On the Statistics of Dimension: Fractal Modulation and Quantum Fractional Dynamics

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Abstract

Developing accurate and versatile models for simulating noise is important in many applications of digital signal and image processing. This paper considers a new approach to modelling a non-stationary stochastic field $u(x, t)$ using a fractional partial differential equation of (time dependent) order $q(t)$ given by

$$\left[\frac{\partial^2}{\partial x^2} - \tau^{q(t)}\frac{\partial^{q(t)}}{\partial t^{q(t)}}\right] u(x,t) = -F(x,t), \quad -\infty < q(t) < \infty, \quad \forall t$$

where τ is a constant and F and q are stochastic functions. The theoretical background and ideas upon which this equation is based are presented. This includes a brief overview of random fractal walks, Lévy flights, fractional calculus and fractional dynamics. A general solution to this equation is then considered using a Green's function method from which an asymptotic solution is derived for $x \to 0$. Numerical algorithms for computing this solution are introduced and results presented to illustrate its characteristics which depend on the random behaviour of $q(t)$. One important and interesting aspect of this approach is concerned with the effect of changing the statistics [i.e. the Probability Density Function (PDF) of $q(t)$] used to 'drive' the solution. Since q is a dimension (the 'Fourier dimension' which is related to the fractal dimension), the introduction of a PDF associated with $q(t)$ leads directly to the notion of the 'statistics of dimension'. In the context of $q(t)$ being a random variable, the asymptotic solution for $u(x, t)$ when $x \to 0$ yields special behaviour when $q = 2$. This is characterised by events that are analogous to Lévy-type flights which we call 'Brownian transients'. We also address the inverse problem in which a discrete solution is found for $q(t)$ from a known stochastic field $u(x, t)$ when $x \to 0$. Two applications are considered: (i) fractal modulation, where $q(t)$ is assigned just two states for all t (analogous to frequency modulation but where a bit stream is forced to 'look like' background fractal noise); (ii) financial analysis in which a new macro-economic volatility measure is considered under the assumption that $q(t)$ is a non-stationary Gaussian distributed field.

1 Introduction

Developing mathematical models to simulate and analyse noise has an important role in digital signal and image processing. Computer generated noise is routinely used to test the robustness of different types of algorithms (e.g. algorithms whose principal goal is to extract information from noise), it is used for data encryption and is even used to enhance or amplify signals through 'stochastic resonance'. Accurate statistical models for noise (e.g. the Probability Distribution Function or the Characteristic Function) are particularly important in image restoration using Bayesian estimation [1], Maximum Entropy methods for signal and image reconstruction [2] and in image segmentation of coherent images in which 'speckle' (arguably a special type of noise, i.e. coherent Gaussian noise) is a prominent feature [3]. The noise characteristics of a given imaging system often dictate the type of filters that are used to process and analyse the data. Noise simulation is also important in the synthesis of images used in computer graphics and computer animation systems in which fractal noise has a special place (e.g. [4] and [5]).

The application of fractal geometry for modelling naturally occurring signals and images is well known. This is due to the fact that the 'statistics' and spectral characteristics of Random Scaling Fractals (RSFs) are consistent with many objects found in nature, a characteristic that is compounded in the term 'statistical self-affinity'. This term refers to random processes whose statistics are scale invariant. A RSF signal is one whose Probability Density Function (PDF) remains the same irrespective of the scale over which the signal is sampled. Thus, as we zoom into a RSF signal, although the time signature changes, the PDF of the signal remains the same (a scaled down version of the original) - a concept that is aptly compounded in the Chinese proverb: 'In every way one can see the shape of the sea'.

Many signals found in nature are statistically self-affine. These include a wide range of noise sources including background cosmic radiation (at most frequencies) for example. In addition, certain speech signals (representative of fricatives) exhibit the characteristics of RSFs as do other signals such as financial time series, seismic signals and so on. The incredible range of vastly different systems which exhibit random fractal behaviour is leading more and more researchers to consider statistical self-affinity to be a universal law, a law that is particularly evident in systems which are undergoing a phase transition (e.g. [6], [7]).

In a stable state, the behaviour of the elements from which a system is composed depends primarily on their local neighbours and the statistics of the system is not self-affine. In a critical state, the elements become connected, propagating 'order' throughout the system in the sense that the statistical characteristics of the system are self-affine with 'system wide' correlations. This is more to do with the connectivity of the elements than the elements themselves [8]. (Critical states can of course be stable in the dynamical sense.) Moreover, critical states appear to be governed by the universal power law

$$\text{System(size)} \propto \frac{1}{\text{size}^q}$$

where q is a non-integer value. Here, the term 'System' is a generic term representative of some definable parameter that can be measured experimentally over different scales of a certain 'size'. This power law is the principal 'signature' that the system is behaving in a statistically self-affine way. There are a wide variety of examples which demonstrate this power law. For example, the growth rate of companies tends to diminishes with size, irrespective of the type of business being conducted; typical US companies are characterised by $q \in [0.7, 0.9]$. The frequency of the creation and extinction of species (as revealed through a growing number of fossil records) is starting to indicate that the pattern of fitness for survival is statistically self-affine. This result can be simulated by relatively simple iteration function systems such as in the Bak-Sneppen model which yields self-affine distributions for the fitness of different species over time [8]. The distribution of base pairs in DNA is statistically self-affine, i.e. the frequency of occurrence of Adenine-Thymine and Cytosine-Guanine in a DNA molecule is the same at different scales. DNA is in effect, a self-affine bit stream.

Conventional RSF models are based on stationary processes in which the 'statistics' of the signal are invariant of time. However, many signals exhibit non-stationary behaviour. In addition, many signals exhibit episodes which are rare but extreme (sudden changes in amplitude and/or frequency), events which are statistically inconsistent with the 'normal' behaviour of the signal. These episodes include so-called Lévy-type flights in cases when the statistics of the signal conform to that of a Lévy-type distribution. Lévy statistics is one of a number of power-law based stochastic models that are currently being used to investigate the 'strange kinetics' of systems undergoing phase transitions including hot plasmas, super-fluids, super-conducting materials and economic systems (e.g. [9] and [10]).

In this paper a completely new model is considered which attempts to unify these features of stochasticism using a phenomenological approach. The model is based on a modification to the diffusion equation in which a fractional temporal derivative to an order $q(t)$ is introduced. By considering a model for the PDF of $q(t)$, a solution is derived which allows a stochastic field to be computed that is fractal, non-stationary and where the likelihood of events we call 'Brownian flights' (effects which are analogous to Lévy-type flights) can be altered via the PDF. Section 2 gives a brief overview of the different aspects of stochasticism which form the background to the postulation and analysis of this model.

2 Stochastic modelling (briefly) revisited

There are two principal criteria used to define the characteristics of a stochastic field:

(i) The PDF or equivalently the Characteristic Function (i.e. the Fourier transform of the PDF).

(ii) The Power Spectral Density Function (PSDF).

The PSDF is the function that describes the envelope or shape of the power spectrum of the field and is related to the autocorrelation function of a signal through the autocorrelation theorem. In this sense the PSDF measures the time correlations of a signal. For example, zero-mean white Gaussian noise is a stochastic field characterised by a PSDF that is effectively constant over all frequencies and has a PDF with a Gaussian profile whose mean is zero.

Stochastic fields can of course be characterised using transforms other than the Fourier transform (from which the PSDF is obtained) such as the Wigner transform [11]. Wavelets are also being used effectively to analyse and process noise in signals and images (e.g. [12] and references therein). However, the conventional PDF-PSDF approach serves many purposes and providing a detailed comparison of other techniques of stochastic analysis is beyond the scope of this publication.

There are two conventional approaches to simulating a stochastic field. The first of these is based on predicting the PDF (or the Characteristic Function) theoretically (if possible). A pseudo random number generator is then designed whose output provides a discrete stochastic field that is characteristic of the predicted PDF. For example, a Gaussian pseudo random number generator can be designed using the Box-Muller transformation operating on uniform deviates [13]. The second approach is based on considering the PSDF of a signal which, like the PDF, is ideally derived theoretically. The stochastic field is then typically simulated by filtering white Gaussian noise.

Many stochastic fields observed in nature have two fundamental properties:

(i) the PSDF is determined by irrational power laws, i.e. $1/\mid\omega\mid^q$ noise where ω is the (angular) frequency and q is a non-integer number;

(ii) the field is statistical self-affine.

2.1 What is a good stochastic model?

A 'good' stochastic model is one that accurately predicts both the PDF and the PSDF of the data. It should take into account the fact that in general, stochastic processes are non-stationary. In addition, it should ideally include behaviour that is characteristic of fractal walks and be able to produce rare but extreme events in which large deviations from the norm occur - effects that might be considered analogous to, but not necessarily related to Lévy-type flights.

Remark: Although we refer to Lévy-type flights and/or distributions throughout this paper, such references should be taken to be qualitative in nature and are being used only in terms of introducing an analogy to a strictly well defined process or term (i.e. Lévy flight or Lévy distribution respectively) which can yield analogous or similar effects.

Lévy Flights Named after the French mathematician Paul Lévy (1886-1971), Lévy fights are random walks whose distribution has infinite moments. The statistics of (conventional) physical systems is usually concerned with stochastic fields which have PDFs where (at least) the first two moments (the mean and variance) are well defined and finite. Lévy statistics is concerned with statistical systems where all the moments (starting with the mean) are infinite [14].

Lévy Distributions Many distributions exist where the mean and variance are finite but are not representative of the process, e.g. the tail of the distribution is significant - where rare, but extreme events occur. These distributions include Lévy-type distributions. Lévy's original approach to deriving such distributions is based on the following question: under what circumstances does the distribution associated with a random walk of a few steps look the same as the distribution after many steps (except for scaling)? This question is effectively the same as asking under what circumstances do we obtain a random walk that is statistically self-affine.

For a 1D random walk, the Characteristic Function $P(k)$ of such a distribution $p(x)$ was first shown by Lévy to be given by (for symmetric distributions only)

$$P(k) = \exp(-a \mid k \mid^q), \quad 0 < q \leq 2 \tag{2.1}$$

where a is a (positive) constant. When $q = 0$

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-a) \exp(ikx) dk = \exp(-a)\delta(x)$$

and the distribution is concentrated solely at the origin as described by the delta function $\delta(x)$. When $q = 1$ the Cauchy distribution

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-a \mid k \mid) \exp(ikx) dk = \frac{1}{\pi} \frac{a}{a^2 + x^2}$$

is obtained and when $q = 2$, $p(x)$ is characterised by the Gaussian distribution

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-ak^2) \exp(ikx) dk = \frac{1}{2\pi} \sqrt{\frac{\pi}{a}} \exp[-x^2/(4a)].$$

This is the only Lévy distribution for which the second moment is finite. The Cauchy distribution has a relatively long tail compared with the Gaussian distribution indicating that a stochastic field described by a Cauchy distribution is likely to have more extreme variations when compared to a Gaussian distributed field. In general, the characteristic function given by equation (2.1) corresponds to an asymptotic PDF of the form

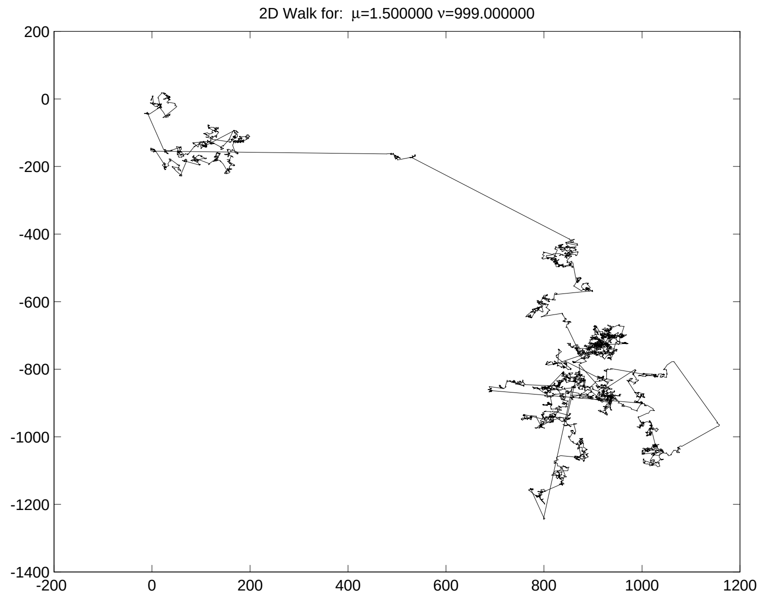


Figure 1. Example of a Lévy Flight - a random walk with long range correlations described by the PDF $\mid x \mid^{-q-1}$ with $q = 1.5$.

$$p(x) \sim \frac{1}{\mid x \mid^{1+q}}, \quad x \to \infty$$

The stochastic fields described by this asymptotic result therefore have long range correlations and do not conform to the classical Central Limit Theorem. Also, the trace of the sites visited by a random field described by a PDF of this type forms a set of fractal dimension q. Hence, Lévy distributed fields are statistically self-affine. An example of a Lévy flight is given in Figure 1 for $q = 1.5$ which illustrates the effect of random walks with long range correlations.

