Fractal Vector-Measures on Fractals

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Space of Vector-Valued measures

We consider measures with values in $\mathbb{R}^n$.

Let X be a compact metric space and $\mathcal{M}(X, \mathbb{R}^n)$ be the space of Borel $\mathbb{R}^n$-valued measures on X.

For our purposes, we can think of a measure $\mu \in \mathcal{M}(X, \mathbb{R}^n)$ as being the product of n signed, scalar valued measures.

Thus, things like the Riesz Representation Theorem and other measure theory properties are easy to derive.

Total variation and semivariation of a vector measure

Corresponding the the total variation norm on the space of scalar-valued measures, there are two "variations" – the total variation and the semivariation.

The total variation of a measure $\mu \in \mathcal{M}(X, I\!R^n)$ on a Borel set $A \subset X$ is defined as

$$|\mu|(A) = \sup_{\pi} \sum_{i} \|\mu(E_i)\|$$

where the sup is over all finite partitions $\pi = \{E_1, E_2, \ldots, E_m\}$ of A.

$|\mu|(X)$ is a norm on $\mathcal{M}(X, I\!R^n)$.

For fixed $\mu \in \mathcal{M}(X, I\!R^n)$, the total variation is a monotone finitely additive function of A.

The semivariation of a measure $\mu \in \mathcal{M}(X, I\!R^n)$ on a Borel set $A \subset X$ is defined as

$$\|\mu\|(A) = \sup\{|x \cdot \mu|(A) \ : \ \|x\| \leq 1\}$$

where $x \cdot \mu$ is the scalar-valued measure obtained by projecting the measure μ in the direction of x and $|x \cdot \mu|$ is the total variation of this scalar-valued measure.

For fixed $\mu \in \mathcal{M}(X, I\!R^n)$, the semivariation is a monotone subadditive function on the Borel subsets of X.

Comparing the definitions, it is easy to show that for any Borel set A and any $\mu \in \mathcal{M}(X, I\!R^n)$ we have

$$\|\mu\|(A) \leq |\mu|(A).$$

The semivariation can also be computed as

$$\|\mu\|(A) = \sup \|\sum_i \alpha_i \mu(E_i)\|$$

where the supremum is taken over all finite partitions of A and all scalars $|\alpha_i| \leq 1$.

Furthermore, it can be shown that

$$\sup\{\|\mu(B)\| \,:\, B \subset A\} \leq \|\mu\|(A) \leq 4 \sup\{\|\mu(B)\| \,:\, B \subset A\}.$$

Thus, a measure $\mu \in \mathcal{M}(X, \mathbb{R}^n)$ is of bounded semivariation iff its range is bounded.

Integration of vector functions with respect to vector measures

In general, it is possible to define the integral of a vector-valued function with respect to a vector-valued measure.

For a simple function $\phi = \Sigma_i a_i \chi_{A_i}$, the integral is defined as

$$\int \phi \, d\mu = \sum_i \langle a_i, \mu(A_i) \rangle$$

where $< \cdot, \cdot >$ is a bilinear form.

The integral is extended to measurable functions in the usual way.

For us, the situation is much simpler. A measure $\mu \in \mathcal{M}(X, I\!R^n)$ is thought of as a product of n scalar-valued measures μ_i. Thus, for any measurable vector-valued function $f = (f_1, f_2, \ldots, f_n)$ we define

$$\int f \, d\mu = \int f_1 \, d\mu_1 + \int f_2 \, d\mu_2 + \cdots + \int f_n \, d\mu_n.$$

Metric on bounded subsets of $\mathcal{M}(X, I\!R^n)$

In analogy with the standard case of probability measures, we wish to define a metric on $\mathcal{M}(X, I\!R^n)$ by

$$d(\mu, \eta) = \sup\{\int_X f(x) \cdot d(\mu - \eta)(x) \ : \ f \in \mathcal{L}ip(X, I\!R^n)\}.$$

It is clear that $d(\mu, \eta) = d(\eta, \mu) \geq 0$. Furthermore, $d(\mu, \eta) = 0$ iff $\mu = \eta$.

