A Two Phase Flow Model Including Dynamics of Interfacial Area

Hans Fredrik Nordhaug, William G. Gray,
Helge K. Dahle, M.A. Celia and Magne S. Espedal

Outline

- Motivation
- Assumptions/basic equations
- Fractional-flow formulation
- Parametrization of:
  - resitances
  - Capillary pressure-saturation-interfacial area
- Prelimenary numerical results

Motivation

- Standard descriptions of porous media flow do not explicitly account for interface dynamics
- More complete models will allow us to assess the level of detail needed in the models
- Practical reasons for calculating amount of interfacial area

Conservation equations

Basic assumptions:

- two-phase flow
- isothermal system
- immobile, non-deforming solid
- common lines are neglected
- the solid-fluid interface dynamics are negligible (i.e no film flow)
- inertial terms are negligible in the momentum equations
- interfacial tensions are all constant and specified

Mass conservation for the w-phase

$$\frac{D^w(\varepsilon^w\rho^w)}{Dt} + \varepsilon^w\rho^w\nabla\cdot\mathbf{v}^w = 0. \tag{1}$$

Momentum conservation for the w-phase

$$-\varepsilon^w\nabla p^w + \varepsilon^w\rho^w\mathbf{g} = (\mathbf{R}^w_{wn} + \mathbf{R}^w_{ws})\cdot\mathbf{v}^w - \mathbf{R}^w_{wn}\cdot\mathbf{v}^{wn}. \tag{2}$$

Mass conservation for the n-phase

$$\frac{D^n(\varepsilon^n\rho^n)}{Dt} + \varepsilon^n\rho^n\nabla\cdot\mathbf{v}^n = 0. \tag{3}$$

Momentum conservation for the n-phase

$$-\varepsilon^n\nabla p^n + \varepsilon^n\rho^n\mathbf{g} = (\mathbf{R}^n_{wn} + \mathbf{R}^n_{ns})\cdot\mathbf{v}^n - \mathbf{R}^n_{wn}\cdot\mathbf{v}^{wn}. \tag{4}$$

Momentum conservation for a massless wn-interface

$$(\mathbf{R}^w_{wn} + \mathbf{R}^n_{wn}) \cdot \mathbf{v}^{wn} = \mathbf{R}^w_{wn} \cdot \mathbf{v}^w + \mathbf{R}^n_{wn} \cdot \mathbf{v}^n. \tag{5}$$

Geometry equation for Interfacial Area

$$\frac{\partial a^{wn}}{\partial t} - \varepsilon J^w_{wn} \frac{\partial s^w}{\partial t} - a^s \cos \Phi^w \frac{\partial x^{ws}_s}{\partial t} = -\nabla \cdot [\mathbf{G}^{wn} a^{wn} \cdot \mathbf{v}^{wn}]. \tag{6}$$

Dynamic capillary pressure equation

$$\varepsilon \frac{\partial s^w}{\partial t} = -L_p [p^n - p^w + \gamma^{wn} J^w_{wn}]. \tag{7}$$

Linearized constitutive equation for surface area fraction

$$a^s \frac{\partial x^{ws}_s}{\partial t} = -L_x [\gamma^{ws} - \gamma^{ns} + \gamma^{wn} \cos \Phi^w]. \tag{8}$$

Additional assumptions:

- No production of interfacial area. Consequently, we will disregard Equation (8) and rewrite Equation (6):

$$\frac{\partial a^{wn}}{\partial t} + \nabla \cdot [\mathbf{G}^{wn} a^{wn} \cdot \mathbf{v}^{wn}] = 0 \qquad (9)$$

- Setting $L_p = \infty$ reduces Equation (7) to

$$\gamma^{wn} J_{wn}^{w} = p^w - p^n = -P_c. \qquad (10)$$

- Incompressible flow gives constant total Darcy velocity in 1D.

Fractional flow formulation

Step 1:
Combine the momentum equations to obtain generalized forms of the Darcy equation for the phases.

$$\mathbf{u}^w = -\ (\mathbf{M}^w)^{-1} \cdot [\varepsilon^w \rho^w \nabla \psi^w + \varepsilon^n \rho^n \mathbf{C}^n \cdot \nabla \psi^n], \tag{11}$$

$$\mathbf{u}^n = -\ (\mathbf{M}^n)^{-1} \cdot [\varepsilon^n \rho^n \nabla \psi^n + \varepsilon^w \rho^w \mathbf{C}^w \cdot \nabla \psi^w], \tag{12}$$

where

$$\psi^\alpha = p^\alpha/\rho^\alpha - gz$$

$$\mathbf{A}^\alpha = (\mathbf{R}_{wn}^w + \mathbf{R}_{wn}^n)^{-1} \cdot \mathbf{R}_{wn}^\alpha/\varepsilon^\alpha, \quad \alpha = w, n$$

$$\mathbf{B}^\alpha = \mathbf{R}_{wn}^\alpha \cdot (\frac{\mathbf{I}}{\varepsilon^\alpha} - \mathbf{A}^\alpha) + \frac{1}{\varepsilon^\alpha}\mathbf{R}_{\alpha s}^\alpha, \quad \alpha = w, n$$

$$\mathbf{C}^n = \mathbf{R}_{wn}^w \cdot \mathbf{A}^n \cdot (\mathbf{B}^n)^{-1}$$

$$\mathbf{C}^w = \mathbf{R}_{wn}^n \cdot \mathbf{A}^w \cdot (\mathbf{B}^w)^{-1}$$

$$\mathbf{M}^w = \mathbf{B}^w - \mathbf{C}^n \cdot \mathbf{R}_{wn}^n \cdot \mathbf{A}^w$$

$$\mathbf{M}^n = \mathbf{B}^n - \mathbf{C}^w \cdot \mathbf{R}_{wn}^w \cdot \mathbf{A}^n$$