2.2 Fractional Calculus and Fractional Dynamics

Fractional Calculus: In a famous letter from l'Hospital to Leibnitz written in 1695, l'Hospital asked the following question: 'Given that $d^n f/dx^n$ exists for all integer n, what if n be $\frac{1}{2}$'. The reply from Leibnitz was all the more interesting: 'It will lead to a paradox ... From this paradox, one day useful consequences will be drawn'.

Fractional calculus has been studied for many years by some of the great names of mathematics since the development of (integer) calculus in the late seventeenth century. Relatively few papers and books exist on such a naturally important subject. However, a study of the works in this area of mathematics clearly show that the ideas used to define a fractional differential and a fractional integral are based on definitions which are in effect, little more than generalizations of results obtained using integer calculus (e.g. [15], [16], [17] and [18]). The classical fractional integral operators are the Riemann-Liouville transform

$$\hat{I}^{-q}f(x)=\frac{1}{\Gamma(q)}\int\limits_0^x\frac{f(y)}{(x-y)^{1-q}}dy,\quad q>0 \tag{2.2}$$

and the Weyl transform

$$\hat{I}^{-q}f(x)=\frac{1}{\Gamma(q)}\int\limits_x^\infty\frac{f(y)}{(x-y)^{1-q}}dy,\quad q>0$$

where

$$\Gamma(q)=\int\limits_0^\infty x^{q-1}\exp(-x)dx$$

For integer values of q (i.e. when $q=n$ where n is a non-negative integer), equation (2.2) reduces to the standard Riemann integral. Equation (2.2) is just a (causal) convolution of the function $f(x)$ with $x^{q-1}/\Gamma(q)$. For fractional differentiation, we perform a fractional integration of appropriate order and then differentiate to an appropriate integer order. The reason for this is that direct fractional differentiation can lead to divergent integrals. Thus, the fractional differential operator $\hat{D}^q$ for $q>0$ is given by

$$\hat{D}^q f(x)\equiv\frac{d^q}{dx^q}f(x)=\frac{d^n}{dx^n}[\hat{I}^{q-n}f(x)],\quad q-n<0 \tag{2.3}$$

Another (conventional) approach to defining a fractional differentiation is based on using the formula for n^{th} order differentiation (obtained by considering the definitions for the first, second, third etc. differentials using backward differencing) and then generalising the formula by replacing n with q. This approach provides us with the result [15]

$$\hat{D}^q f(x)=\lim_{N\to\infty}\left[\frac{(x/N)^{-q}}{\Gamma(-q)}\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(j+1)}f\left(x-j\frac{x}{N}\right)\right]$$

A review of this result shows that for $q=1$, this is a point process but for other values it is not, i.e. the evaluation of a fractional derivative depends on the history of the function in question. Thus, unlike an integer differential, a fractional differential has 'memory'. Although the memory of this process fades, it does not so quickly enough to allow truncation of the series in order to retain acceptable accuracy. The concept of memory association can also be seen from the definition given in equation (2.3) which depends on equation (2.2) in which the value of $\hat{I}^{-q}f(x)$ at a point x depends on the behaviour of $f(x)$ from 0 to

x via a convolution with the kernel $x^{q-1}/\Gamma(q)$. The convolution process is of course dependent on the history of the function $f(x)$ for a given kernel and thus, in this context, we can consider a fractional derivative defined via equation (2.3) to have memory.

In addition to the conventional and classical definitions of fractional derivatives and integrals, more general definitions have recently been developed including the Erdélyi-Kober operators, hypergeometric operators and operators involving other special functions such as the Maijer G-function and the Fox H-function [19]. Moreover, all such operators leading to a fractional integral of the Riemann-Liouville type and the Weyl type would appear (through induction) to have the general forms

$$\hat{I}^{-q}f(x) = x^{-q-1}\int\limits_0^x \Phi\left(\frac{y}{x}\right) y^q f(y)dy$$

and

$$\hat{I}^{-q}f(x) = x^{q}\int\limits_x^\infty \Phi\left(\frac{x}{y}\right) y^{-q-1} f(y)dy$$

respectively, where the kernel Φ is an arbitrary continuous function so that the integrals above make sense in sufficiently large functional spaces.

One of the most useful definitions of a fractional derivative, in terms of its 'ease of use' and wide ranging applications, is based on the Fourier transform, i.e.

$$\frac{d^q}{dx^q}f(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}(ik)^q F(k)\exp(ikx)dk, \quad q>0 \tag{2.4}$$

where F is the Fourier transform of f. When $q = 1, 2, 3...$, this definition reduces to a well known result that is trivial to derive in which for example, the 'filter' ik is referred to as a 'differentiator'. We can consider an extension of this result by letting $q < 0$ which provides a definition for a fractional integral where in the case of $q = -1$, the filter $(ik)^{-1}$ is an 'integrator'. When $q = 0$ we just have $f(x)$ expressed in terms of its Fourier transform $F(k)$. This Fourier based definition of a fractional derivative can be extended further to include a definition for a 'fractional Laplacean' ∇^q where for d dimensions

$$\nabla^q = -\frac{1}{(2\pi)^d}\int d^n\mathbf{k}\exp(i\mathbf{k}\cdot\mathbf{r})k^q, \quad k = \mid \mathbf{k} \mid \tag{2.5}$$

are $\mathbf{r}$ is a d-dimensional vector. This is the fractional Riesz operator. It is designed to provide a result that is compatible with the case of $q = 2$ for $d > 1$ (which is the reason for introducing the negative sign). Another equally valid generalisation is

$$\nabla^q = \frac{1}{(2\pi)^d} \int d^n \mathbf{k} \exp(i\mathbf{k} \cdot \mathbf{r})(ik)^q, \quad k = \mid \mathbf{k} \mid$$

which introduces a q dependent phase factor of $\pi q/2$ into the operator leading to compatibility with equation (2.4) for $d = 1$.

In general terms, although it is possible to compute fractional integrals and differentials using the results discussed above, neither a fractional differential (or a fractional integral) appear to have a geometric and/or physical interpretation unlike an integer differential, which can be considered to be a map of the gradient of a piecewise continuous function (for which a picture can be drawn at any point by zooming into the function and arguing that over a small enough portion of the function, a straight line can be considered which has a definable gradient). However, when generalised functions are introduced, the concept of definable gradients requires attention and generalised functions such as the delta function $\delta(x)$ are only defined properly in terms of the role they play in certain transforms [20], e.g. the sampling property of the delta function in which $\delta(x)$ is defined as that function for which

$$\int_{-\infty}^{\infty} \delta(x_0 - x) f(x) dx = f(x_0)$$

There have been some attempts to develop a geometric interpretation of a fractional differential and/or integral. In the few published works that consider this problem, relationships have been explored between fractional calculus and fractal geometry. For example, Nigmatullin [21] discusses the possible connection between a fractional integral and a Cantor set. Rutman [22] on the other hand, concludes that no direct relationship between fractional calculus and fractal geometry has yet been established. It is arguable that there may never be a proper geometrical interpretation of a derivative in the same way that generalised functions do not have a proper geometrical interpretation (at least in the Euclidean sense). On the other hand, the use of non-standard analysis [23] could provide a way forward by attempting to define a fractional differential in an appropriate hyper-space in a way that is analogous to the hyper-space definitions of integer differentials of generalised functions such as the delta function. A generalisation of the well known result

$$\int_{-\infty}^{\infty} \delta^{(n)}(x - y) f(y) dy = \frac{d^n}{dx^n} f(x)$$

to

$$\int_{-\infty}^{\infty} \delta^{(q)}(x - y) f(y) dy = \frac{d^q}{dx^q} f(x)$$

could be of value in formulating a hyper-space theory of fractional calculus.

An open question Although there has been a rapid increase in the number of publications on fractional calculus over the past ten years, there still remains a fundamental and open question: Is there a geometrical representation of a fractional derivative? If not, can one prove that a graphical representation of a fractional derivative does not exist? The general consensus of opinion is that there is no geometrical interpretation of a derivative of fractional order and that if there is, then as Virginia Kiryakova concludes in her book on 'Generalised Fractional Calculus and Applications' [19], '... it is likely to be found in our fractal world'. A simple illustration of this 'likelihood' can be obtined by using a generalised dimensional analysis. For example, consider the following equations:

$$\frac{d}{dx}f(x) = g(x) \quad \text{and} \quad \frac{d^2}{dx^2}f(x) = g(x).$$

Suppose that g is dimensionless and that x is taken to be length. Then provided both equations have a solution, these solutions are of dimensions 1 and 2 respectively. Thus, if the equation

$$\frac{d^q}{dx^q}f(x) = g(x), \quad q \in [1, 2]$$

has a solution, then the solution should be of dimension q. In this sense, we can consider a definition of a fractal in terms of a solution to a fractional differential equation. Although self consistent, this argument suffers from the same problem as all the others associated with fractional calculus in that all definitions and results appear to be based on a generalisation of those associated with integer calculus. On the other hand, one might argue that decimal numbers are based on a generalisation of integer numbers but this does not mean that decimal numbers are not useful!

Fractional dynamics The lack of any consistent geometrical interpretation of a fractional derivative is unfortunate because the application of ordinary and partial differential equations of fractional order for modelling stochastic processes is starting to be investigated by a growing number of mathematicians and theoretical physicists. Mathematical modelling using (time dependent) fractional Partial Differential Equations (PDEs) is generally known as fractional dynamics [24] and is currently a 'hot topic' in mathematics and theoretical physics. A number of recent publications (e.g. [25]) have shown a close relationship between fractional diffusion equations of the type (where p is the space-time dependent PDF and τ is the generalised diffusivity)

$$\nabla^2 p - \tau\frac{\partial^q}{\partial t^q}p = 0, \quad 0 < q \leq 1$$

and

$$\nabla^q p - \tau\frac{\partial}{\partial t}p = 0, \quad 0 < q \leq 2 \tag{2.6}$$

and continuous time random walks with either temporal or spatial scale invariance (fractal walks). Fractional diffusion equations of this type have been shown to produce a framework for the description of anomalous diffusion phenomena and Lévy-type behaviour. In addition, certain classes of fractional differential equations are known to yield Lévy-type distributions. For example, the normalised one-sided Lévy-type PDF

$$p(x) = \frac{a^q}{\Gamma(q)} \frac{\exp(-a/x)}{x^{1+q}}, \quad a > 0, \quad x > 0$$

is a solution of the fractional integral equation [26]

$$x^{2q} p(x) = a^q \hat{I}^{-q} p(x)$$

where

$$\hat{I}^{-q} p(x) = \frac{1}{\Gamma(q)} \int_0^x \frac{p(y)}{(x-y)^{1-q}} dy, \quad q > 0$$

Another example involves the solution to the anomalous diffusion equation given by equation (2.6). Fourier transforming this equation and using the fractional Riesz operator defined in equation (2.5), we have [27]

$$\frac{\partial}{\partial t} P(k,t) = -\frac{1}{\tau} k^q P(k,t)$$

which has a solution of the form

$$P(k,t) = \exp(-tk^q/\tau)$$

Comparing this result with equation (2.1), we recognise the Characteristic Function of a Lévy distribution with $a = t/\tau$. Some authors have extended the problem further by considering a fractal based generalisation of the Fokker-Planck-Kolomogorov (FPK) equation [28]

$$\frac{\partial^q}{\partial t^q} p(x,t) = \frac{\partial^\beta}{\partial x^\beta} [s(x) p(x,t)]$$

where s is an arbitrary function and $0 < q \leq 1$, $0 < \beta \leq 2$. This equation is referred to as the fractal FPK equation; the standard FPK equation is of course recovered for $q = 1$ and $\beta = 2$. The Characteristic Function associated with $p(x,t)$ is given by

$$P(k,t) = \exp(-ak^\beta t^q)$$

where a is a constant which again, defines a Lévy distribution. Finally, d-dimensional fractional master equations of the type ([29] and [30] for example)

$$\frac{\partial^q}{\partial t^q}p(\mathbf{r},t)=\sum_{\mathbf{s}}w(\mathbf{r}-\mathbf{s})p(\mathbf{s},t),\quad 0<q\leq 1$$

are being used to model non-equilibrium phase transitions where p denotes the probability of finding the diffusing entity at a position $\mathbf{r}\in R^d$ at time t (assuming that it was at the origin $\mathbf{r}=\mathbf{0}$ at time $t=0$) and w are the fractional transition rates which measure the propensity for a displacement $\mathbf{r}$ in units of $1/(\text{time})^q$. These equations conform to the general theory of continuous time random walks and provide models for random walks of fractal time.