The triangle inequality also holds since

$$\begin{aligned} \sup \int_X f \cdot d(\mu - \nu) &= \sup \int_X f \cdot d(\mu - \eta + \eta - \nu) \\ &\leq \sup \int_X f \cdot d(\mu - \eta) + \sup \int_X f \cdot d(\eta - \nu). \end{aligned}$$

The problem is that this "distance" can be infinite.

Problem of infinite "distance"

If $\mu = 2\eta$ then

$$\int_X f(x) \cdot d(\mu - \eta) = \int_X f(x) \cdot d(\eta).$$

Clearly this last integral can be arbitrarily large, since for any $w \in I\!R^n$, $f + w \in \mathcal{L}ip(X, I\!R^n)$ for any $f \in \mathcal{L}ip(X, I\!R^n)$.

The solution in the case of scalar-valued measures was to restrict to probability measures, *i.e.* those positive measures μ for which $\mu(X) = 1$.

The corresponding restriction in the vector measure case is to restrict to the affine subspace

$$\mathcal{S}_v(X, I\!R^n) = \{\mu \in \mathcal{M}(X, I\!R^n) \ : \ \mu(X) = v\}$$

where $v \in I\!R^n$ is a fixed vector.

Does this restriction solve the problem? Not quite.

For $\mu, \nu \in \mathcal{S}_v(X, I\!R^n)$ and $f \in \mathcal{L}ip(X, I\!R^n)$ we have

$$\begin{aligned} \left|\int_X f(x) \cdot d(\mu - \eta)\right| &= \left|\int_X (f(x) - f(x_0)) \cdot d(\mu - \eta) + f(x_0) \cdot \int_X \ d(\mu - \eta)\right| \\ &= \left|\int_X (f(x) - f(x_0)) \cdot d(\mu - \eta)\right|. \end{aligned}$$

To bound this, it is necessary to restrict μ and ν to some "bounded" subset of $\mathcal{S}_v(X, I\!R^n)$.

If $\|\mu\| \leq C$ and $\|\nu\| \leq C$ then

$$\left|\int_X (f(x) - f(x_0)) \cdot d(\mu - \eta)\right| \leq 2C diam(X)$$

since $\|f(x) - f(x_0)\| \leq diam(X)$ because $f \in \mathcal{L}ip(X, I\!R^n)$.

We denote by $\mathcal{S}_v^c(X, I\!R^n)$ the subset

$$\{\mu \in \mathcal{M}(X, I\!R^n) \ : \ \mu(X) = v, \|\mu\| \leq C\}.$$

IFS Operator on vector measures

Let w_i be contractive maps on X (with contractivity factors s_i), p_i be real numbers and R_i be linear operators on $I\!R^n$.

The IFS operator T from $\mathcal{M}(X, I\!R^n)$ to $\mathcal{M}(X, I\!R^n)$ is defined as

$$T(\mu)(B) = \sum_i p_i R_i \mu \left(w_i^{-1}(B) \right).$$

This operator is an extension, to the vector-valued case, of the Markov operator from standard IFS theory.

If for some vector $\vec{v} \in I\!R^n$, we have

$$\sum_i p_i R_i \vec{v} = \vec{v}$$

then if $\mu(X) = \vec{v}$ we have $T(\mu)(X) = \vec{v}$.

This means that T leaves $\mathcal{S}_v(X, I\!R^n)$ invariant.

Contractivity of the IFS

Theorem 1 *Suppose that for some $C > 0$ and $v \in \mathbb{R}^n$ we have that T leaves $\mathcal{S}_v^c(X, \mathbb{R}^n)$ invariant. Then*

$$d(T(\mu), T(\eta)) \leq (\sum_i s_i p_i \|R_i^t\|) \;\; d(\mu, \eta)$$

for all $\mu, \eta \in \mathcal{S}_v^c(X, \mathbb{R}^n)$.

As a Corollary, we get that T is contractive if

$$\sum_i s_i p_i \|R_i^t\| < 1.$$

Notice that if $v = 0$ and T is contractive the only possible fixed point is the measure $\mu(B) = 0$ for all sets B.

