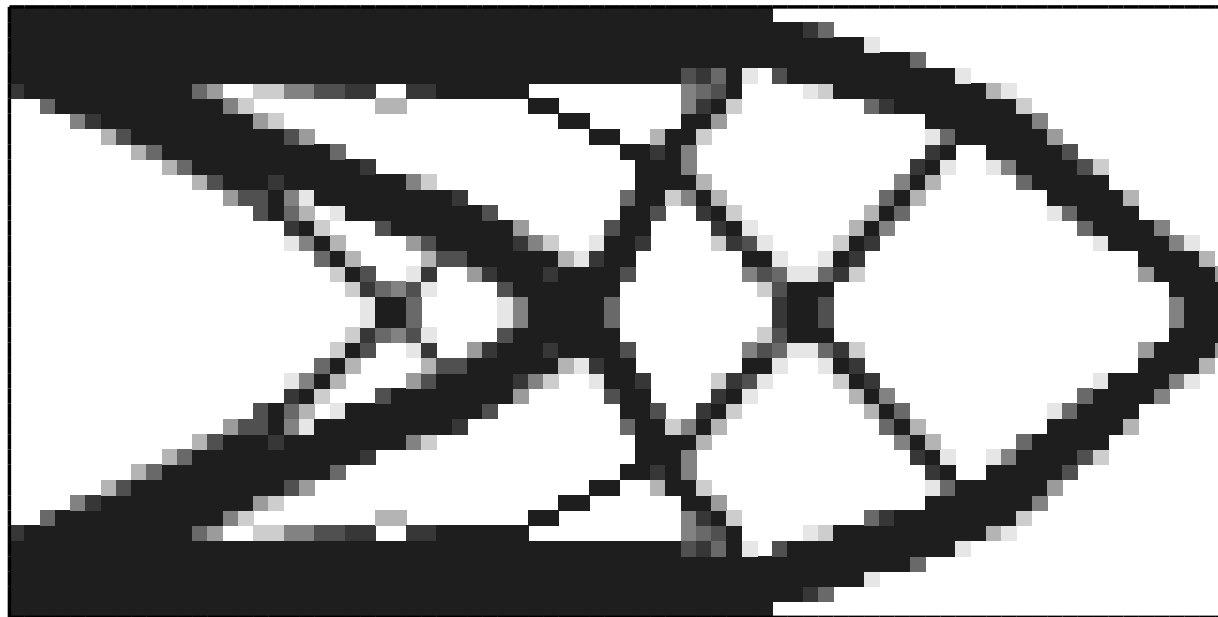
Optimal short cantilever

Minimization of compliance



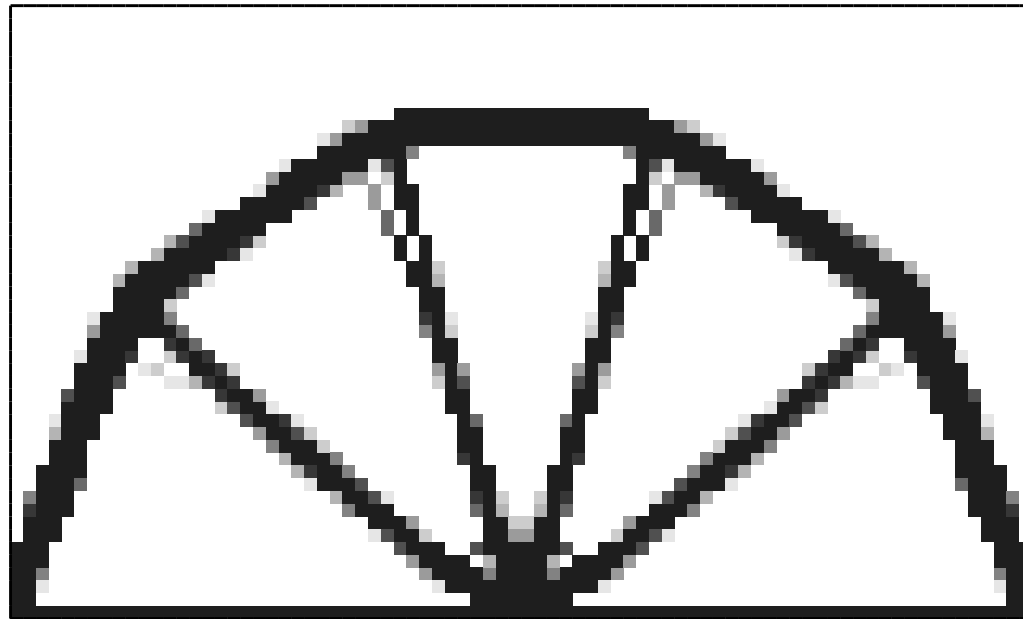
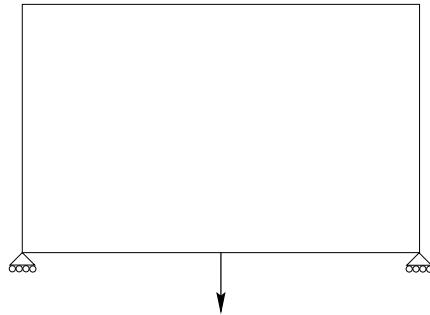
Optimal cantilever