A short defence by Erwin Schrödinger A study of the current publications on fractional dynamics reveals that the fractional PDEs proposed are being solved through application of the Fourier based definition of a fractional differential. One could therefore argue that such equations are being 'invented' to yield a desired result in Fourier space (e.g. the Characteristic Function for a Lévy distribution). Moreover, definitions of fractional derivatives are being considered in such a way that they lead to self-consistent results for the integer case [e.g. the Riesz definition of a fractional Laplacean given in equation (2.5)]. In this sense, taking well known PDEs and 'fractionalising' them in such a way that a Fourier transform yields the desired result might justifiably be considered as an example of phenomenology at its worst! It is clearly more desirable to derive a PDE from basic principles (based on the known physical laws) and then solve the equation using appropriate solution methods (some of which are based on certain integral transforms including the Fourier transform!). On the other hand, there are a number of important PDEs whose form is based on postulation alone and cannot be derived or proved but nevertheless yield remarkably accurate models for the behaviour of a physical system informed by experiment. One such equation is the Schrödinger equation which is based on two fundamental ideas that were later confirmed experimentally for particles of size $<10^{-10}$m and provides an accurate model for the behaviour of particles of size down to 10^{-15}m. The first of these was that energy E is proportional to frequency ν or that $E=h\nu$ where $h\sim 6.6\times 10^{-34}$ joule-seconds is Planck's constant. This simple but non-intuitive relationship was proposed by Max Planck in 1900 and verified experimentally through Einstein's theory for the photoelectric effect (published in 1905) and the black body radiation law. The second idea was due to de Broglie who in 1922 first proposed that momentum p is inversely proportional to wavelength λ or $p=h/\lambda$; a result that was verified experimentally in 1927 by observing Bragg diffraction patterns for particles of different mass. Thus, writing $\bar{h}=h/2\pi$, $\omega=2\pi\nu$ and $k=2\pi/\lambda$ we have

$$E=\bar{h}\omega\quad\text{and}\quad p=\bar{h}k\quad\text{or}\quad\mathbf{p}=\bar{h}\mathbf{k}$$

since momentum is a vector quantity in general. In classical mechanics, the energy of a particle of mass m under the influence of a potential $V(\mathbf{r})$ is given by

$$E = \frac{p^2}{2m} + V$$

where $p =| \mathbf{p} |$ and $\mathbf{r}$ is a 3D vector. Hence, from the equations for E and p above, it follows that we can construct the dispersion relation

$$\bar{h}\omega = \frac{\bar{h}^2 k^2}{2m} + V \tag{2.7}$$

where $k =| \mathbf{k} |$. The main point here is that Schrödinger's equation is based on postulating a particular type PDE which yields this relationship upon Fourier transformation. In particular, if we define the 4D inverse Fourier transform to be

$$u(\mathbf{r}, t) = \hat{F}^{-1}[U(\mathbf{k}, \omega)] = \frac{1}{(2\pi)^4} \int U(\mathbf{k}, \omega) \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)] d^3\mathbf{k} d\omega$$

then

$$\frac{\partial u}{\partial t} = \hat{F}^{-1}[(-i\omega)U] \quad \text{and} \quad \nabla^2 u = \hat{F}^{-1}[(-k^2)U]$$

Equation (2.7) can be written as

$$\hat{F}^{-1}\left[\bar{h}i(-i\omega)U\right] = \hat{F}^{-1}\left[\frac{\bar{h}^2 k^2}{2m} U\right] + \hat{F}^{-1}[VU]$$

which is equivalent to writing

$$i\bar{h}\frac{\partial u}{\partial t} = -\frac{\bar{h}^2}{2m}\nabla^2 u + Vu$$

In this sense, Schrödinger's equation is based on the introduction of the wave operators

$$E \to i\bar{h}\frac{\partial}{\partial t} \quad \text{and} \quad \mathbf{p} \to i\bar{h}\nabla$$

The physical basis for introducing these operators is that in quantum mechanics, particles can be considered to be wave-packets described by the wave function u in which $| u |^2$ is taken to be the probability of the particle existing at a position $\mathbf{r}$ at a time t. This leads to the so called wave-particle duality principle of matter which has and continues to be a significant philosophical problem. Nevertheless, the use of Schrödinger's equation for modelling the behaviour of matter, and in particular, its applications to solid state physics has arguably done more for the IT revolution (in terms of understanding the behaviour of semiconductors) than any other PDE.

3 A new non-stationary fractional dynamic model

A common theme associated with the fractional dynamic models discused in the previous section is that they describe stationary random processes in terms of solutions to a PDF. In this paper, we postulate a PDE whose characteristics incorporate behaviour that describes non-stationary fractal walks and Lévy-type flights in terms of a solution to the stochastic field itself (and **not** its PDF). Also, as shall be discussed later, within the context of the solution derived in this paper, these so called Lévy-type flights are actually the result of randomly introducing Brownian noise over a short period of time into an otherwise non-stationary fractal signal. In this sense, they have nothing to do with Lévy flights as such, but produce results that may be considered to be analogous to them. We call these effects 'Brownian transients' which have longer time correlations than fractal noise.

Suppose we consider an inhomogeneous fractal diffusion equation of the form

$$\left[\frac{\partial^2}{\partial x^2} - \tau\frac{\partial^q}{\partial t^q}\right] u(x,t) = -F(x,t), \quad 0 < q \leq 1$$

where τ is a constant, F is a stochastic source term with some PDF and u is the stochastic field whose solution we require. When $q = 1$ we obtain the diffusion equation but in general, a solution to this equation will provide stationary temporal fractal walks - random walks of fractal time. One way of introducing a (first order) non-stationary process is to let $F(x,t)$ be a functional, i.e. consider a source term of the form $F[x,t,\alpha(x),\beta(t)]$ where α and β are arbitrary functions. For example, suppose that we write F in separable form $F(x,t) = f(x)n(t)$ and that $n(t)$ is a random variable of time with a normal PDF given by

$$\Pr[n(t)] = \frac{1}{\sigma\sqrt{2\pi}} \exp[-(\mu - n)^2/2\sigma^2], \quad -\infty < n < \infty$$

where μ and σ are the mean and standard deviation respectively. By letting μ and/or σ be functions of t, we can introduce time variations in the mean and/or standard deviation respectively. In this case, varying the mean will cause the range of $n(t)$ to change with time and varying the standard deviation will change the variance of $n(t)$ with time. Note that in this case, the form of the distribution of the field remains the same - it is a time varying Gaussian field. A more general statement of a non-stationary process is one in which the distribution itself changes with time.

Another way of introducing non-stationarity is through q by letting it become a function of time t. Suppose that in addition to this, we extend the range of q to include the values 0 and 2 so that $0 \leq q \leq 2$. This idea immediately leads us to an interesting consequence because with q in this range, we can choose $q = 1$ to get the diffusion equation but also choose $q = 2$ to obtain an entirely different equation, namely, the wave equation. Choosing (quite arbitrarily) q to be in this range, leads to control over the basic physical characteristics of the equation so that we can define a statistic mode when $q = 0$, a diffusive mode when $q = 1$ and a propagative mode when $q = 2$. In this case, non-stationarity is introduced through the use of a time varying fractional derivative whose values can change the physical meaning of the equation. Since the range of q has been chosen arbitrarily, we generalise further and consider the equation

$$\left[\frac{\partial^2}{\partial x^2} - \tau^{q(t)}\frac{\partial^{q(t)}}{\partial t^{q(t)}}\right] u(x,t) = -F(x,t), \quad -\infty < q(t) < \infty, \quad \forall t \tag{3.1}$$

When $q = 0$ $\forall t$, the time dependent behaviour is determined by the source function alone; when $q = 1$ $\forall t$, u describes a diffusive process where τ is the 'diffusivity' (the inverse of the coefficient of diffusion); when $q = 2$ we have a propagative process where τ is the 'slowness' (the inverse of the wave speed). The latter process should be expected to 'propagate information' more rapidly than a diffusive process leading to transients or 'flights' of some type. We shall refer to the parameter q as the Fourier dimension which, for a continuous fractal signal, is related to the conventional definition of the fractal (i.e. the 'Similarity', 'Minkowksi' or 'Box Counting') dimension D by [31]

$$q \leq D + \frac{1}{2}, \quad 1 < D < 2 \tag{3.2}$$

How should we choose $q(t)$**?** Since $q(t)$ 'drives' the non-stationary behaviour of u, the way in which we model $q(t)$ is crucial. It is arguable that the changes in the statistical characteristics of u which lead to its non-stationary behaviour should in themselves be random. Thus, suppose that we let the Fourier dimension at a time t be chosen randomly, a randomness that is determined by some PDF. In this case, the non-stationary characteristics of u will be determined by the PDF (and associated parameters) alone. Also, since q is a dimension, we can consider our model to be based on the 'statistics of dimension'.

There are a variety of PDFs that could be applied (including a Lévy distribution) which will in turn effect the range of q. By varying the exact nature of the distribution considered, we can 'drive' the non-stationary behaviour of u in different ways. For example, suppose we consider a system which is assumed to be primarily diffusive; then a 'normal' PDF of the type

$$\Pr[q(t)] = \frac{1}{\sigma\sqrt{2\pi}} \exp[-(q-1)^2/2\sigma^2], \quad -\infty < q < \infty$$

will ensure that u is entirely diffusive when $\sigma \to 0$. However, as σ is increased in value, the likelihood of $q = 2$ (and $q = 0$) becomes larger. In other words, the standard deviation provides control over the likelihood of the process becoming propagative. If for example, we consider a Gamma distribution given by

$$\Pr[q(t)] = \begin{cases} \frac{1}{\beta^\alpha}\frac{1}{\Gamma(\alpha)} q^{\alpha-1}\exp(-q/\beta), & q > 0 \\ 0, & q \leq 0 \end{cases}$$

where $\alpha > 0$ and $\beta > 0$, then q lies in the positive half space alone with mean and variance given by

$$\mu = \alpha\beta \quad \text{and} \quad \sigma^2 = \alpha\beta^2$$

respectively. PDFs could also be considered which are of compact support such as the Beta distribution given by

$$\Pr[q(t)] = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} q^{\alpha-1}(1-q)^{\beta-1}, & 0 < q < 1 \\ 0, & \text{otherwise} \end{cases}$$

where α and β are positive constants. Here, the mean and variance are

$$\mu = \frac{\alpha}{\alpha+\beta} \quad \text{and} \quad \sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

respectively and for $\alpha > 1$ and $\beta > 1$ there is a unique mode at the value

$$x_{\text{mode}} = \frac{\alpha-1}{\alpha+\beta+2}$$

Irrespective of the type of distribution that is considered, equation (3.1) poses a fundamental problem which is how to define and work with the term

$$\frac{\partial^{q(t)}}{\partial t^{q(t)}} u(x,t)$$

Given the result (for constant q)

$$\frac{\partial^q}{\partial t^q} u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\omega)^q U(x,\omega) \exp(i\omega t) d\omega, \quad -\infty < q < \infty$$

we might generalise as follows:

$$\frac{\partial^{q(t)}}{\partial t^{q(t)}} u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (i\omega)^{q(t)} U(x,\omega) \exp(i\omega t) d\omega$$

However, if we consider the case where the Fourier dimension is a relatively slowly varying function of t, then we can legitimately consider $q(t)$ to be composed of a sequence of different states $q_i = q(t_i)$. This allows us to develop a stationary solution for a fixed q over a fixed period of time. Non-stationary behaviour can then be introduced by using the same solution for different values or 'quanta' q_i over fixed (or varying) periods of time and concatenating the solutions for all q_i. Quantizing $q(t)$ in this way also allows us to define a process that we call 'Fractal Modulation' (FM) whereby $q(t)$ is assigned two states, q_1 and q_2 where $q_1 \neq q_2$. Typically the probability of obtaining q_1 or q_2 is considered to be 0.5 so that no weighting can be attributed to either state. By letting these states correspond to 0 and 1 in a bit stream, we can consider the application of FM to a digital communications systems. This is one of the applications considered later in this paper.

4 Green's function solution

We shall consider a Green's function solution to equation (3.1) for constant q when $F(x,t) = f(x)n(t)$ where $f(x)$ and $n(t)$ are both stochastic functions and where $n(t)$ is taken to be white Gaussian noise. Applying a separation of variables here is not strictly necessary. However, it yields a solution in which the terms affecting the temporal behaviour of $u(x,t)$ are clearly indentifiable. Thus, we require a general solution to the equation

$$\left(\frac{\partial^2}{\partial x^2} - \tau^q \frac{\partial^q}{\partial t^q}\right) u(x,t) = -f(x)n(t)$$

Let

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(x,\omega)\exp(i\omega t)d\omega, \quad \text{and} \quad n(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} N(\omega)\exp(i\omega t)d\omega$$

Then, using the result

$$\frac{\partial^q}{\partial t^q}u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(x,\omega)(i\omega)^q exp(i\omega t)d\omega$$

this fractional PDE transforms to

$$\left(\frac{\partial^2}{\partial x^2} + \Omega_q^2\right) U(x,\omega) = -f(x)N(\omega)$$

where we shall take

$$\Omega_q = i(i\omega\tau)^{\frac{q}{2}}$$

and ignore the case for $\Omega_q = -i(i\omega\tau)^{\frac{q}{2}}$. Defining the Green's function g to be the solution of

$$\left(\frac{\partial^2}{\partial x^2} + \Omega_q^2\right) g(x \mid x_0,\omega) = -\delta(x-x_0)N(\omega)$$

where δ is the delta function, we obtain the following solution:

$$U(x_0,\omega) = N(\omega)\int_{-\infty}^{\infty} g(x \mid x_0,k)f(x)dx \tag{4.1}$$

where

$$g(X,\omega)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\exp(iuX)}{(u+\Omega_q)(u-\Omega_q)}du, \quad X=\mid x-x_0\mid$$

The contour integral

$$\oint_C \frac{\exp(izX)}{(z+\Omega_q)(z-\Omega_q)}dz$$

has complex poles at $z=\pm\Omega_q$ which are q dependent (varying from $\pm i$ when $q=0$, through to $\pm i(i\omega\tau)^{1/2}$ when $q=1$ and on to $\mp\omega\tau$ when $q=2$ for example). For any value of q, we can compute this contour integral using the Residue Theorem. The contour must of course be chosen in such a way that it runs along the real axis in order to evaluate g. By chosing to evaluate the integral for a q-dependent pole in the z plane for $-\infty<x<\infty$ and $0\leq iy<\infty$ where $z=x+iy$ we obtain (through application of a semi-circular contour C)

$$g(x\mid x_0,k)=\frac{i}{2\Omega_q}\exp(i\Omega_q\mid x-x_0\mid) \tag{4.2}$$

under the assumption that Ω_q is finite. This result reduces to conventional solutions for cases when $q=1$ (diffusion equation) and $q=2$ (wave equation) as shall now be shown.