Proof of Theorem

Let $f \in \mathcal{L}ip(X, \mathbb{R}^n)$. We calculate for μ and η in $\mathcal{M}(X, \mathbb{R}^n)$

$$\begin{aligned}
\int_X f(x) \cdot d(T(\mu) - T(\eta)))\, dx &= \sum_i p_i \int_X f(x) \cdot d\left(R_i(\mu(w_i^{-1}(x)) - \eta(w_i^{-1}(x))) \right) \\
&= \sum_i p_i \int_X f(w_i(y))^t R_i d(\mu(y) - \eta(y)) \\
&= \int_X \left(\sum_i p_i \left(R_i^t f(w_i(y)) \right) \right)^t d(\mu(x) - \eta(x)) \\
&= (\sum_i p_i s_i \|R_i^t\|) \int_X \phi(y)^t\ d(\mu(y) - \eta(y))
\end{aligned}$$

where $\phi(y) = \frac{1}{\sum_i p_i s_i \|R_i^t\|} \Sigma_i\, p_i R_i^t f(w_i(x)) \in \mathcal{L}ip(X, \mathbb{R}^n)$ by definition of s_i. Taking the supremum, we get

$$d(T(\mu), T(\eta)) \leq (\sum_i p_i s_i \|R_i^t\|)\ \ d(\mu, \eta)$$

and the result follows.

Motivation: A color chaos game

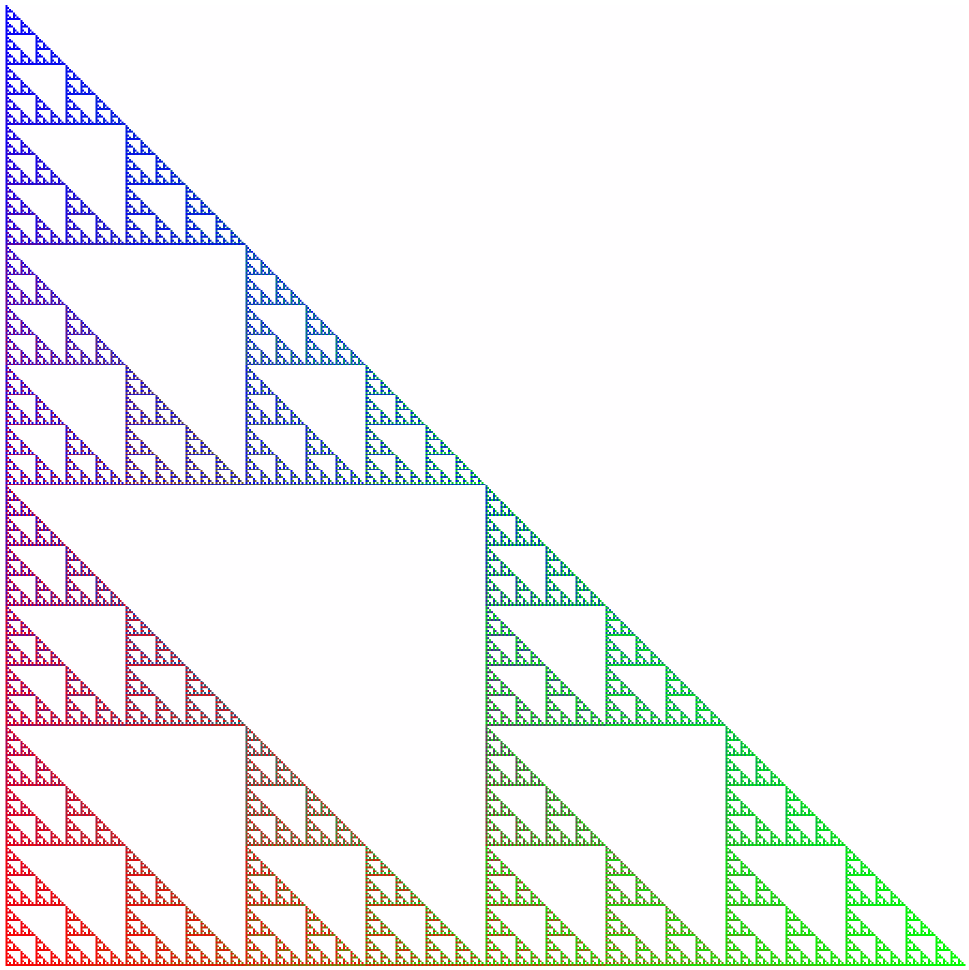
This variation of the chaos game was relayed to me by Juaquin Anderson.