($\mathbf{M}^\alpha$ is the total mobility for the α-phase)

Step 2:

Express phase velocities $\mathbf{u}^w$ and $\mathbf{u}^n$, and interface velocity $\mathbf{v}^{wn}$, as a function of the total Darcy velocity $\mathbf{u} = \mathbf{u}^w + \mathbf{u}^n$ and the capillary pressure $P_c = p^n - p^w$:

$$\mathbf{u}^w = \mathbf{F}_w \cdot \mathbf{u} + \mathbf{D}_w \cdot (\mathbf{g}^{wn} - \nabla P_c)$$

$$\mathbf{u}^n = \mathbf{F}_n \cdot \mathbf{u} + \mathbf{D}_n \cdot (\mathbf{g}^{wn} + \nabla P_c)$$

$$\mathbf{v}^{wn} = \mathbf{F}_{wn} \cdot \mathbf{u} + \mathbf{D}_{wn} \cdot (\mathbf{g}^{wn} - \nabla P_c)$$

where

$$\begin{aligned}
\mathbf{g}^{wn} &= (\rho^n - \rho^w)\mathbf{g}, \\
\mathbf{F}_w &= (\varepsilon^w + \varepsilon^n \mathbf{C}^n) \cdot \mathbf{M}^n \cdot \mathrm{N}^{-1}, \\
\mathbf{F}_n &= (\varepsilon^w \mathbf{C}^w + \varepsilon^n) \cdot \mathbf{M}^w \cdot \mathrm{N}^{-1}, \\
\mathbf{D}_w &= \varepsilon^n \varepsilon^w (1 - \mathbf{C}^w \cdot \mathbf{C}^n) \cdot \mathrm{N}^{-1} = -\mathbf{D}_n,
\end{aligned}$$

$$\mathrm{N} = \varepsilon^n \mathbf{M}^w + \varepsilon^n \mathbf{C}^n \cdot \mathbf{M}^n + \varepsilon^w \mathbf{C}^w \cdot \mathbf{M}^w + \varepsilon^w \mathbf{M}^n.$$

$$\mathbf{F}^{wn} = \mathbf{A}^w_{wn} \cdot \mathbf{F}_w + \mathbf{A}^n_{wn} \cdot \mathbf{F}_n$$

$$\mathbf{D}^{wn} = (\mathbf{A}^w_{wn} - \mathbf{A}^n_{wn}) \cdot \mathbf{D}_w$$

Step 3:

Substitute velocity terms back into mass conservation equation for w-phase and geometrical equation for interfacial area:

$$\epsilon \frac{\partial s^w}{\partial t} + \mathbf{u} \cdot \nabla \cdot (\mathbf{F}_w) + \mathbf{g}^{wn} \cdot \nabla \cdot (\mathbf{D}_w) = \nabla \cdot (\mathbf{D}_w \cdot \nabla P_c)$$

$$\begin{aligned} \frac{\partial a^{wn}}{\partial t} + \frac{1}{3} \left(\mathbf{u} \cdot \nabla \cdot (\mathbf{F}_{wn} a^{wn}) + \mathbf{g}^{wn} \cdot \nabla \cdot (\mathbf{D}_{wn} a^{wn}) \right) = \\ \frac{1}{3} \nabla \cdot (a^{wn} \mathbf{D}_{wn} \cdot \nabla P_c)) \end{aligned}$$

Parametrization of resistances

We assume the functional form of the resistances $\mathbf{R}^{\alpha}_{\alpha\beta}$ to be

$$\mathbf{R}^{w}_{wn} = \mu^{w} g^{w}_{wn}(a^{wn}) h^{w}_{wn}(s^{w}) \mathbf{K}^{-1} {\varepsilon^{w}}^{2}$$

$$\mathbf{R}^{w}_{ws} = \mu^{w} g^{w}_{ws}(a^{ws}) h_{w}(s^{w}) \mathbf{K}^{-1} {\varepsilon^{w}}^{2}$$

$$\mathbf{R}^{n}_{wn} = \mu^{n} g^{n}_{wn}(a^{wn}) h^{n}_{wn}(s^{n}) \mathbf{K}^{-1} {\varepsilon^{n}}^{2},$$

$$\mathbf{R}^{n}_{ns} = \mu^{n} g^{n}_{ns}(a^{ns}) h_{n}(s^{n}) \mathbf{K}^{-1} {\varepsilon^{n}}^{2}.$$

The motivation for these forms are as follows:
Equation (2) and (4) are generalized forms of Darcy's law. We therefore expect μ^{α} and $\mathbf{K}$ to be terms in the resistances. In addition, to make the dimensions on the left and right side of the equations to match ${\varepsilon^{\alpha}}^{2}$ must be a factor too. The g's and h's must be non-dimensional (and positive).

To get expressions for the g's and h's we look at the limit cases where we have only one phase. Equation (2) and (4) should then reduce to the usual Darcy's law.