Wave equation solution When $q=2$, equation (4.1) provides a solution for the outgoing Green's function [32]. Thus with $\Omega_2=-\omega\tau$, we have

$$U(x_0,\omega)=\frac{N(\omega)}{2i\omega\tau}\int\limits_{-\infty}^{\infty}\exp(-i\omega\tau\mid x-x_0\mid)f(x)dx$$

and Fourier inverting we get

$$\begin{aligned} u(x_0,t) &= \frac{1}{2\tau}\int\limits_{-\infty}^{\infty}dx f(x)\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{N(\omega)}{i\omega}\exp(-i\omega\tau\mid x-x_0\mid)\exp(i\omega t)d\omega \\ &= \frac{1}{2\tau}\int\limits_{-\infty}^{\infty}dx f(x)\int\limits_{-\infty}^{t}n(t-\tau\mid x-x_0\mid)dt \end{aligned}$$

which describes the propagation of a wave travelling at velocity $1/\tau$ subject to variations in space and time as defined by $f(x)$ and $n(t)$ respectively. For example, when f and n are both delta functions,

$$u(x_0,t)=\frac{1}{2\tau}H(t-\tau\mid x-x_0\mid)$$

where H is the Heaviside step function defined by

$$H(y) = \begin{cases} 1, & y > 0; \\ 0, & y < 0. \end{cases}$$

This is a d'Alembertian type solution to the wave equation where the wavefront occurs at $t = \tau \mid x - x_0 \mid$ in the causal case.

Diffusion equation solution When $q = 1$ and $\Omega_1 = i\sqrt{i\omega\tau}$,

$$u(x_0, t) = \frac{1}{2} \int_{-\infty}^{\infty} dx f(x) \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(-\sqrt{i\omega\tau} \mid x - x_0 \mid)}{\sqrt{i\omega\tau}} N(\omega) \exp(i\omega t) d\omega$$

For $p = i\omega$, we can write this result in terms of a Bromwich integral (i.e. an inverse Laplace transform) and using the convolution theorem for Laplace transforms with the result [33]

$$\int_{c-i\infty}^{c+i\infty} \frac{\exp(-a\sqrt{p})}{\sqrt{p}} \exp(pt) dp = \frac{1}{\sqrt{\pi t}} \exp[-a^2/(4t)]$$

we obtain

$$u(x_0, t) = \frac{1}{2\sqrt{\tau}} \int_{-\infty}^{\infty} dx f(x) \int_{0}^{t} \frac{\exp[-\tau(x_0 - x)^2/(4t_0)]}{\sqrt{\pi t_0}} n(t - t_0) dt_0$$

Thus, if for example we consider the case when n is a delta function, the result reduces to

$$u(x_0, t) = \frac{1}{2\sqrt{\pi\tau t}} \int_{-\infty}^{\infty} f(x) H(t) \exp[-\tau(x_0 - x)^2/(4t)] dx, \quad t \to \infty$$

which describes classical diffusion in terms of the convolution of an initial source $f(x)$ (introduced at time $t = 0$) with a Gaussian function.

General series solution The evaluation of $u(x_0, t)$ from equation (4.1) via direct Fourier inversion for arbitrary values of q is not possible due to the irrational nature of the exponential function $\exp(i\Omega_q \mid x - x_0 \mid)$ with respect to ω. To obtain a general solution, we use the series representation of the exponential function and write equation (4.1) in the form

$$U(x_0, \omega) = \frac{iM_0 N(\omega)}{2\Omega_q} \left[1 + \sum_{m=1}^{\infty} \frac{(i\Omega_q)^m}{m!} \frac{M_m(x_0)}{M_0} \right]$$

where

$$M_m(x_0) = \int_{-\infty}^{\infty} f(x) \mid x - x_0 \mid^m dx$$

We can now Fourier invert term by term to develop a series solution. This requires us to consider three distinct cases.
Case 1: $q = 0$ Evaluation of $u(x_0, t)$ in this case is trivial since from equation (4.1)

$$U(x_0, \omega) = \frac{M(x_0)}{2} N(\omega) \quad \text{or} \quad u(x_0, t) = \frac{M(x_0)}{2} n(t) \tag{4.3}$$

where

$$M(x_0) = \int_{-\infty}^{\infty} \exp(- \mid x - x_0 \mid) f(x) dx$$

Case 2: $q > 0$ Fourier inverting, the first term in this series becomes

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{iN(\omega)M_0}{2\Omega_q} \exp(i\omega t) d\omega &= \frac{M_0}{2\tau^{\frac{q}{2}}} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{N(\omega)}{(i\omega)^{\frac{q}{2}}} \exp(i\omega t) d\omega \\ &= \frac{M_0}{2\tau^{\frac{q}{2}}} \frac{1}{\Gamma(q/2)} \int_{0}^{t} \frac{n(\xi)}{(t-\xi)^{1-(q/2)}} d\xi, \quad \mathrm{Re}[q] > 0 \end{aligned}$$

The second term is

$$-\frac{M_1}{2} \frac{1}{2\pi} \int_{-\infty}^{\infty} N(\omega) \exp(i\omega t) d\omega = -\frac{M_1}{2} n(t)$$

The third term is

$$-\frac{iM_2}{2.2!} \frac{1}{2\pi} \int_{-\infty}^{\infty} N(\omega) i (i\omega\tau)^{\frac{q}{2}} \exp(i\omega t) d\omega = \frac{M_2 \tau^{\frac{q}{2}}}{2.2!} \frac{d^{\frac{q}{2}}}{dt^{\frac{q}{2}}} n(t)$$

and the fourth and fifth terms become

$$\frac{M_3}{2.3!} \frac{1}{2\pi} \int_{-\infty}^{\infty} N(\omega) i^2 (i\omega\tau)^q \exp(i\omega t) d\omega = -\frac{M_3 \tau^q}{2.3!} \frac{d^q}{dt^q} n(t)$$

and

$$i \frac{M_4}{2.4!} \frac{1}{2\pi} \int_{-\infty}^{\infty} N(\omega) i^3 (i\omega\tau)^{\frac{3q}{2}} \exp(i\omega t) d\omega = \frac{M_4 \tau^{\frac{3q}{2}}}{2.4!} \frac{d^{\frac{3q}{2}}}{dt^{\frac{3q}{2}}} n(t)$$

respectively with similar results for all other terms. Thus, through induction, we can write $u(x_0,t)$ as a series of the form

$$
\begin{aligned}
u(x_0,t) &= \frac{M_0(x_0)}{2\tau^{q/2}}\frac{1}{\Gamma(q/2)}\int\limits_0^t \frac{n(\xi)}{(t-\xi)^{1-(q/2)}}d\xi - \frac{M_1(x_0)}{2}n(t) + \\
&\quad \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(k+1)!}M_{k+1}(x_0)\tau^{kq/2}\frac{d^{kq/2}}{dt^{kq/2}}n(t) \qquad (4.4)
\end{aligned}
$$

Observe that the first term involves a fractional integral, the second term is composed of the source function $n(t)$ alone (apart from scaling) and the third term is an infinite series composed of fractional differentials of increasing order $kq/2$. Note also that the first term is scaled by a factor involving $\tau^{-q/2}$ whereas the third term is scaled by a factor that includes $\tau^{kq/2}$

Case 3: $q<0$**.** In this case, the first term becomes

$$
\begin{aligned}
\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{iN(\omega)M_0}{2\Omega_q}\exp(i\omega t)d\omega &= \frac{M_0}{2}\tau^{\frac{q}{2}}\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}N(\omega)(i\omega)^{\frac{q}{2}}\exp(i\omega t)d\omega \\
&= \frac{M_0}{2}\tau^{\frac{q}{2}}\frac{d^{\frac{q}{2}}}{dt^{\frac{q}{2}}}n(t)
\end{aligned}
$$

The second term remains the same and the third term is

$$
-\frac{iM_2}{2.2!}\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{N(\omega)i}{(i\omega\tau)^{\frac{q}{2}}}\exp(i\omega t)d\omega = \frac{M_2}{2.2!}\frac{1}{\tau^{q/2}}\frac{1}{\Gamma(q/2)}\int\limits_0^t\frac{n(\xi)}{(t-\xi)^{1-(q/2)}}d\xi
$$

Evaluating the other terms, by induction we obtain

$$
\begin{aligned}
u(x_0,t) &= \frac{M_0(x_0)\tau^{q/2}}{2}\frac{d^{q/2}}{dt^{q/2}}n(t) - \frac{M_1(x_0)}{2}n(t) + \\
&\quad \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(k+1)!}\frac{M_{k+1}(x_0)}{\tau^{kq/2}}\frac{1}{\Gamma(kq/2)}\int\limits_0^t\frac{n(\xi)}{(t-\xi)^{1-(kq/2)}}d\xi \qquad (4.5)
\end{aligned}
$$

where $q\equiv\mid q\mid,\ \ q<0$. Here, the solution is composed of three terms: a fractional differential, the source term and an infinite series of fractional integrals of order $kq/2$. Thus, the roles of fractional differentiation and fractional integration are reversed as q changes from being greater to less than zero. N.B. all fractional differentials associated with the equations above and hence forth should be considered in terms of equation (2.3).

Asymptotic forms for $f(x) = \delta(x)$. Equations (4.3), (4.4) and (4.5) warrant further theoretical investigation which is beyond the scope of this paper. Instead, we consider a special case in which the source function $f(x)$ is an impulse so that

$$M_m(x_0) = \int_{-\infty}^{\infty} \delta(x) \mid x - x_0 \mid^m dx = \mid x_0 \mid^m$$

This result immediately suggests a study of the asymptotic solution

$$u(t) = \lim_{x_0 \to 0} u(x_0, t) = \begin{cases} \frac{1}{2\tau^{q/2}} \frac{1}{\Gamma(q/2)} \int\limits_0^x \frac{n(\xi)}{(t-\xi)^{1-(q/2)}} d\xi, & q > 0 \\ \frac{n(t)}{2}, & q = 0 \\ \frac{\tau^{q/2}}{2} \frac{d^{q/2}}{dt^{q/2}} n(t), & q < 0 \end{cases}$$

The solution for the time variations of the stochastic field u for $q > 0$ are then given by a fractional integral alone and for $q < 0$ by a fractional differential alone. In particular, for $q > 0$, we see that the solution is based on a causal convolution. Thus in t-space

$$u(t) = \frac{1}{2\tau^{q/2}\Gamma(q)} \frac{1}{t^{1-q/2}} \otimes n(t), \quad q > 0$$

where $\otimes$ denotes (causal) convolution and in ω-space

$$U(\omega) = \frac{N(\omega)}{2\tau^{q/2}(i\omega)^{q/2}} \tag{4.6}$$

This result is the conventional fractal noise model where for a fractal signal, q is related to the fractal dimension by equation (3.2). Table 1 quantifies the results for different values of q with conventional name associations. Note that u has the following fundamental property

$$\lambda^q \Pr[u_\lambda(t)] = \Pr[u(\lambda t)]$$

where

$$u_\lambda(t) = \frac{1}{2\tau^{q/2}\Gamma(q)} \frac{1}{t^{1-q/2}} \otimes n(\lambda t), \quad \lambda > 0$$

This property describes the statistical self-affinity of u. Thus, the asymptotic solution considered here, yields a result that describes a RSF signal characterised by a PSDF of the form $1/ \mid \omega \mid^q$ which is a measure of the time correlations in the signal.

Other asymptotic forms Another interesting asymptotic form of equation (4.4) is

q-value	t-space	ω-space (PSDF)	Name
$q=0$	$\frac{1}{2}n(t)$	$\frac{1}{4}$	White noise
$q=1$	$\frac{1}{2\sqrt{\tau}\Gamma(1/2)}\frac{1}{\sqrt{t}}\otimes n(t)$	$\frac{1}{4\tau\|\omega\|}$	Pink noise
$q=2$	$\frac{1}{2\tau\Gamma(1)}\int\limits_0^t n(t)dt$	$\frac{1}{4\tau^2\omega^2}$	Brown noise
$q>2$	$\frac{1}{2\tau^{q/2}\Gamma(q/2)}t^{(q/2)-1}\otimes n(t)$	$\frac{1}{4\tau^q\|\omega\|^q}$	Black noise

Table 1. Noise characteristics for different values of q [note that $\Gamma(1/2)=\sqrt{\pi}$ and $\Gamma(1)=1$].

$$u(x_0,t)=\frac{M_0(x_0)}{2\tau^{q/2}}\frac{1}{\Gamma(q/2)}\int\limits_0^t\frac{n(\xi)}{(t-\xi)^{1-(q/2)}}d\xi-\frac{M_1(x_0)}{2}n(t),\quad \tau\rightarrow 0 \tag{4.7}$$

Here, the solution is the sum of fractal noise and white noise. By relaxing the condition $\tau\rightarrow 0$ we can consider the approximation

$$\begin{aligned}u(x_0,t)\quad\simeq\quad&\frac{M_0(x_0)}{2\tau^{q/2}}\frac{1}{\Gamma(q/2)}\int\limits_0^t\frac{n(\xi)}{(t-\xi)^{1-(q/2)}}d\xi-\\&\frac{M_1(x_0)}{2}n(t)+\frac{M_2(x_0)}{2.2!}\tau^{q/2}\frac{d^{q/2}}{dt^{q/2}}n(t),\quad\tau<<1\end{aligned}$$

in which the solution is expressed in terms of the sum of fractal noise, white noise and the fractional differentiation of white noise.