Start with an IFS with probabilities $\{w_i, p_i\}$.

For each map, choose a color C_i and a probability of changing to this color, $0 \leq \hat{p}_i \leq 1$.

1. Pick x_0 to be the fixed point of w_0 and set the current color to $c_0 = C_0$.
2. Choose map w_j according to the probabilities p_i.
3. Change the current color to color C_j with probability $\hat{p}_j$. That is, with probability $\hat{p}_j$ set $c_{n+1} = C_j$ else $c_{n+1} = c_n$.
4. Set $x_{n+1} = w_j(x_n)$ and plot x_{n+1} using color c_{n+1}.
5. If enough points have been generated, stop. Else, go to step 2.

This algorithm will yield a plot of the attractor "colored" in a self-similar way by the colors associated with the maps.



First application: IFS on probability measure-valued measures

Let Λ be a finite set with n elements and M_i be Markov transition matrices on Λ (*i.e.* M_i is column stochastic).

Let $\{w_i, p_i\}$ be an IFS with probabilities.

Let $v \in I\!R^n$ be the probability vector so that $(\sum_i p_i M_i)v = v$ (notice that $\sum_i p_i M_i$ is also column stochastic).

Define

$$\mathcal{P}(X, I\!R^n) = \{\mu \in \mathcal{M}(X, I\!R^n) \,:\, \mu(B) \geq 0, \, \forall B \subset X, \mu(X) = v\}.$$

Finally, define the operator $M : \mathcal{P}(X, I\!R^n) \to \mathcal{P}(X, I\!R^n)$ by

$$M(\mu)(B) = \sum_i p_i M_i \mu \left(w_i^{-1}(B) \right)$$

for all Borel $B \subset X$.

It is clear that M maps into $\mathcal{P}(X, I\!R^n)$ since each M_i is a stochastic matrix.

Let $\vec{\mu}$ be the invariant measure for M. Then $|\vec{\mu}|(B) = |M(\vec{\mu})|(B)$.

Thus, letting $\mu = |\vec{\mu}|$ we know that μ is the attractor of the IFS with probabilities $\{w_i, p_i\}$ or that

$$\mu(B) = \sum_i p_i \mu\left(w_i^{-1}(B)\right).$$

Heuristically, we can think of $\vec{\mu}$ as

$$d\vec{\mu}(x) = \vec{f}(x)\ d\mu(x)$$

where $\vec{f}(x)$ is the "attractor" of the "IFS"

$$\vec{f}(x) \longrightarrow \sum_i M_i \vec{f}\left(w_i^{-1}(x)\right).$$

Since $\vec{\mu}$ is absolutely continuous with respect to μ we know that $\vec{f}(x)$ exists and is the Radon-Nikodym derivative of $\vec{\mu}$ with respect to μ.

Back to color chaos game

For the color chaos game, the matrices M_i are

$$M_i = (1 - \hat{p}_i)I + \hat{p}_i\delta_i$$

where

$$\delta_i = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and the i^{th} row is the row of ones.

Given a set $B \subset X$ with $\mu(B) > 0$, the probability measure on Λ defined by

$$\frac{\vec{\mu}(B)}{\mu(B)}$$

describes the distribution of the colors on the set B.

Application to vector calculus

Suppose that we specify the IFS operator with the restriction that each R_i is a rotation.

In this case, the contractivity condition becomes

$$\sum_i s_i p_i < 1.$$

Suppose that $T : \mathcal{S}_v(X, \mathbb{R}^n) \to \mathcal{S}_v(X, \mathbb{R}^n)$ and $0 \neq w \in \mathbb{R}^n$.