One set of functions that satifies such conditions are:

$$
\begin{aligned}
g^w_{wn}(a^{wn}) &= \frac{a^{wn}}{a^s}, & h^w_{wn}(s^w) &= 1 - s^w \\
g^n_{wn}(a^{wn}) &= \frac{a^{wn}}{a^s}, & h^n_{wn}(s^n) &= 1 - s^n = s^w \\
g^w_{ws}(a^{ws}) &= \frac{a^{ws}}{a^s}, & h_w(s^w) &= k^{-1}_{rw}(s^w) = ({s^w}^2)^{-1} \\
g^n_{ns}(a^{ns}) &= \frac{a^{ns}}{a^s}, & h_n(s^n) &= k^{-1}_{rn}(s^n) = ({s^n}^2)^{-1} = ((1 - s^w)^2)^{-1}
\end{aligned}
$$

Fractional flow functions and diffusion functions in 1D:

$$F_w = \frac{\mu^n {s^w}^2({s^w}^3 a^{wn}\mu^w - 2a^{wn}\mu^w {s^w}^2 + (a^{wn}\mu^w - \mu^n a^{ns} + a^{ns}\mu^w)s^w + \mu^n a}{N}$$

$$F_n = -\frac{\mu^w(1-s^w)^2(a^{wn}\mu^n {s^w}^3 - a^{wn}\mu^n {s^w}^2 - (\mu^w a^{ws} - a^{ws}\mu^n)s^w - a^{ws}\mu^n)}{N}$$

and

$$D_w = -\frac{Ka^s((\mu^w - \mu^n)s^w + \mu^n)}{N} {s^w}^2 (1-s^w)^2 = -D_n$$

where N is given by

$$N = \mu^w a^{wn}\mu^n {s^w}^4 + ({\mu^w}^2 a^{ws} - 2\mu^w a^{wn}\mu^n - {\mu^n}^2 a^{ns} - \mu^w a^{ws}\mu^n + \mu^n a^{ns}\mu^w){s^w}^3 + (3\mu^w a^{ws}\mu^n - 2{\mu^w}^2 a^{ws} + {\mu^n}^2 a^{ns} + \mu^w a^{wn}\mu^n){s^w}^2 + (-3\mu^w a^{ws}\mu^n + {\mu^w}^2 a^{ws})s^w + \mu^w a^{ws}\mu^n,$$

Parametrization of capillary pressure

To fit $P_c - S$ relationships, let (van Genuchten):

$$P_c = \frac{(s^{-\frac{n}{n-1}} - 1)^{\frac{1}{n}}}{\alpha_G}. \qquad (13)$$

We fix n and vary α_G to get three curves satisfying

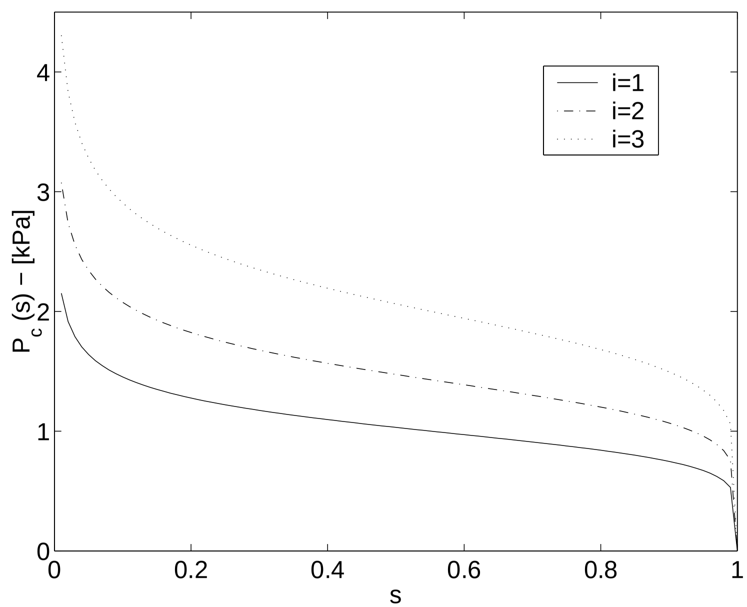
$$P_c^1 \le P_c^2 \le P_c^3$$

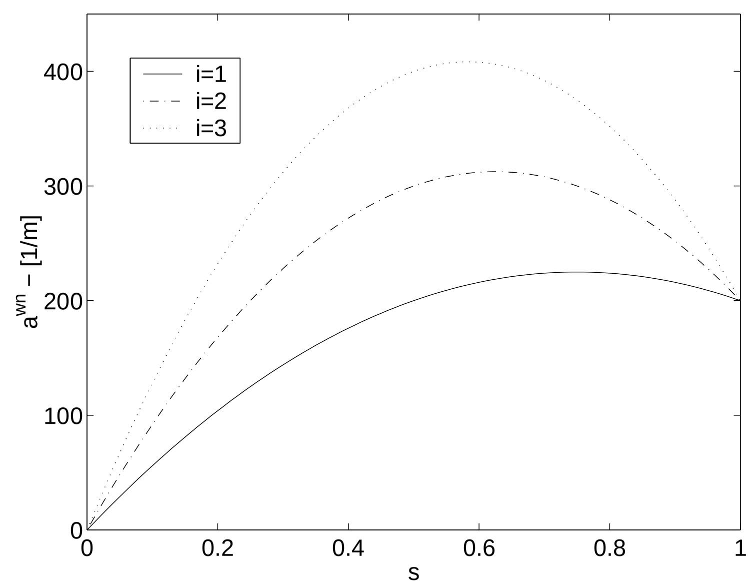
where P_c^1 and P_c^3 are approximations to the inner and outer envelopes for the $P_c - S$ curves. Along these three curves we specify the interfacial area:

$$a_i^{wn}(s) = \alpha_i s(1-s) + \beta s, \quad i = 1, 2, 3, \qquad (14)$$

where α_i is chosen such that $a_1^{wn} \le a_2^{wn} \le a_3^{wn}$.

Ex.