5 Digital algorithms

There are two principal algorithms that are required to investigate the results given in the previous section using a digital computer. The first of these concerns the computation of discrete fractal noise u_j given q which is as follows:

(i) Compute a pseudo random zero mean (Gaussian) distributed array $n_j, j=0,1,...,N-1$.

(ii) Compute the Discrete Fourier Transform (DFT) of n_j giving N_j using a Fast Fourier Transform (FFT).

(iii) Filter N_j with $1/(i\omega_j)^{q/2}$.

(iv) Inverse DFT the result using an FFT to give u_j (real part).

The second algorithm is an inversion algorithm. Given the digital algorithm described above, the inverse problem can be defined as given u_j compute q. A suitable approach to solving this problem, which is consistent with the algorithm given above is to estimate q from the power spectrum of u_j whose expected form (considering the positive half space only and excluding the DC component which is singular) is

$$\hat{P}_j = \frac{A}{\omega_j^q}; \quad j = 1, 2, ...(N/2) - 1$$

where A is a constant. Here, we assume that the FFT provides data in 'standard form' and that the DC or zero frequency component occurs at $j = 0$.

If we now consider the error function

$$e(A, q) = \| \ln P_j - \ln \hat{P}_j \|_2^2$$

where P_j is the power spectrum of u_j, then the solution of the equations (least squares method)

$$\frac{\partial e}{\partial q} = 0; \quad \frac{\partial e}{\partial A} = 0$$

gives

$$q = \frac{\left(\frac{N}{2} - 1\right) \sum\limits_{j=1}^{(N/2)-1} [(\ln P_j)(\ln \omega_j)] - \left(\sum\limits_{j=1}^{(N/2)-1} \ln \omega_j \right) \left(\sum\limits_{j=1}^{(N/2)-1} \ln P_j \right)}{\left(\frac{N}{2} - 1\right) \sum\limits_{j=1}^{(N/2)-1} (\ln \omega_j)^2 - \left(\sum\limits_{j=1}^{(N/2)-1} \ln \omega_j \right)^2} \tag{5.1}$$

and

$$A = \exp \left(\frac{\sum\limits_{j=1}^{(N/2)-1} \ln P_j + q \sum\limits_{j=1}^{(N/2)-1} \ln \omega_j}{\frac{N}{2} - 1} \right)$$

The algorithm required to implement this inverse solution is as follows:

(i) Compute the power spectrum P_j of the fractal noise u_j using an FFT.

(ii) Extract the positive half space data (excluding the DC term).

(iii) Compute q using equation (5.1)

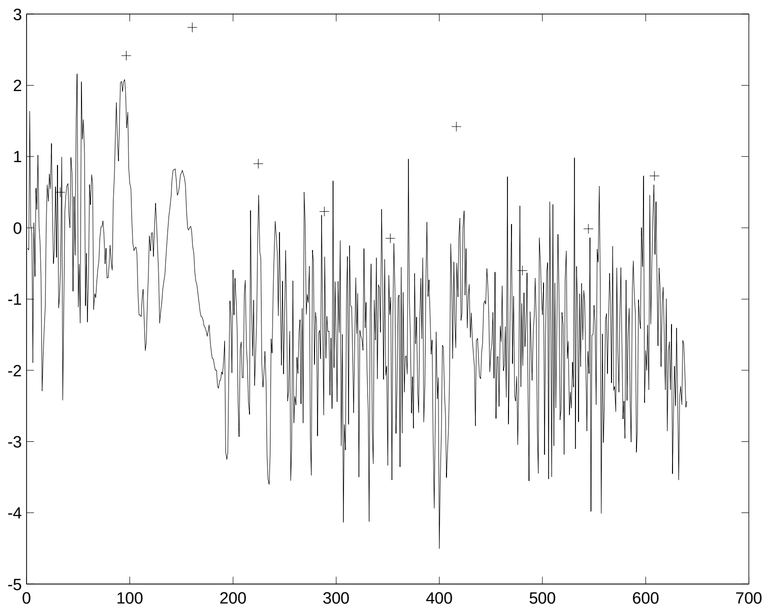


Figure 2. Example of a non-stationary signal formed by changing the values of the Fourier dimension given by the points marked with a +

This algorithm (which is commonly known as the Power Spectrum Method) provides a reconstruction for q which is (on average) accurate to 2 decimal places for $N \geq 64$ [5].

Non-stationary algorithm The results considered so far have been based on the assumption that the Fourier dimension is constant. In Section 3, we considered the case of $q(t)$ having discrete states $q_i = q(t_i)$ assuming that $q(t)$ is a slowly varying function. In this case, the solutions and algorithms discussed so far are valid for any $q_i, i = 0, 1, 2, ...M-1$ over a window of time Δt_i say, which is taken to be composed of $N-1$ elements. A non-stationary signal can therefore be generated by computing the matrix $u_{ij}; i = 0, 1, 2, ...M-1; j = 0, 1, 2, ...N-1$ for each q_i and concatenating all the rows of this matrix together to form a contiguous stream of data of size $(M-1) \times (N-1)$. An example of such a non-stationary signal is given in Figure 2 where Δt_i is composed of 64 elements and the Fourier dimension has 10 different values for $q_i \in (-1, 3)$ providing a signal that is composed of 640 elements. This result has been generated using the MS-Windows'98 MATLAB V5 DSP Toolbox as have all the other results discussed in this Section (and Section 7).

The result shown in Figure 2, is an example in which Δt_i remains fixed for all values of i. In other words, changes in the Fourier dimension occur at regular intervals. A further generalisation can be made by choosing Δt_i randomly. An interesting approach to this is based on letting Δt_i be a fractal signal so that the stochastic behaviour of $u(t)$ is governed by fractal time. The effect of introducing fractal time in this way will be published elsewhere.

Following the ideas discussed in Section 3 (on 'how should we choose $q(t)$?'), q_i is taken to be a discrete random variable which is chosen to conform to a discrete PDF or histogram. In this paper, we consider the 'normal' distribution given by

$$\Pr[q(t)] = \frac{1}{\sigma\sqrt{2\pi}} \exp[-(q-1)^2/2\sigma^2], \quad -\infty < q < \infty \quad \forall \quad t$$

Continuity condition The short time integration of white noise that occurs when $q = 2$ can lead to a significant change in the range of values of u_j. This type of behaviour shall be called a Brownian transient. The time series that will be generated by application of equation (4.6) for different values q will be composed of Brownian transients that 'look like' spikes. These spikes will be of arbitrary polarity given that $n(t)$ is **zero-mean** white Gaussian noise - simulations of such signals being based on the assumption that the mean is always zero.

Suppose that we wish to simulate a non-stationary signal whose mean value changes from time to time. One way of doing this is to let the mean value vary according to the amplitude obtained at the end of a Brownian transient. This provides continuity in the mean value from one Brownian transient to another. The effect of introducing this continuity condition is shown in Figures 3 - 7 which illustrates the effect of increasing the standard deviation of q_i from 0 to 2 in steps of 0.5. In these Figures, we have plotted the non-stationary fractal signal (top graph), the argand diagram or complex-plane map (middle graph) and a density filter map (lower graph). The complex plane map is obtained by computing the Hilbert transform of the signal to give the quadrature component [3]. This provides an image illustrating fractal walks in the complex plane. The density filter maps provide a measure of the density or compactness of points over which fractal walks have occurred (fractal dust) which is not possible to see in the complex plane maps due to the density of data. (The fine structure of the fractal signals is also not visible in these figures due to the density of data but on closer inspection are seen to be of the type given in Figure 2.) Observe that as the standard deviation increases, the number of Brownian transients increases. The positions at which these flights occur together with their directions and amplitudes are entirely arbitrary and 'driven' by the Gaussian pseudo random number generators used to compute n_j and q_i which in turn depends on the seeds used to initiate these generators. What is important is that the likelihood of generating a Brownian transient is determined by the standard deviation of q_i alone.

6 Application 1: Fractal Modulation

Embedding information in data whose properties and characteristics resemble those of the background noise of a transmission system is of particular interest

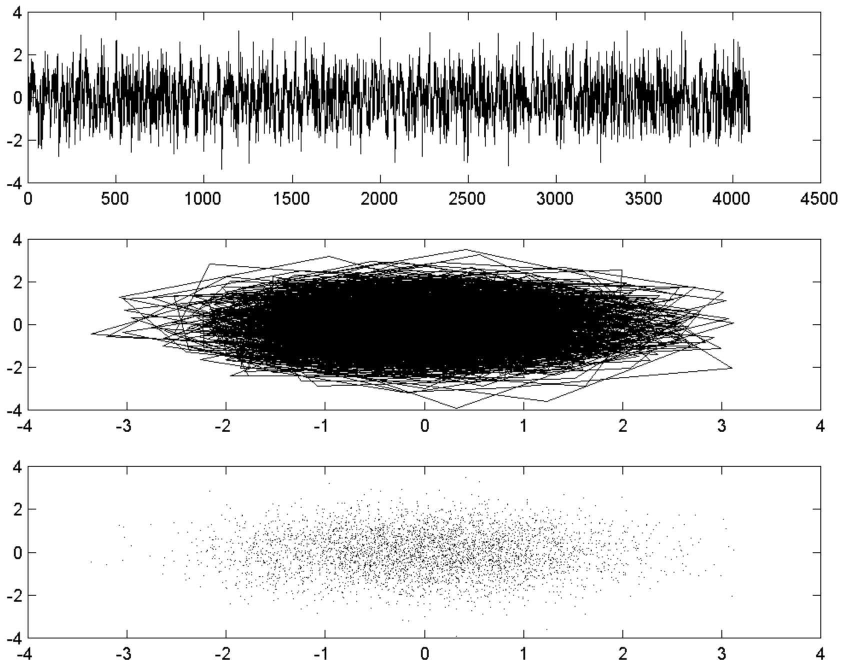


Figure 3. Non-stationary fractal signal for a standard deviation $\sigma = 0.1$ showing the time series (top), complex-plane map (middle) and the density filter map (bottom)

in covert digital communications which are of natural interest to the military. In the first of the applications considered in this paper, we explore a method of coding bit streams by modulating the fractal dimension of a fractal noise generator. Techniques for reconstructing the bit streams (i.e. solving the inverse problem) in the presence of additive noise (assumed to be introduced during the transmission phase) are considered and some results on the robustness of these inverse solutions presented. This form of 'embedding information in noise' is of value in the transmission of information in situations when a communications link requires an extra level of security or when an increase in communications traffic needs to be hidden in a covert sense by coupling an increase in the background noise of a given area with appropriate disinformation (e.g. increased sun spot activity). Alternatively, the method can be considered to be just another layer of covert technology used for military communications in general. In principle, the method we develop here can be applied to any communications system and in the following section, a brief overview of existing techniques is discussed.

Secure Digital Communications A digital communications systems is one that is based on transmitting and receiving bit streams. The basic processes involved are as follows: (i) a digital signal is obtained from sampling an analogue

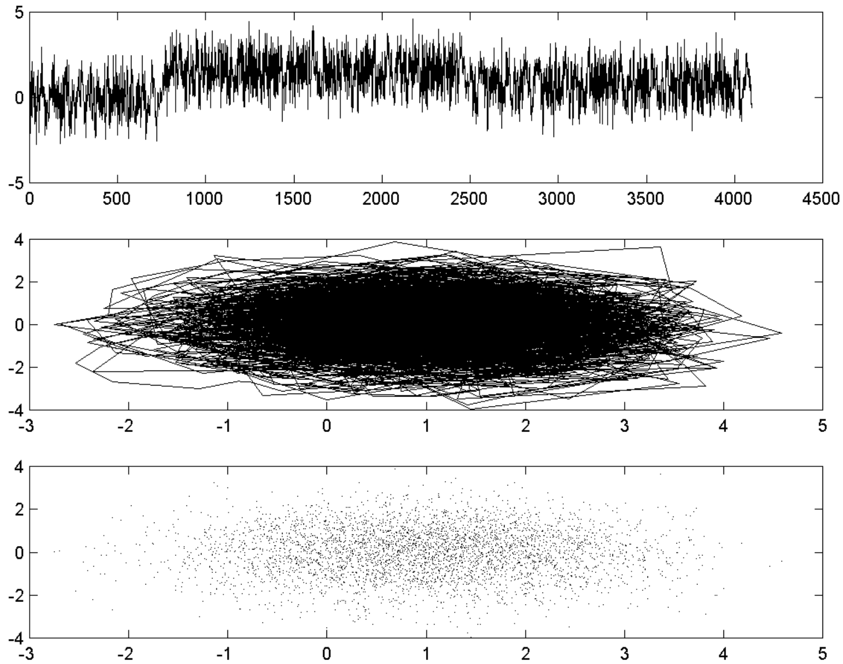


Figure 4. Non-stationary fractal signal for a standard deviation $\sigma = 0.5$

signal derived from some speech and/or video system; (ii) this signal (floating point stream) is converted into a binary signal consisting of 0s and 1s (bit stream); (iii) the bit stream is then modulated and transmitted; (iv) at reception, the transmitted signal is demodulated to recover the transmitted bit stream; (v) the (floating point) digital signal is reconstructed. Digital to analogue conversion may then be required depending on the type of technology being used.