Let R be a rotation of R^n so that $R(v) = (\|v\|/\|w\|)w$.

Then $\hat{T} : \mathcal{S}_w(X, \mathbb{R}^n) \to \mathcal{S}_w(X, \mathbb{R}^n)$ where

$$\hat{T}(\mu)(B) = \frac{\|w\|}{\|v\|} \sum_i p_i (R \circ R_i \circ R^{-1}) \mu(w_i^{-1}(B)).$$

Thus we can modify T to preserve any other vector we like.

If $\sum_i p_i R_i v = v$ for some $v \in \mathbb{R}^2$, then, in fact, $\sum_i p_i R_i w = w$ for all $w \in \mathbb{R}^2$ since rotations in $\mathbb{R}^2$ commute.

In this case, $\sum_i p_i R_i = I$.

Application to fractal curves: Tangent and Normal vectors

Let C be a continuous curve which is the attractor of the affine IFS $W = \{w_i\}$.

Let v be the vector from the initial point of C to the final point of C.

The image of v under the IFS will be the piecewise linear approximation of the curve.

Let $v_i = w_i(v)$, then $\Sigma_i v_i = v$.

Let R_i be the rotation that takes v into the direction of v_i and $p_i = \|v_i\|/\|v\|$.

Then $p_i R_i(v) = v_i$ so

$$\sum_i p_i R_i(v) = \sum_i v_i = v.$$

Thus, the IFS operator T (using the above R_i and p_i) leaves $\mathcal{S}_v(X, I\!R^n)$ invariant.

Tangent Vector Measure

Assuming that T is contractive, we interpret the invariant measure μ as the *tangent vector measure* to the curve C.

WHY??

Let L be the line segment from the initial point of C to the final point of C.

Let λ be the normalized arclength measure on L and define $\eta \in \mathcal{S}_v(X, I\!R^n)$ by

$$\eta(B) = v\lambda(B).$$

η is thought of as the tangent vector measure to L.

$T^n(\eta)$ is the tangent vector measure to $W^n(L)$, the n^{th} order piecewise linear approximation to C.

We think of μ as $d\mu(s) = \vec{T}(s)ds$ where $\vec{T}$ is the "unit tangent" vector field for C and ds is the "arclength" element for C.

The problem is that neither one of these exist!

However, if $B \subset C$ is some segment then

$$\int_B \, d\mu = \mu(B) = B_{final} - B_{initial}$$

is the displacement vector, just as in the smooth curve case.

Normal Vector Measure

Assume that $C \subset I\!R^2$.

Let w be a rotation of v by an angle of $\pi/2$.

Then T leaves $\mathcal{S}_w(X, I\!R^n)$ invariant as well (recall we are in the plane).

If ν is the invariant measure of T in $\mathcal{S}_w(X, I\!R^n)$ we think of ν as the normal vector measure to C.

The fly in the ointment: measures of unbounded semivariation

We assumed that the operator T (defined for the tangent vector measure) was contractive, is it?

The problem is that $\|T(\nu)\| \geq \|\nu\|$, so semivariation might increase.

As an example, we use the Koch curve.

We get $\|\mu_0\| = 1$, $\|\mu_1\| = 1$, $\|\mu_2\| = 11/9$.

In general,

$$\|\mu_n\| = \lfloor 4^n/3 \rfloor (2)(1/3^n) + 1/2(1/3^n).$$

Thus, $\|\mu_n\| \to \infty$.

CAN'T USE THE METRIC!!!!!!!!!!!

Solution – unbounded linear functional

Our salvation is that we only need to integrate against Lipschitz functions.

Suppose that $\Sigma_i s_i p_i = \sigma < 1$.

For $f \in \mathcal{L}ip(X, I\!R^n)$ we have

$$\| (T^*)^n f(x) - (T^*)^n f(y)\| \leq \sigma^n d(x,y) \leq \sigma^n diam(X) \to 0.$$

Thus, $\lim_n (T^*)^n f$ is a constant function.