For a given P_c we specify $a^{wn}(s, P_c)$ as

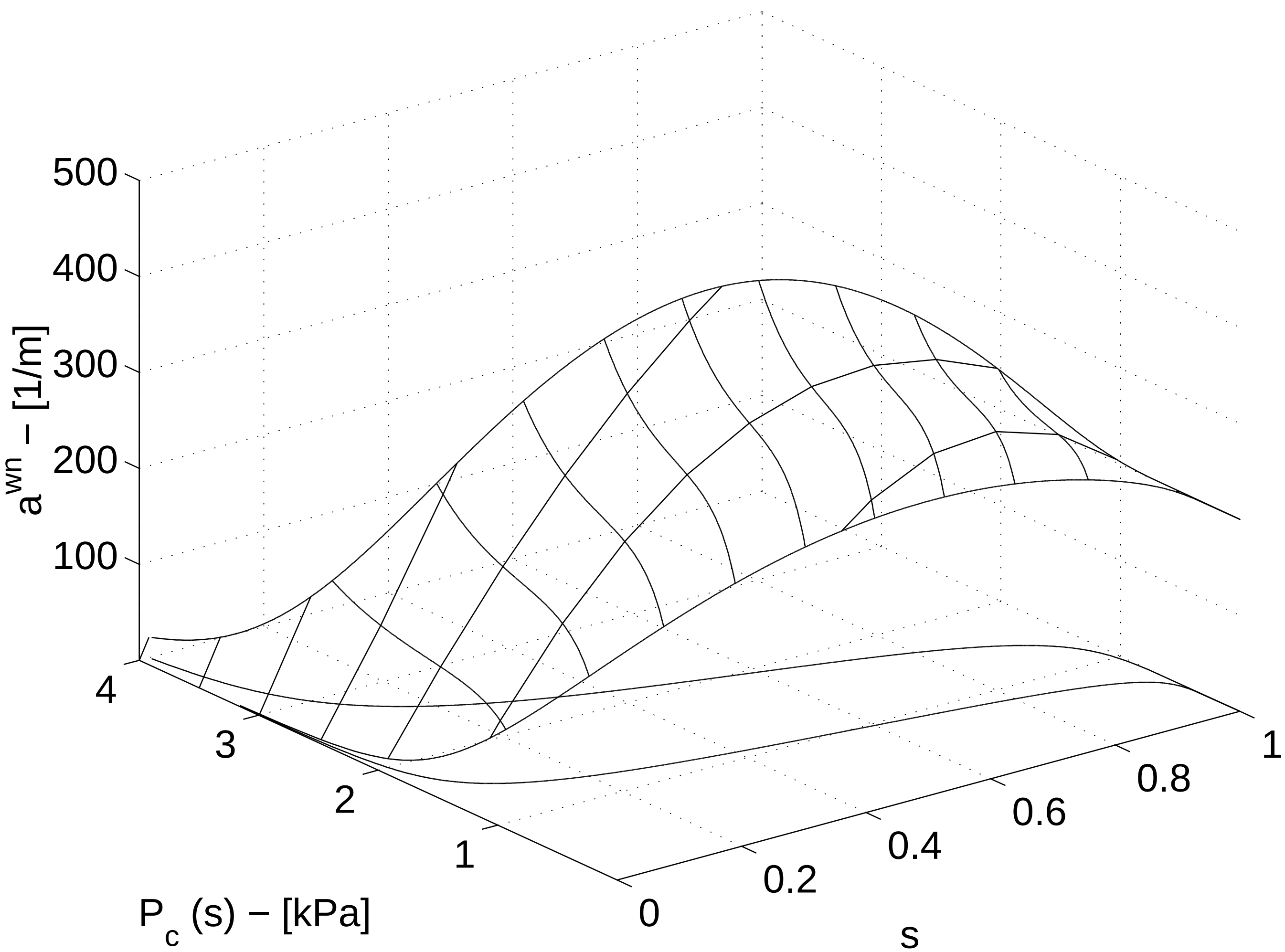
$$a^{wn}(s, P_c) = c_1 s^2 + c_2 s + c_3. \tag{15}$$

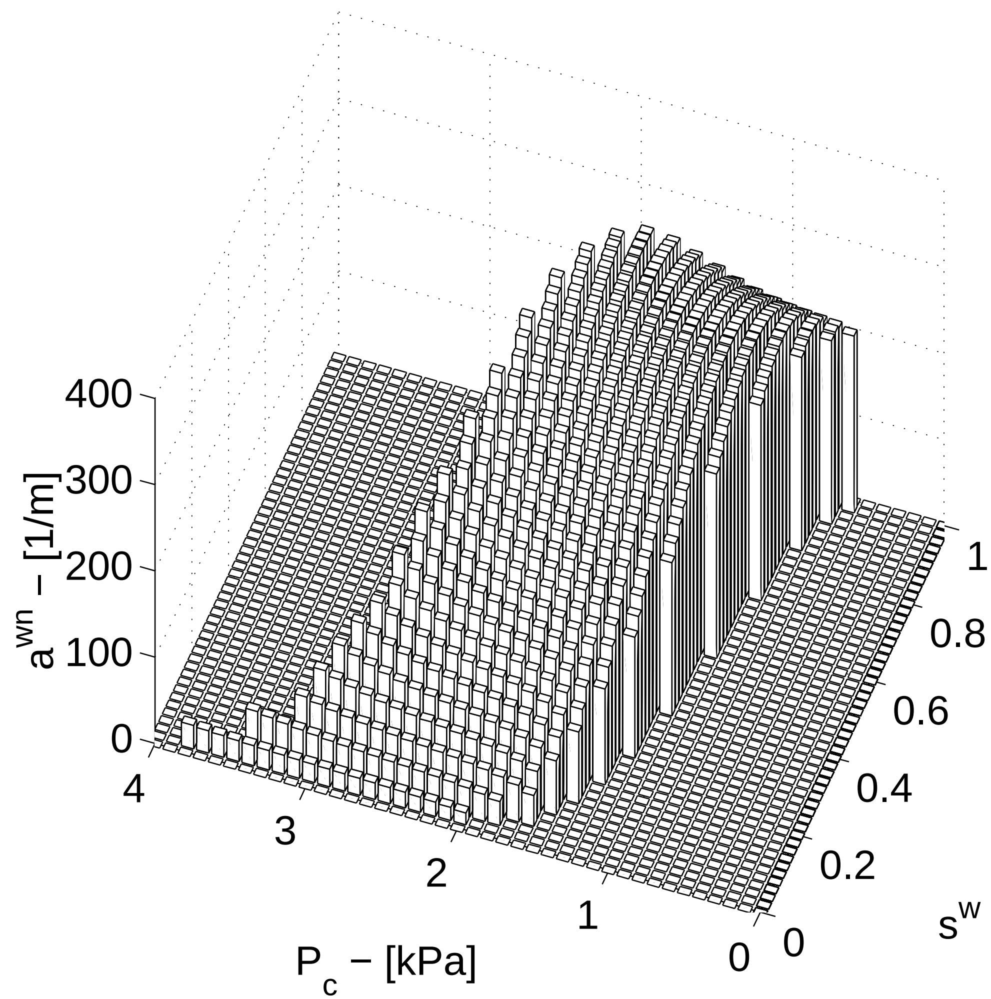
Using Lagrangian interpolation we get the following expression for the interfacial area as a function of capillary pressure and saturation:

$$\begin{aligned} a^{wn}(P_c, s) = & a_1^{wn}(P_c)\frac{(s - s_2(P_c))(s - s_3(P_c))}{(s_1(P_c) - s_2(P_c))(s_1(P_c) - s_3(P_c))} \\ & + a_2^{wn}(P_c)\frac{(s - s_1(P_c))(s - s_3(P_c))}{(s_2(P_c) - s_1(P_c))(s_2(P_c) - s_3(P_c))} \\ & + a_3^{wn}(P_c)\frac{(s - s_1(P_c))(s - s_2(P_c))}{(s_3(P_c) - s_1(P_c))(s_3(P_c) - s_2(P_c))} \end{aligned} \tag{16}$$

Table of parameters:

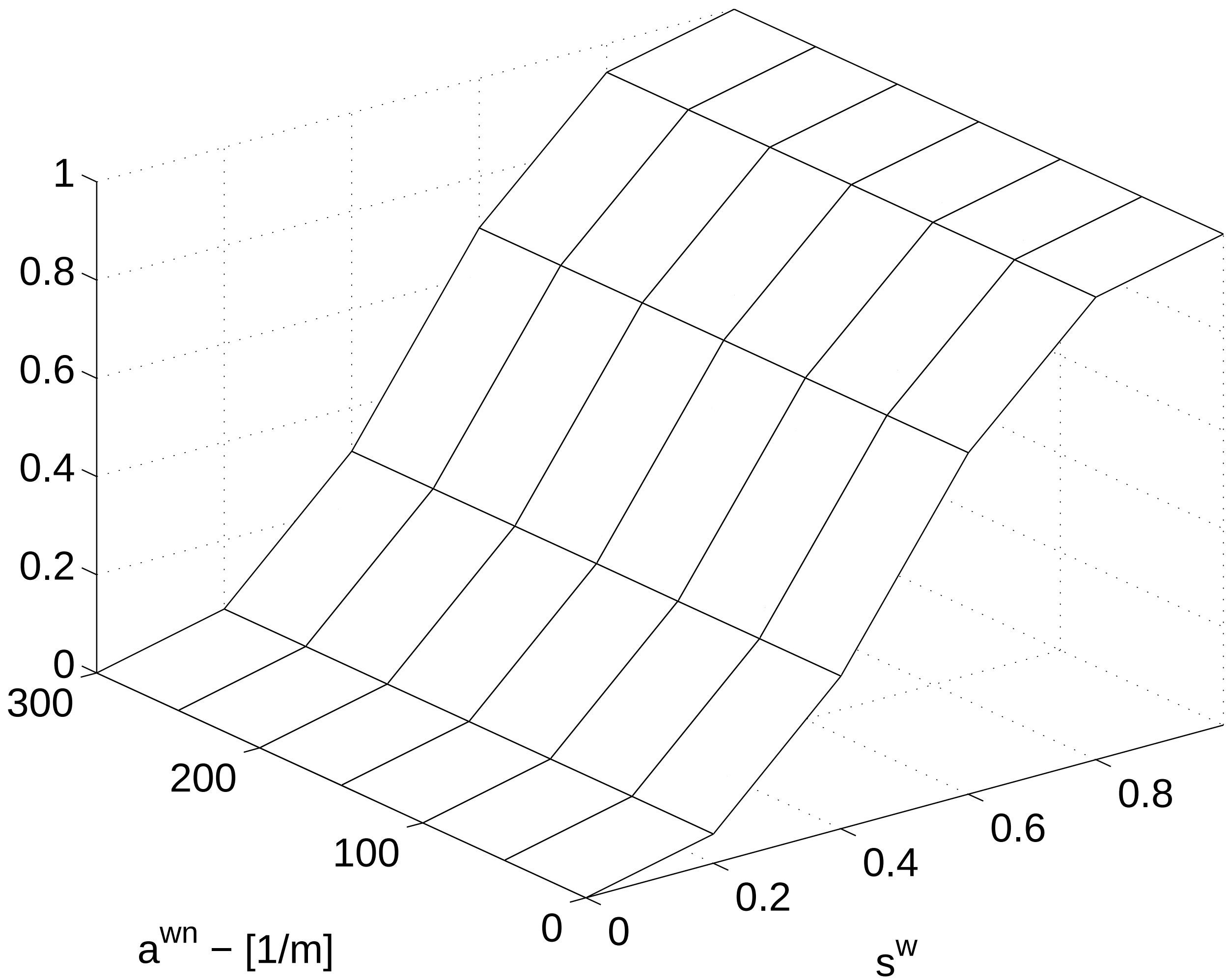
$$
\begin{aligned}
\mu^w &= 10^{-3} && [Ns/m^2] \\
\mu^n &= 10^{-2} && [Ns/m^2] \\
\epsilon &= 0.27 && \\
K &= 1.3 \times 10^{-9} && [m^2] \\
u &= 1.3 \times 10^{-3} && [m/s] \\
a^s &= 7.6 \times 10^4 && [1/m] \\
a^{ns} &= 0.7 \times 10^4 && [1/m] \\
a^{ws} &= 6.9 \times 10^4 && [1/m] \\
n &= 7 && \\
\alpha_G &= 10^{-3},\ 0.7 \times 10^{-3},\ 0.5 \times 10^{-3} && \\
\alpha_i &= 400,\ 800,\ 1200 && \\
\beta &= 0 &&
\end{aligned}
$$