In the case of sensitive information, an additional step is required between stages (ii) and (iii) above where the bit stream is coded according to some classified algorithm. Appropriate decoding is then introduced between stages (iv) and (v) with suitable pre-processing to reduce the effects of transmission noise for example which introduces bit errors. The bit stream coding algorithm is typically based on a pseudo random number generator using linear congruential methods or iteration function sequences in chaotic regions of their phase spaces (chaotic number generation). The modulation technique is typically either Frequency Modulation or Phase Modulation. Frequency modulation involves assigning a specific frequency to each 0 in the bit stream and another higher (or lower) frequency to each 1 in the stream. The difference between the two frequencies is minimised to provide space for other channels within the available bandwidth. Phase modulation involves assigning a phase value to one of four possible combination that occur in a bit stream (i.e. 00, 11, 01 or 10).

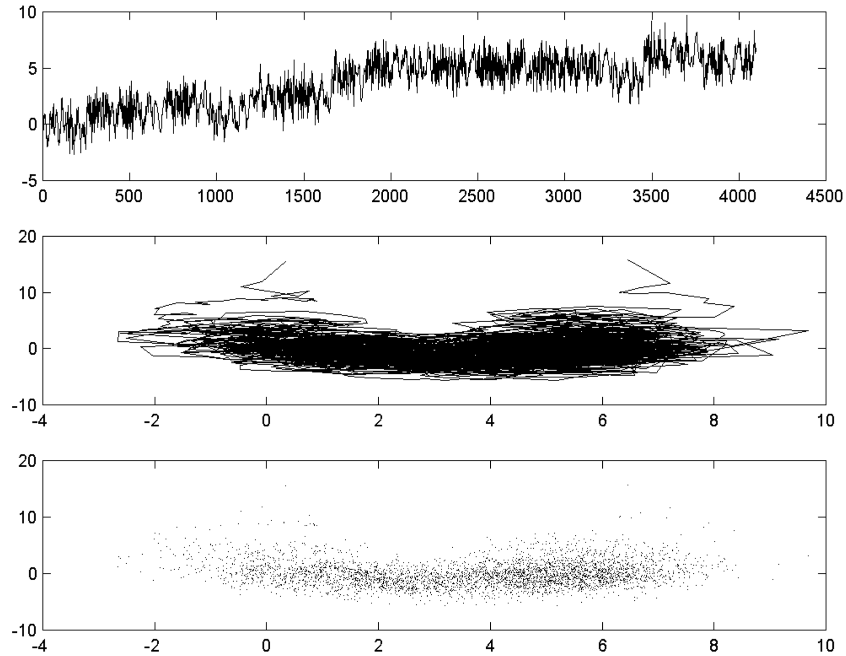


Figure 5. Non-stationary fractal signal for a standard deviation $\sigma = 1.0$

Scrambling methods can be introduced before binarization. A conventional approach to this is to distort the digital signal by adding random numbers to the out-of-band components of its spectrum. The original signal is then recovered by lowpass filtering. This approach requires an enhanced bandwidth but is effective in the sense that the signal can be recovered from data with a relatively low signal-to-noise ratio. 'Spread-spectrum' or 'frequency hopping' is used to spread the transmitted (e.g. frequency modulated) information over many different frequencies. Although spread-spectrum communications use more bandwidth than necessary, by doing so, each communications system avoids interference with another because the transmissions are at such minimal power, with only spurts of data at any one frequency. The emitted signals are so weak that they are almost imperceptible above background noise. This feature results in an added benefit of spread spectrum which is that eaves-dropping on a transmission is very difficult and in general, only the intended receiver may ever known that a transmission is taking place - the frequency hopping sequence being known only to the intended party. Direct sequencing, in which the transmitted information is mixed with a coded signal, is based on transmitting each bit of data at several different frequencies simultaneously, with both the transmitter and receiver synchronised to the same coded sequence. More sophisticated spread spectrum techniques are now being used including hybrid ones that leverage the best features of frequency hopping and direct sequencing as well as other ways

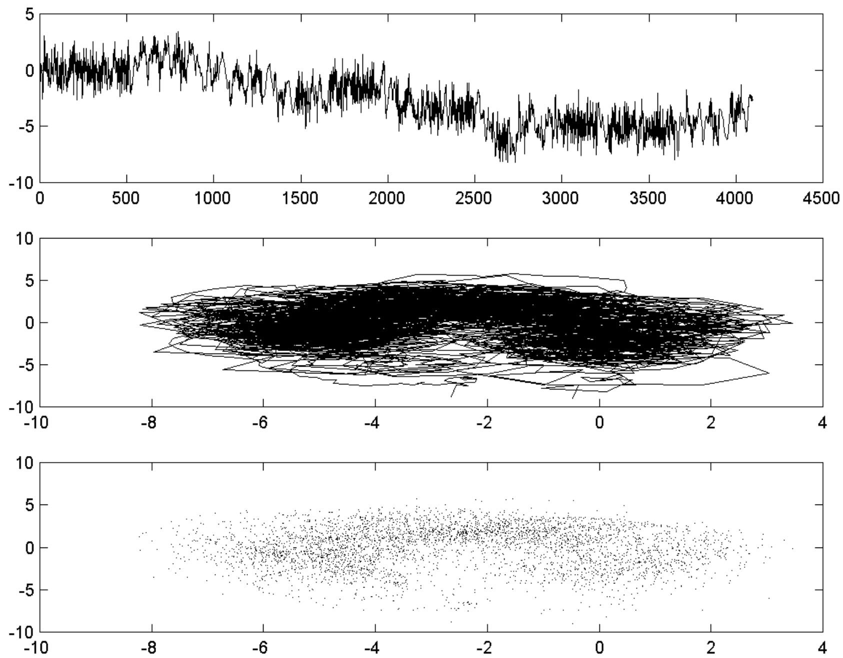


Figure 6. Non-stationary fractal signal for a standard deviation $\sigma = 1.5$.

to code data. These new methods are particularly resistant to jamming, noise and multipath anomalies - a frequency dependent effect in which the signal is reflected from objects in urban and/or rural environments and from different atmospheric layers, introducing delays in the transmission that can confuse any unauthorised reception of the transmission.

The purpose of Fractal Modulation is to try and make a bit stream 'look like' transmission noise (assumed to be fractal). The technique considered here has focused on the design of algorithms which encode a bit stream in terms of two fractal dimensions that can be combined to produce a fractal signal characteristic of transmission noise. Ultimately, Fractal Modulation could be considered to be an alternative to Frequency Modulation although the technological demands associated with this idea have not yet been investigated and lie beyond the scope of this paper. However, Fractal Modulation could relatively easily be used as an additional pre-processing security measure before transmission. The fractal modulated signal would then be binarized and the new bit stream fed into a conventional frequency modulated digital communications system albeit with a considerably reduced information throughput for a given bit rate [34]. The problem is as follows: given an arbitrary binary code, convert it into a non-stationary fractal signal by modulating the fractal dimension in such a way that the original binary code can be recovered in the presence of additive noise with minimal bit errors.

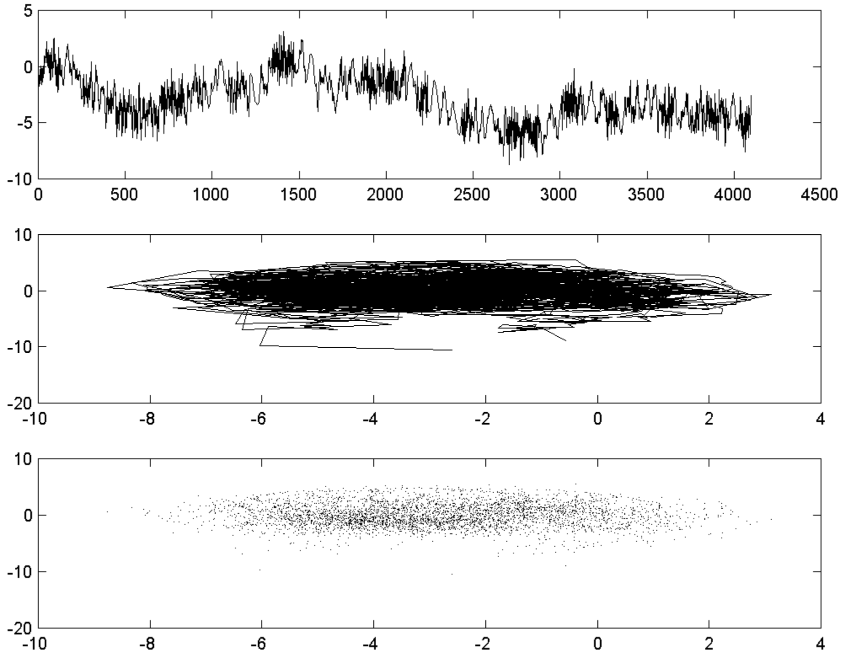


Figure 7. Non-stationary fractal signal for a standard deviation $\sigma = 2.0$

In terms of the theory discussed in Section 3, we consider

$$\left[\frac{\partial^2}{\partial x^2} - \tau^{q(t)}\frac{\partial^q(t)}{\partial t^{q(t)}}\right] u(x,t) = -\delta(x)n(t), \quad q > 0, \quad x \to 0$$

where $q(t)$ is assigned two states, namely q_1 and q_2 (which correspond to 0 and 1 in a bit stream respectively) for a fixed period of time. The forward problem (Fractal Modulation) is then defined as: given $q(t)$ compute $u(t) \equiv u(0,t)$. The inverse problem (Fractal Demodulation) is defined as: given $u(t)$ compute $q(t)$.

Fractal Modulation and Demodulation

Instead of working in terms of the Fourier dimension q, we shall consider the fractal dimension where, from equation (3.2)

$$D = q - \frac{1}{2}, \quad 1 < D < 2$$

The technique is outlined below:

(i) For a given bit stream, allocate $D_{\min}$ to bit=0 and $D_{\max}$ to bit=1.

(ii) Compute a fractal signal of length N for each bit in the stream.

(iii) Concatenate the results to produce a contiguous stream of fractal noise.

The total number of samples can be increased through N (the number of samples per fractal) and/or increasing the number of fractals per bit. This results in improved estimates of the fractal dimensions leading to a more accurate reconstruction. Fractal demodulation is achieved by computing the fractal dimensions via the Power Spectrum Method discussed in Section 5 [equation (5.1)] using a conventional moving window to provide the fractal dimension signature $D_i, \quad i = 0, 1, 2, \ldots.$ The bit stream is then obtained from the following algorithm:

If $D_k \leq \Delta$ then bit =0;

If $D_k > \Delta$ then bit =1;

where

$$\Delta = D_{\min} + \frac{1}{2}(D_{\max} - D_{\min}).$$

The principal criteria for the optimization of this modulation/demodulation technique is to minimize $(D_{\max} - D_{\min})$ subject to accurate reconstructions for D_i in the presence of (real) transmission noise.

The system developed to investigate this modulation technique has been written using the Borland Turbo C++ (Version 3) making use of the graphics functions available with this compiler and providing options on: (i) fractal size - number of samples used to compute a fractal signal; (ii) fractals per bit - number of fractal signals used to represent one bit; (iii) $D_{\min}$ - fractal dimension for bit=0; (iv) $D_{\max}$ - fractal dimension for bit=1; (v) addition of Gaussian noise before reconstruction for a given Signal-to -Noise Ratio (SNR). Option (v) is based on the result compounded in equation (4.7) in which the signal is taken to be the sum of fractal noise and white Gaussian noise.

An example of a fractal modulated signal is given in Figure 8 in which the binary code 0....1....0.... has been considered in order to illustrate the basic principle. This figure shows the original binary code (top window) the (clipped) fractal signal (middle window) and the fractal dimension signature D_i (lower window - dotted line) using 1 fractal per bit, 64 samples per fractal for a 'Low dimension' ($D_{\min}$) and a 'Hi dimension' ($D_{\max}$) of 1.1 and 1.9 respectively. The reconstructed code is superimposed on the original code (top window - dotted line) and the original and estimated codes are displayed on the right hand side. In this example, there is a 2% bit error. By increasing the number of fractals per bit so that the bit stream is represented by an increased number of samples, greater accuracy can be achieved. This is shown in Figure 9 where for 3 fractals/bit there are no bit errors. In this example, each bit is represented by concatenating 3 fractal signals each with 64 samples. The reconstruction is based on a moving window of size 64.