Minimization of compliance



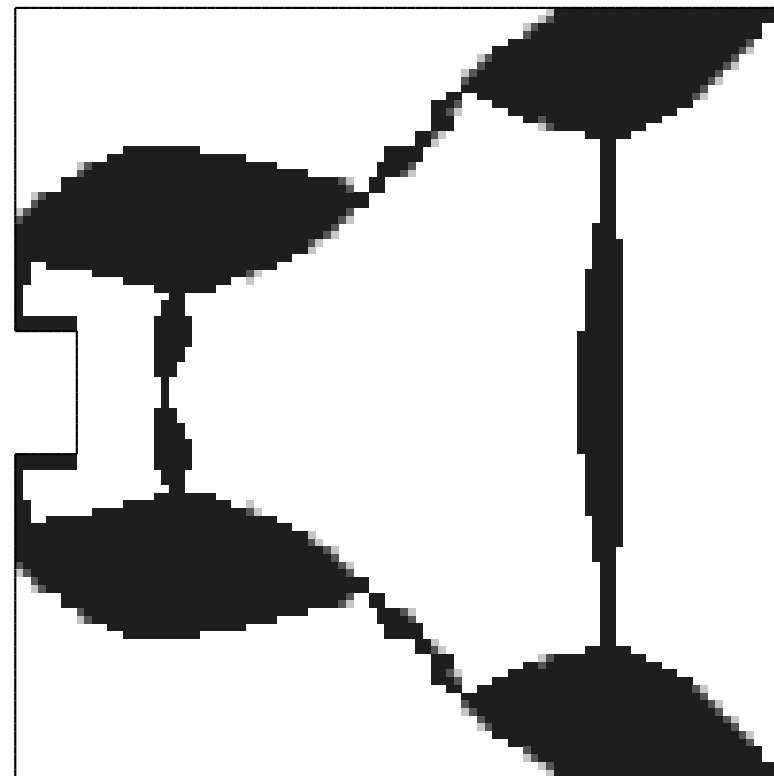
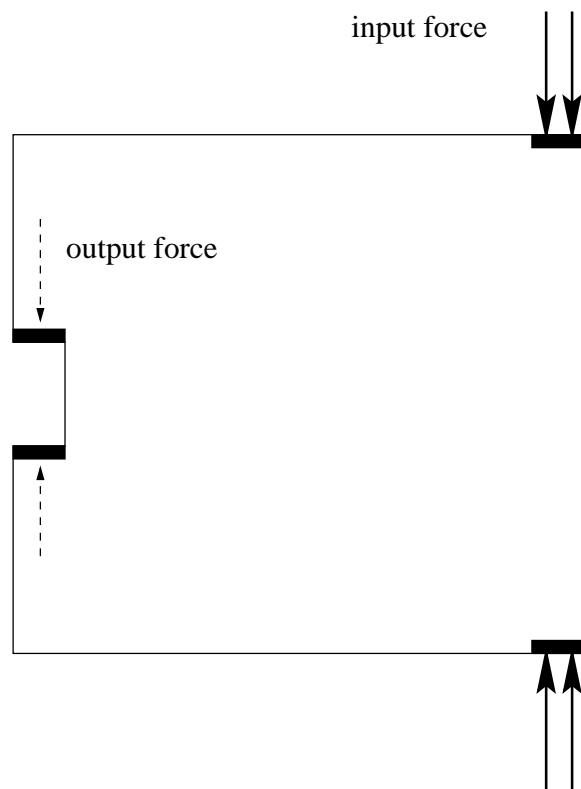
Optimal bridge