500
400
300
200
100
a^{wn} – [1/m]
4
3
2
1
P_c (s) – [kPa]
0
0.2
0.4
0.6
0.8
1
s

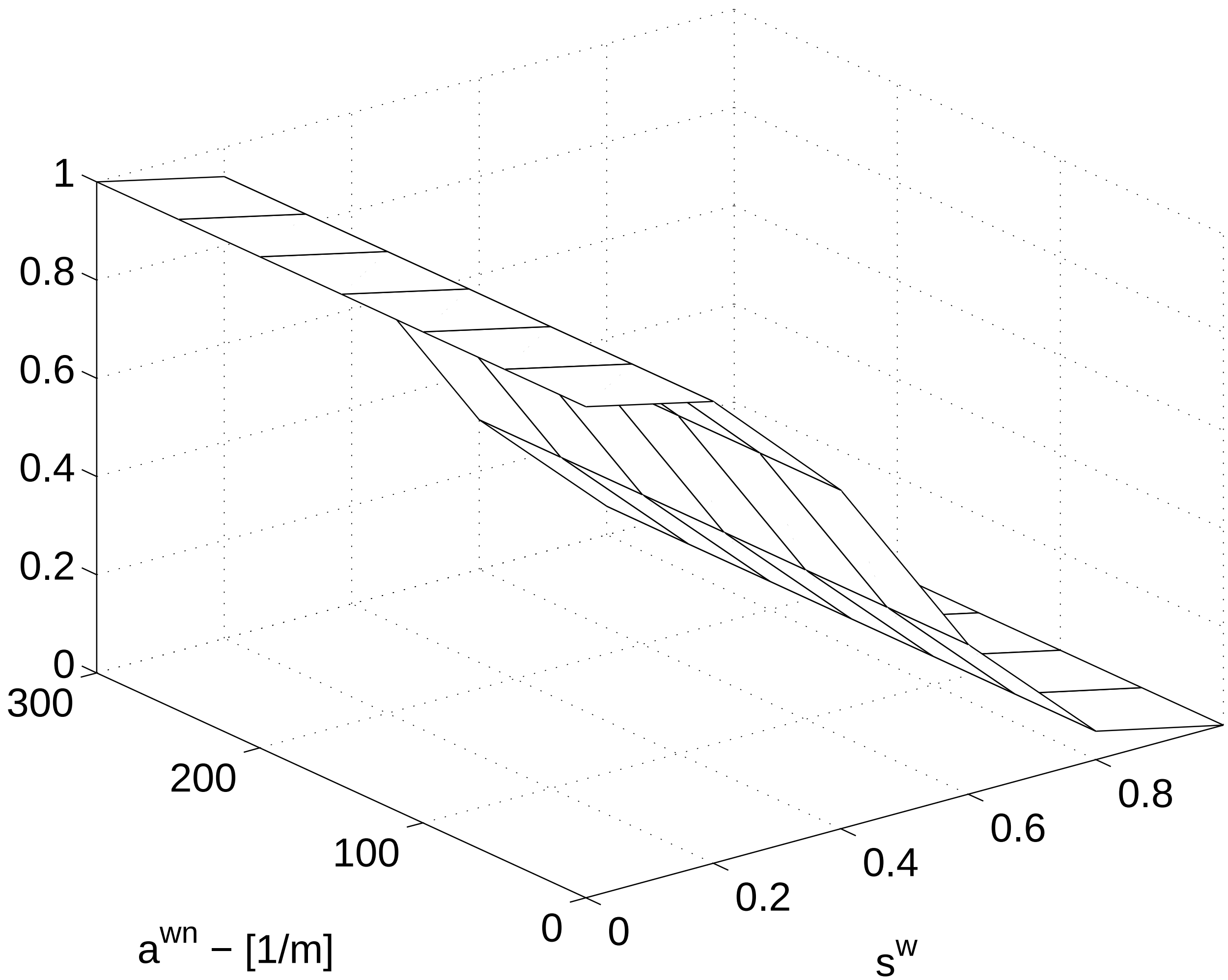


400
300
200
100
0
a^{wn} – [1/m]
4
3