In Figures 8 and 9, the change in signal texture from 0 to 1 and from 1 to 0 is relatively clear because $(D_{\min}, D_{\max}) = (1.1, 1.9)$. By reducing the difference in fractal dimension, the textural changes across the signal can be reduced. This is shown in Figure 10 for $(D_{\min}, D_{\max}) = (1.6, 1.9)$ and a random 64 bit pattern

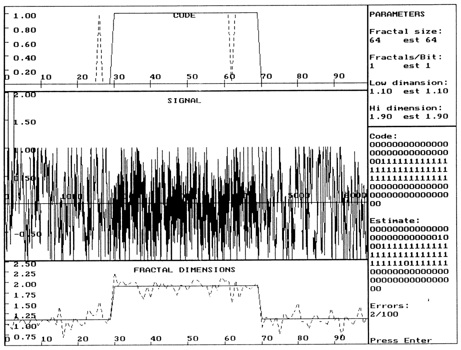


Figure 8. Fractal modulation of the code 0...1...0... for one fractal per bit.

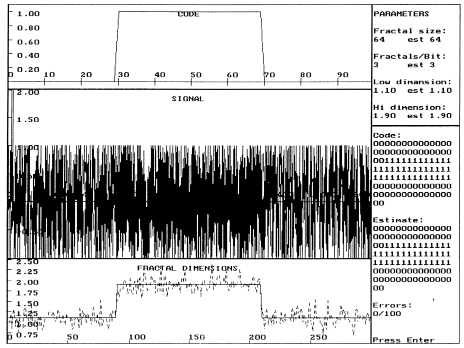


Figure 9. Fractal modulation of the code 0...1...0... for three fractals per bit.

where there is 1 bit in error. Figure 11 shows the same result but with 10% white Gaussian noise added to the fractal modulated signal before demodulation. Note that the bit errors have not been increased as a result of adding 10% noise.

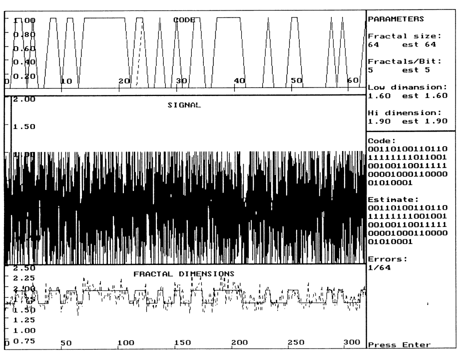


Figure 10. Fractal modulation of a random bit stream without additive noise.

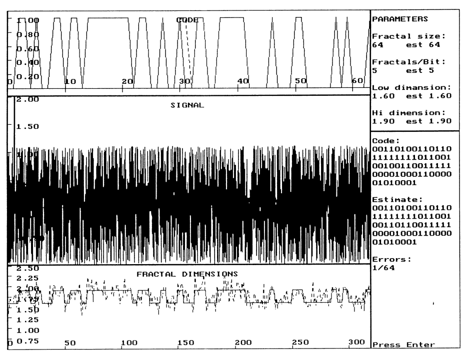
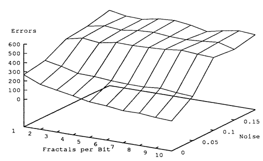


Figure 11. Fractal modulation of a random bit stream with 10% additive noise.

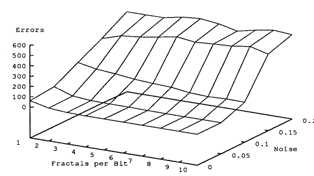
Performance measures In order to obtain a quantitative picture of the accuracy of fractal demodulation subject to changes in the fractal generating parameters and additive noise, a bit stream of 1000 randomly chosen bits was used. The average number of errors (for 64 samples and then 128 samples) were compared with the number of 'Fractals per bit' and Noise-to-Signal ratio or 'Noise' for different $D_{\min}$ and $D_{\max}$. The noise n_j added to the fractal signal u_j was white Gaussian noise and the noise-to-signal ratio or 'Noise' defined in terms of the ratio

$$\text{Noise} = \frac{\|n_j\|_\infty}{\|u_j\|_\infty}$$

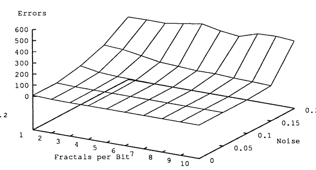
Figure 12 provides surface plots showing the number of bit errors as a function of the number of fractals per bit and the noise, for fractals signals computed using 64 samples and different $(D_{\min}, D_{\max})$. Figure 13 provides similar plots for fractal signals using 128 samples per fractal. As expected, the results show that a combination of wide intervals between the two fractal dimensions with a large number of fractals per bit achieves greater accuracy. For example, the results for $D_{\min} = 1.6$ and $D_{\max} = 1.9$ achieves less than 10% bit errors for 5 or more fractals per bit and 128 samples per fractal with 15% (i.e. 0.15) noise.



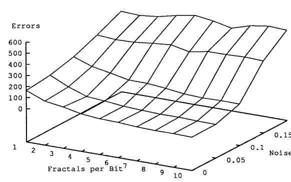
$D_{\min} = 1.1, D_{\max} = 1.3$



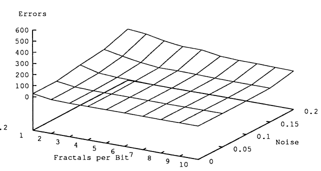
$D_{\min} = 1.1, D_{\max} = 1.6$



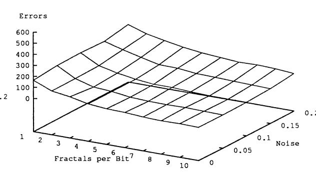
$D_{\min} = 1.1, D_{\max} = 1.9$



$D_{\min} = 1.3, D_{\max} = 1.6$



$D_{\min} = 1.3, D_{\max} = 1.9$

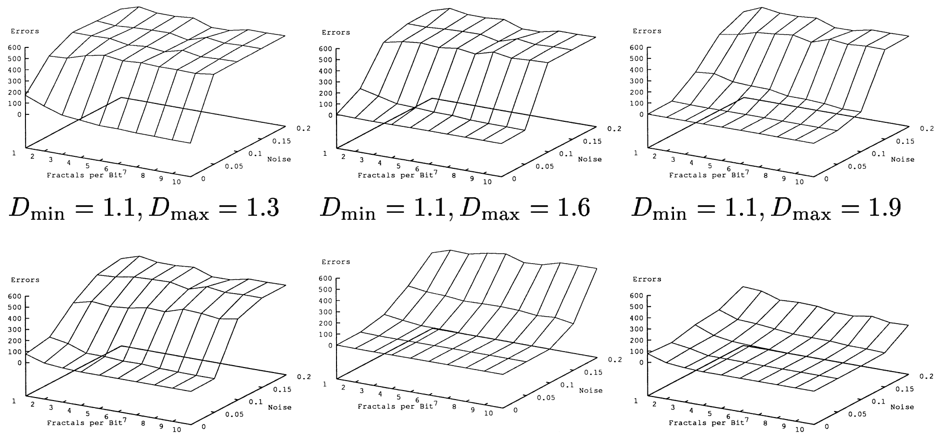


$D_{\min} = 1.6, D_{\max} = 1.9$

Figure 12. Surface plots illustrating the bit errors as a function of the number of Fractals per Bit and the Noise for 64 samples per fractal and different values of $D_{\min}$ ('Low') and $D_{\max}$ ('Hi').

Discussion Fractal modulation is a technique which attempts to embed a bit stream in fractal noise by modulating the fractal dimension. As expected, the errors associated with recovering a bit stream are critically dependent on the SNR. The reconstruction algorithm provides relatively low error rates with a relatively high level of noise, provided the difference in fractal dimension is not too small and that many fractals per bit are used. In any application, the parameter settings would have to optimized with respect to a given transmission environment. The technique could work with lower SNRs if coupled with a suitable inference engine.

The success of the technique (with regard to its covert intent) depends on the appropriateness of the transmission noise model used to embed a bit stream. Here, we have used a model based on $\mid \omega \mid^{-q}$ noise. This power law is consistent with statistically self-affine noise. Another possible PSDF is [35]



$D_{\min} = 1.1, D_{\max} = 1.3$ $D_{\min} = 1.1, D_{\max} = 1.6$ $D_{\min} = 1.1, D_{\max} = 1.9$

$D_{\min} = 1.3, D_{\max} = 1.6$ $D_{\min} = 1.3, D_{\max} = 1.9$ $D_{\min} = 1.6, D_{\max} = 1.9$

Figure 13. Surface plots illustrating the bit errors as a function of the number of Fractals per Bit and the Noise for 128 samples per fractal and different values of $D_{\min}$ ('Low') and $D_{\max}$ ('Hi').

$$| U(\omega) |^2 = \frac{C\omega^{2g}}{(\omega_0^2 + \omega^2)^q}$$

where q and g are positive non-integer numbers. This PSDF represents a more general and possibly, a more versatile model. It is consistent with a wider range of noise than the one considered here and is a generalization of the fractal, Bermann and Ornstein-Uhlenbeck processes [36]. However, it also poses a significantly more difficult inverse problem [37]. Another possible extension to the fractal modulation technique considered in this paper is to choose a larger number of states $q_i, i = 1, 2, ...$ representing a (renormalised) run length code for example. Finally, the application to digital communications in terms of radio and/or microwave communications systems could be extended to include e-commerce in which sensitive information is transmitted through open networks covertly.

7 Application 2: Macro-economic volatility prediction

The application of statistical techniques for analysing economic time series and the financial markets in general is a well established practice. These practices include time series analysis and prediction using a wide range of stochastic modelling methods and the use of certain PDEs for describing specific financial systems (e.g. the Black-Scholes equation for financial derivatives [38]). A seemingly inexhaustible range of statistical parameters (clustered or otherwise) are being developed to provide an increasingly complex portfolio of financial measures.

The financial markets are a rich source of highly complicated data. Changing every second or less, their behaviour is difficult to predict often leaving economists and politicians alike baffled at events such as the UK stock market 'crash' of October 1987 (Black Monday). This event is shown in Figure 14 which is the time series for the UKs Financial Times Stock Exchange (FT-SE) between April 1987 and April 1988.

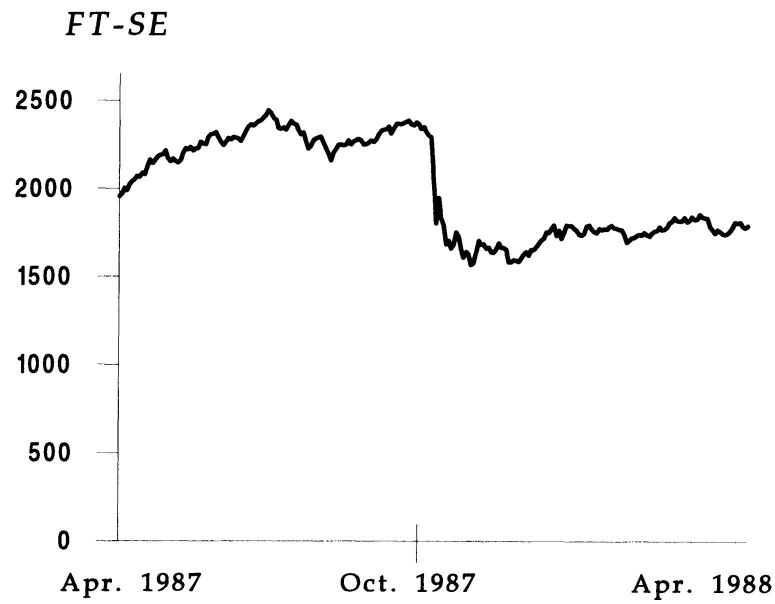


Figure 14. Financial Times Stock Exchange 100 closing prices from April 1987 to April 1988 [38].

Attempts to develop stochastic models for financial time series (which are just digital signals) can be traced back to the late eighteenth century when Louis Bachelier, who studied under Henri Poincaré in Paris, proposed that fluctuations in the prices of stocks and shares could be viewed in terms of random walks. Bachelier's model (like many of those that have been developed since) failed to predict extreme behaviour in financial signals because of its assumption that such signals conform to Gaussian statistics.

A good stochastic financial model should ideally consider all the observable behaviour of the financial system it is attempting to model. It should therefore be able to provide some predictions on the immediate future behaviour of the system within an appropriate confidence level. Predicting the markets has become (for obvious reasons) one of the most important problems in financial mathematics. Although in principle, it might be possible to model the behaviour of each individual agent operating in a financial market, the uncertainty principle should be respected in that one can never be sure of obtaining all the necessary information required on the agents themselves and their modus operandi. The uncertainty principle will of course play an increasingly important role as the scale of the financial system for which a model is required increases. Thus, while quasi-deterministic models may be of value for understanding micro-economic systems (with fixed or slightly fuzzy 'operational conditions' for example), in an ever increasing global or macro economy (in which the operational conditions associated with the fiscal policies of a given nation state are increasingly open) we can take advantage of the scale of the system to describe its behaviour in terms of functions of random variables. However, many economists - who believe that agents are rational and try to optimise their utility function (essentially a trade off between profit and risk) - are reluctant to accept a stochastic approach to modelling the markets claiming that it is 'an insult to the intelligence of the market(s)'. In trying to develop a stochastic financial model based on a macro-economy, it is still important to respect the fact that the so called global economy is actually controlled by three major trading centres (i.e. Tokyo, London and New York). Politicians are also generally reluctant to accept a stochastic approach to financial modelling and forecasting. The idea that statistical self-affinity may be a universal law, and that the evolution of an economy and society in general could be the result of one enormous and continuing phase transition does little to add confidence to the worth of politics (until politics itself is considered in the same light!). Nevertheless, since the early 1990s, fractal market analysis has been developed and used to study a wide range of financial time series [39]. This includes different fractal measures such as the Hurst dimension, Lyaponov exponent, the correlation dimension and multi-fractals [40]. Ultimately, an economy depends on the confidence of those from which it is composed and although governments (and/or central or federal banks) have a variety of control parameters (interest rates being amongst the most important) with which to adjust the pace of economic growth or otherwise, there is no consensus on how to control or accurately define the term confidence.