Minimization of compliance



## Gripping mechanism

Minimization of a least square criterion



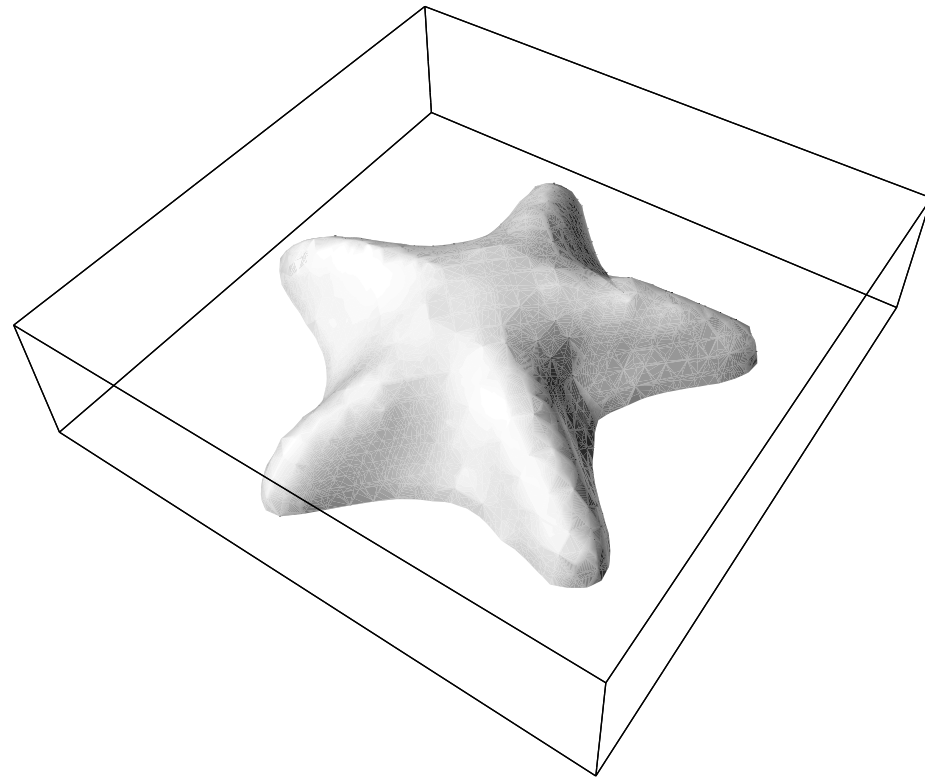
Design dependent loads

$$\begin{cases} -\operatorname{div} (A e(u)) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ (A e(u))n = g = p_0 Id & \text{on } \Gamma_N \cup \Gamma \end{cases}$$

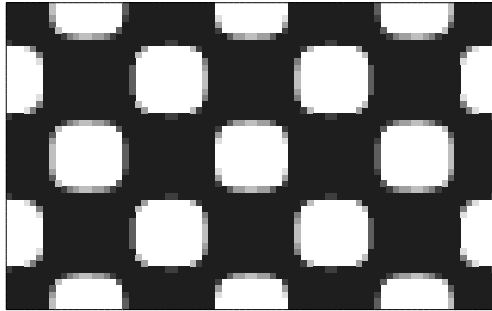
## Compliance minimization

$$J(\Omega) = \int_{\Gamma \cup \Gamma_N} g \cdot u \, ds = \int_{\Omega} A e(u) \cdot e(u) \, dx,$$

$$J'(\Omega_0)(\theta) = \int_{\Gamma_0} \left( 2 \left[ \frac{\partial(g \cdot u)}{\partial n} + Hg \cdot u \right] - Ae(u) \cdot e(u) \right) \theta \cdot n \, ds,$$



Non-linear elasticity



## Conclusion

- ➔ Efficient method.
- ➔ With a **good initialization**, comparable to the homogenization method.
- ➔ Can be pre-processed by the homogenization method.
- ➔ Can handle **non-linear** models, design dependent loads and any smooth objective function.