Define $\mathcal{T} : \mathcal{L}ip(X, I\!R^n) \to I\!R$ by

$$\mathcal{T}(f) = \lim_n (T^*)^n f$$

then $\mathcal{T}$ is a linear functional.

Since T^* is the dual operator to T, we think of $\mathcal{T}$ as being the dual to the "invariant measure" μ.

Notice that $\mathcal{T}$ is only defined on $\mathcal{L}ip(X, I\!R^n)$. It is usually unbounded on the set of all continuous $I\!R^n$-valued function on X.

Green's Theorem for domains with fractal boundaries

Let $D \subset \mathbb{R}^2$ be a compact domain whose boundary is the union of m affine IFS fractal curves C_i.

Suppose further that each C_i has no self-intersections and intersects at most two other $C_j's$ at at most one endpoint.

Finally, suppose that the Lebesgue measure of each C_i is zero.

Let $f : U \to \mathbb{R}^2$ be a smooth function defined on some neighborhood of D.

For each C_i we can compute

$$\int_{C_i} f(x) \cdot \, d\mu_i(x).$$

Thus, we know

$$\int_{\partial D} f(x) \cdot d\mu(x) = \sum_i \int_{C_i} f(x) \cdot \, d\mu_i(x).$$

Let D_1 be a polygonal approximation to D obtained by using line segments from the initial to final points of each C_i.

Under the combined IFS mapping, D_1 maps to some refined polygonal approximation D_2. Continuing, we obtain D_n as a sequence of polygonal approximations to D.

The boundary of D_n, ∂D_n, is an approximation to the boundary of D.

Lemma 1 *Let χ_D and χ_{D_n} be the characteristic functions of D and D_n respectively. Then $\chi_D \to \chi_{D_n}$ pointwise for almost all x.*

By the Bounded Convergence Theorem,

$$\int_{D_n} \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dxdy \to \int_D \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dxdy.$$

For each n, we have

$$\int_{\partial D_n} f(x) \cdot d\mu_n(x) = \int_{D_n} \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dxdy.$$

Since $\mu_n \to \mu$ we have

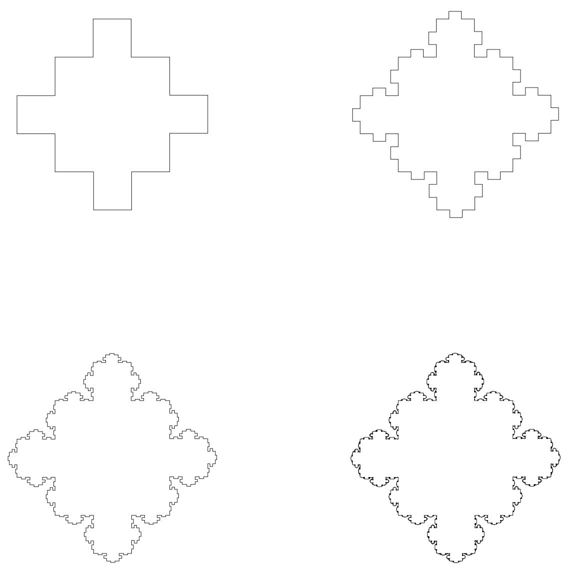
$$\int_{\partial D_n} f(x) \cdot d\mu_n(x) \to \int_{\partial D} f(x) \cdot d\mu(x).$$

Theorem 2 *(Green's Theorem) Let $D \subset \mathbb{R}^2$ be a compact domain with ∂D being the disjoint union of finitely many fractal curves and $dim_H(\partial D) < 2$. If f is a smooth vector field, then*

$$\int_{\partial D} f(x) \cdot d\mu(x) = \int_D \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \, dxdy$$

where μ is the tangent vector measure to ∂D.

Illustration of D_n and D and their boundaries



Extensions and further work

The same framework should work for higher dimensional domains using the differential form formalism.

In this case, the vector values would be in the exterior algebra of the appropriate dimension.

This same general framework could be extended to Banach space-valued measures or Operator-valued measures.

Applications to analysis on fractals are also possible (e.g. ODE's and PDE's).