Volatility and risk analysis Evaluating risk is crucial to investors at every level and in all areas of commerce and finance. In terms of trading on a stock market, risk is inevitably related to the volatility of the markets world wide. In times of high volatility, high risk strategies can lead to short term gains but also considerable losses. In general, the lower the volatility of the markets over a period of time, the lower the risk of investment becomes during this period (assuming that the overall trend is positive of course). In other words, slow but stable growth tends to provide confidence in the markets. It is therefore interesting to consider whether the deviation in the Fourier dimension as a function of time can provide a useful volatility measure. Suppose that we classify a stable market (assumed to have slow but steady growth) to be one that is characterised by a diffusive mode in which $q(t)$ fluctuates only slightly (i.e. $q(t) = 1 \pm \epsilon(t)$ where ϵ is arbitrarily small $\forall t$). In this case, the deviation from a value of 1 is small and we should therefore expect the likelihood of a Brownian transient to be very low. The volatility of the system should therefore also be low. On the other hand, given that when $q = 2$, we should expect a Brownian transient (which in

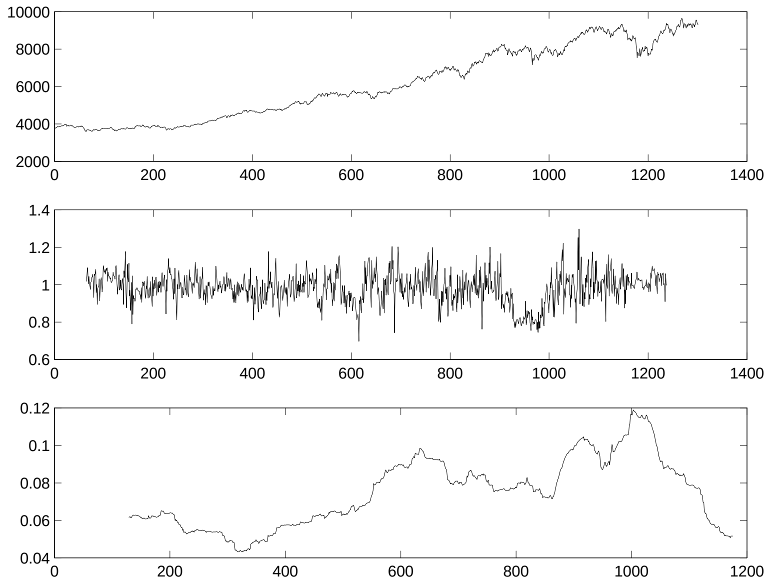


Figure 15. Dow Jones average (daily) from 1994 to 1998 (top graph), the corresponding Fourier dimensions (middle graph) and the standard deviations of the Fourier dimensions obtained using a moving window of size 128 (bottom graph).

stock market terms, we interpret to be a relatively rare but significant change in the financial index), subject to the continuity condition discussed in Section 5 and illustrated in Figures (3 - 7), then if the deviation of $q(t)$ increases, so should the likelihood of Brownian transients. The volatility of the system should therefore be high. From this argument, we can consider the deviation in $q(t)$ (rather than $q(t)$ itself) to provide a measure of volatility.

To progress further with this theme, it is necessary to find the type of PDF to which $q(t)$ is likely to conform for financial data in order to establish a deviation measure. To investigate this, we have considered the Dow Jones industrial average on a daily rate from 1994 to 1998. Figure 15 shows the signature for this financial index (top graph) and the associated q-signature (middle graph). Here, q_i has been computed using the Power Spectrum Algorithm discussed in Section 5 using a moving window of size 64.

The Dow Jones index shows continual growth over the period 1994-1998. However, the growth pattern of the latter part of this period is substantially more irregular than that of the first part. After day 600, the index becomes increasingly more volatile. Also, observe that $q(t)$ fluctuates about 1 over this period and is therefore diffusive in behaviour. Figure 16 shows a least squares fit (smooth line) to the histogram of the q-signature (jagged line) given in Figure 15 where we have assumed that the data conforms to a ‘normal’ distribution. In this

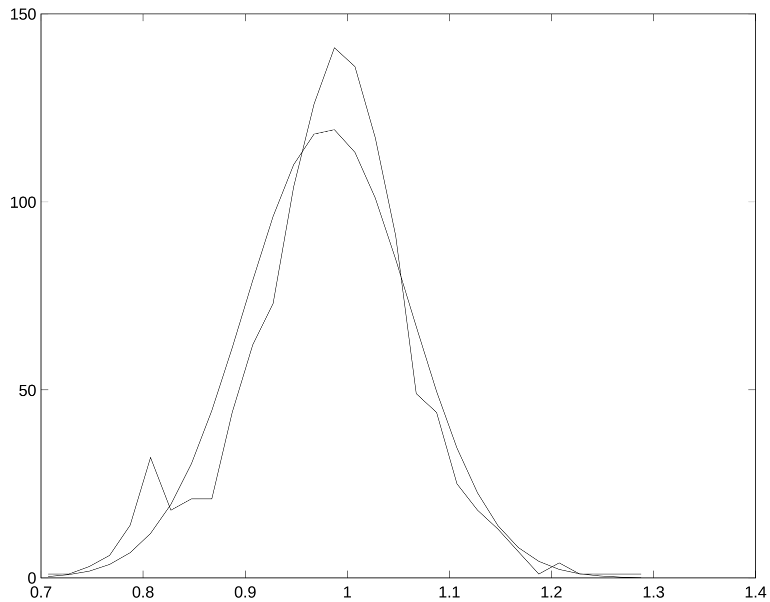


Figure 16. Least squares fit to the the distribution of the Fourier dimensions given in Figure 15 (jagged line) assuming a Gaussian distribution (smooth line).

example, $\mu = 0.93$ and $\sigma = 1.12$. This fit provides confidence in the Gaussian statistics of the q-signature although the mean value has been shifted slightly to the left due to the behaviour of the data between $q = 0.58$ and 0.72. Figure 15 (bottom graph) shows the variations in the standard deviation σ_i which have been obtained using a moving window of size 128 under the assumption that all data is Gaussian distributed.

It is interesting to note that σ_i has a local minimum that occurs over the region in which the Dow Jones index has the most stable pattern of growth (i.e. for $i \simeq 230 - 400$ corresponding to the financial year 1995/96). In this sense, σ_i can be considered to be a measure of the volatility of this market. However, considerably more analysis of this measure using different data and comparisons with other financial index parameters needs to be performed before stating this result with confidence. Nevertheless, the results presented here lead us to consider an analysis of the volatility of the markets in terms of the following theoretical model: Let the stock market fluctuations be determined by the solution of the equation

$$\left[\frac{\partial^2}{\partial x^2} - \tau^{q(t)}\frac{\partial^{q(t)}}{\partial t^{q(t)}}\right] u(x,t) = -\delta(x)n(t), \quad x \to 0 \tag{7.1}$$

where

$$\Pr[n(t)] = \frac{1}{\sigma_n\sqrt{2\pi}} \exp(-n^2/2\sigma_n), \quad -\infty < n < \infty$$

and

$$\Pr[q(t')]_t = \frac{1}{\sigma_q(t)\sqrt{2\pi}} \exp[-(\mu - q)^2/2\sigma_q(t)], \quad -\infty < q < \infty$$

Then,

(i) From the time series data $u(0,t)$, compute $q(t)$.

(ii) From $q(t)$ compute $\sigma_q(t)$.

8 Conclusions, future directions and a short story

We have considered a solution to equation (7.1) and shown that for $q > 0$ we can write

$$u(t) = \frac{1}{2\tau^{q(t)/2}} \frac{1}{\Gamma[q(t)]} \int_0^t \frac{n(\xi)}{(t-\xi)^{1-[q(t)/2]}} d\xi \quad x \to 0$$

where

$$q(t) = \begin{cases} q_1, & t_0 < t \leq t_1 \\ q_2, & t_1 < t \leq t_2 \\ \vdots & \vdots \end{cases}$$

and is taken to conform to some PDF. The non-stationary characteristics of $u(t)$ are then determined by the statistics of the dimension $q(t)$. This result has been used to demonstrate behaviour which is analogous to Lévy-type distributed fields but is actually based on random episodes of Brownian noise embedded in non-stationary fractal noise. We have called these episode 'Brownian transients' which in this paper, have been demonstrated for the case when both $n(t)$ and $q(t)$ are Gaussian distributed and where $\Delta t_i = t_{i+1} - t_i$ is a constant. Fractal Modulation has been defined as that process in which $q(t)$ has just two states, q_1 and q_2 where $q_1 \neq q_2$. All the results presented are ultimately based on generalisations, the most important being the definition of a fractional derivative using the Fourier transform.

The ideas presented in this paper lead to a number of further avenues of investigation. For example, given the solution above, what is the effect of applying different distributions to compute $q(t)$ and how might they qualify in the simulation of different noise types? Also, how would the application of fractal time (as discussed in Section 5) effect the behaviour of the field? From the theoretical point of view, a natural extension of this work is to consider solutions in 2D and 3D (with regard to spatial dependence) and further, to introduce fractal spatial dependence using the Riesz definition for a fractional Laplacean. The stochastic fractionalization of other PDEs is also waiting to be investigated, in particular, the Fokker-Planck-Kolmogorov equation. In Section 2, we provided a short defence of (Fourier based) fractionalization by highlighting through a brief sketch, the Fourier based origins of the Schrödinger equation. It is well known, that the fundamental postulates of quantum mechanics are based on the energy E and momentum $\mathbf{p}$ operators

$$E \to i\bar{h}\frac{\partial}{\partial t} \quad \text{and} \quad \mathbf{p} \to i\bar{h}\nabla$$

It may therefore be of interest to consider a stochastic fractionalization of these operators, e.g.

$$E \to i\bar{h}\frac{\partial^{q(t)}}{\partial t^{q(t)}} \quad \text{and} \quad \mathbf{p} \to i\bar{h}\nabla^{\beta(\mathbf{r})}$$

It would be of interest to know whether operators of this type could be of value for modelling phase transitions in quantum mechanics.

The approach to modelling a non-stationary stochastic field presented in this paper is radically different to that used in conventional fractional dynamics which is based on fractional PDEs that operate on a PDF. The model presented in this paper has two important features:

(i) the fractal PDE operates on the stochastic field itself and not on its PDF;

(ii) the fractional order of the fractional derivative is time varying and is itself a stochastic variable.

Stochastisism is introduced through the application of a stochastic source term and a stochastic fractional differential. This approach has a certain analogy with quantum mechanics for two reasons. First, is that the solution to equation (7.1) cannot be compared directly by experiment. Only the PDF of $u(t)$ can be related to an experiment by computing the histogram of the data and then applying appropriate regression and curve fitting algorithms (as required). Second is the fact that our solution is entirely dependent on the quantization of $q(t)$ into time steps over which the value of q is a constant. In this sense, the ideas presented in this paper fall into the catergory of 'Quantum Fractional Dynamics'. In this paper, quantization of the time steps has been fixed. In a future publication, we shall study the behaviour of $u(t)$ when the time step is a random variable that describes a self-affine signal. The stochastic field then becomes a function of fractal time.

Given the importance of generalization on the results discussed in this paper (through the use of fractional calculus) and in other areas of mathematics, we shall conclude with an appropriate story which is illustrative of the 'problem' (or 'solution'). There was once a group of mathematicians and a group of engineers riding together on a train to joint meetings. All the engineers had tickets, but the mathematicians only had one ticket between them. Inquisitive by nature, the engineers asked the mathematicians how they were going to get away with such a small sample of tickets when the conductor came through. The mathematicians said, 'easy, we have solutions to this problem'. Later, when the conductor came to punch tickets, all the mathematicians slipped away quietly into the bathroom. When the conductor knocked on the door, the head mathematician slipped their one ticket under the door thoroughly fooling the layman conductor. After the joint meetings were over, the mathematicians and the engineers again found themselves on the same train. Always quick to catch on, the engineers had purchased one ticket between them intent on applying the same money saving solution the mathematicians had demonstrated to them earlier. The mathematicians on the other hand had purchased no tickets for the trip back home. Confused, the engineers said 'we understood your solution for the one ticket case, but how can you possibly get away with no tickets at all'. The mathematicians said 'easy, we have another solution to this problem'. Later, when the conductor was in the next car, all the engineers trotted off into the bathroom. Shortly, the head mathematician crept over to where the engineers were hiding and knocked authoritatively on the door. As they had been instructed, the engineers slipped their one ticket under the door. The head mathematician took the engineers' one and only ticket, returned to the mathematicians who then successfully re-applied their one-ticket solution used on the previous train journey. The moral of this story is: do not use mathematics unless you understand the principles behind it.

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