2
1
0
P_c – [kPa]
0
0.2
0.4
0.6
0.8
1
s^w

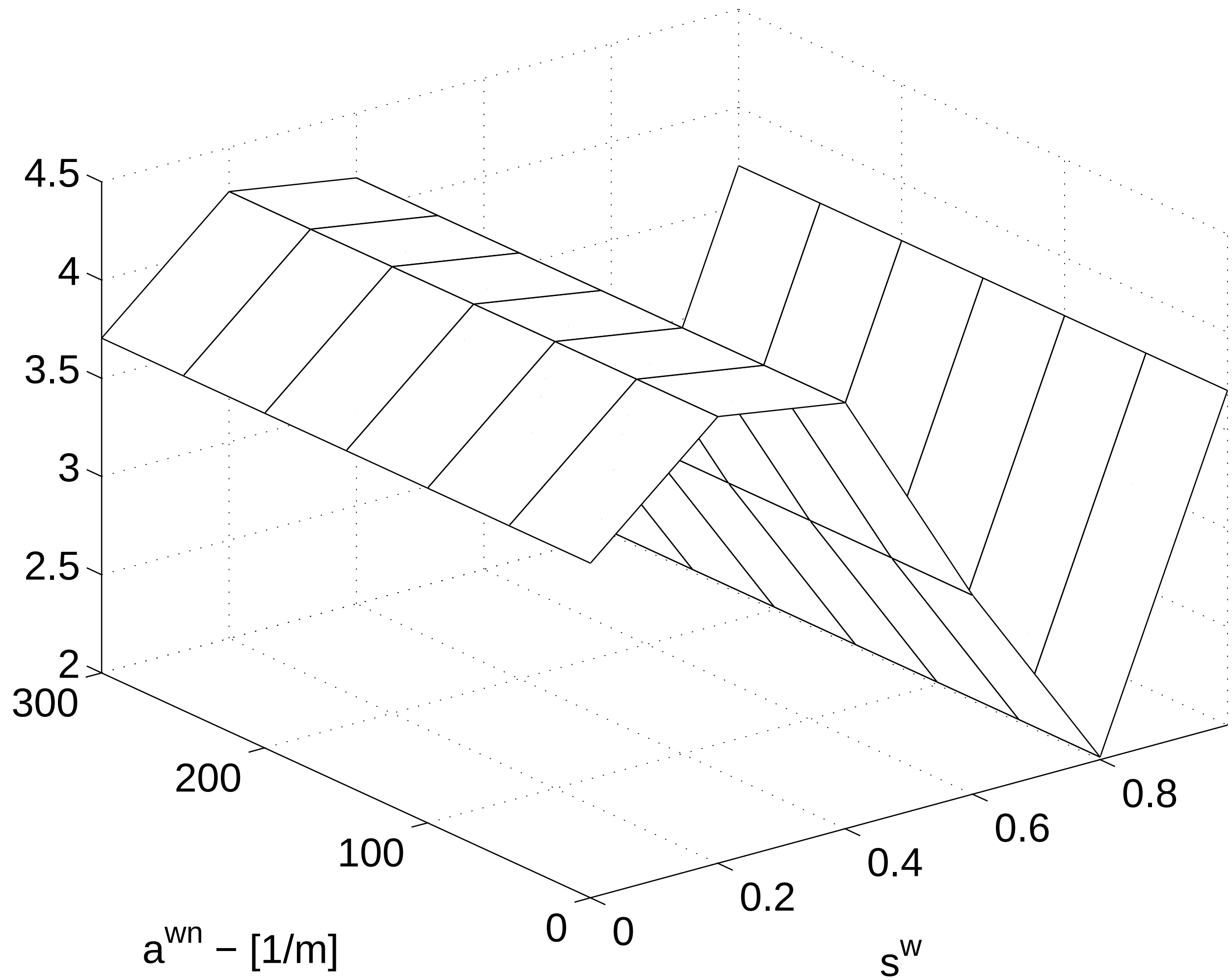
$\mathbf{F}_w$:



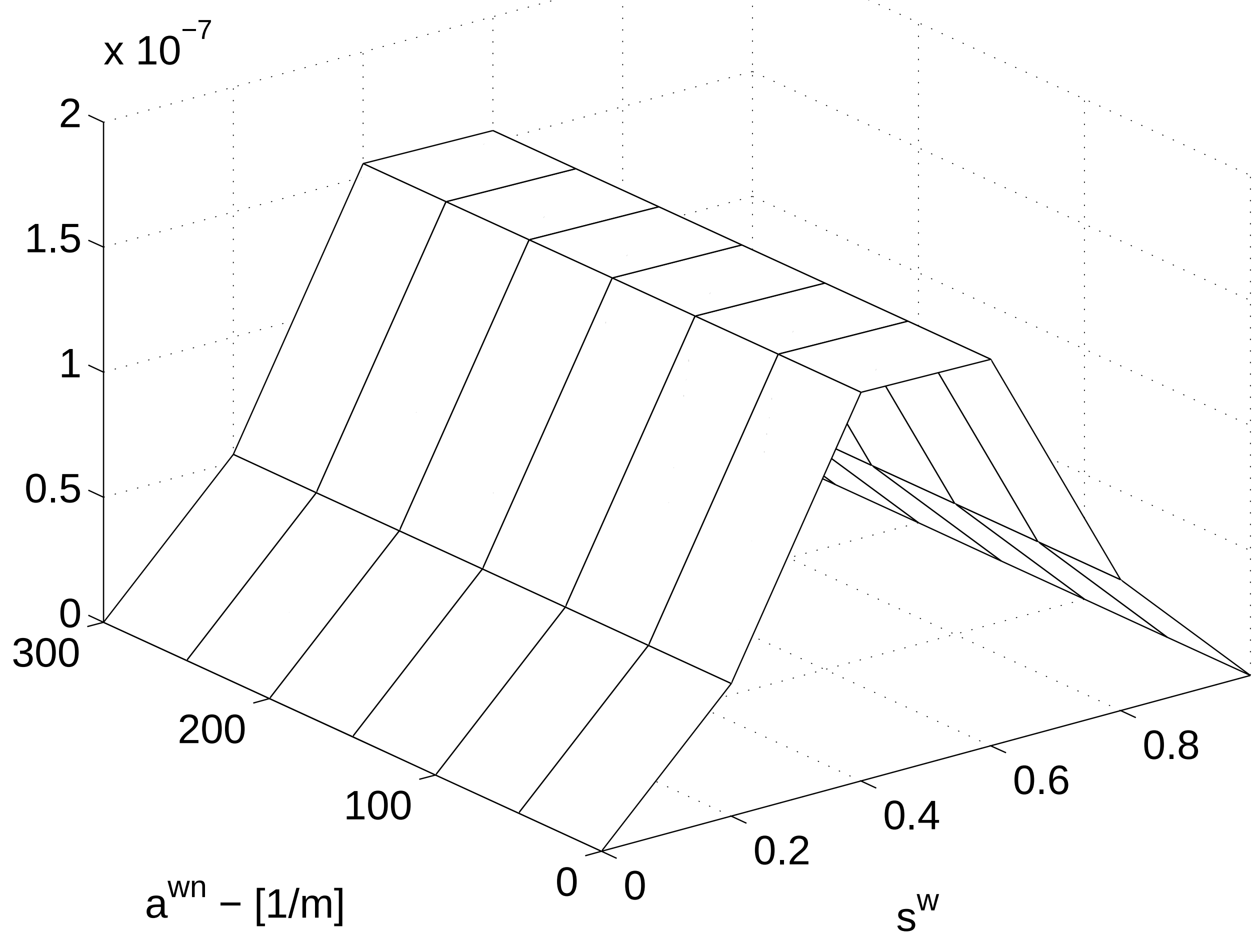
$\mathbf{F}_n$:



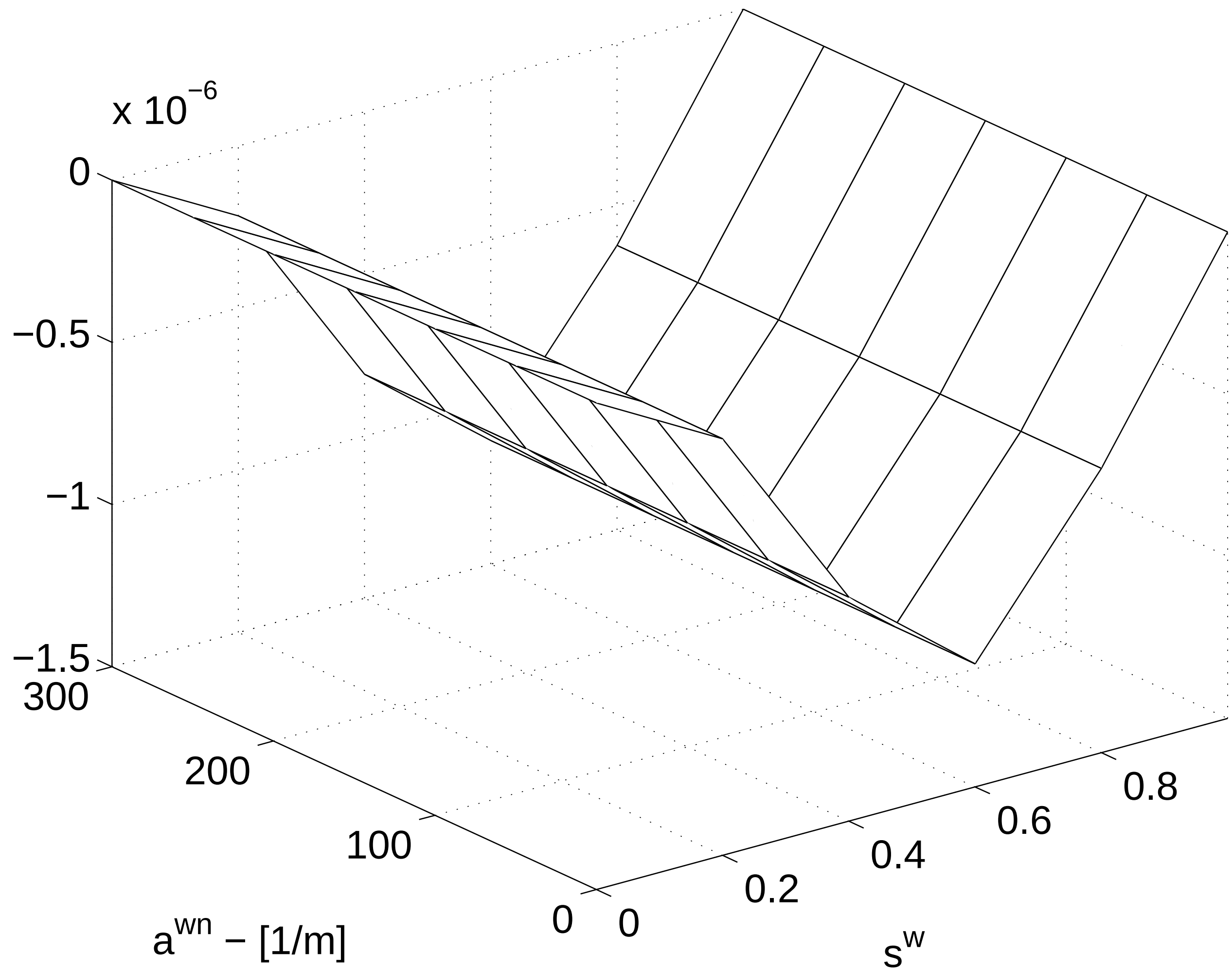
$\mathbf{F}_{wn}$:



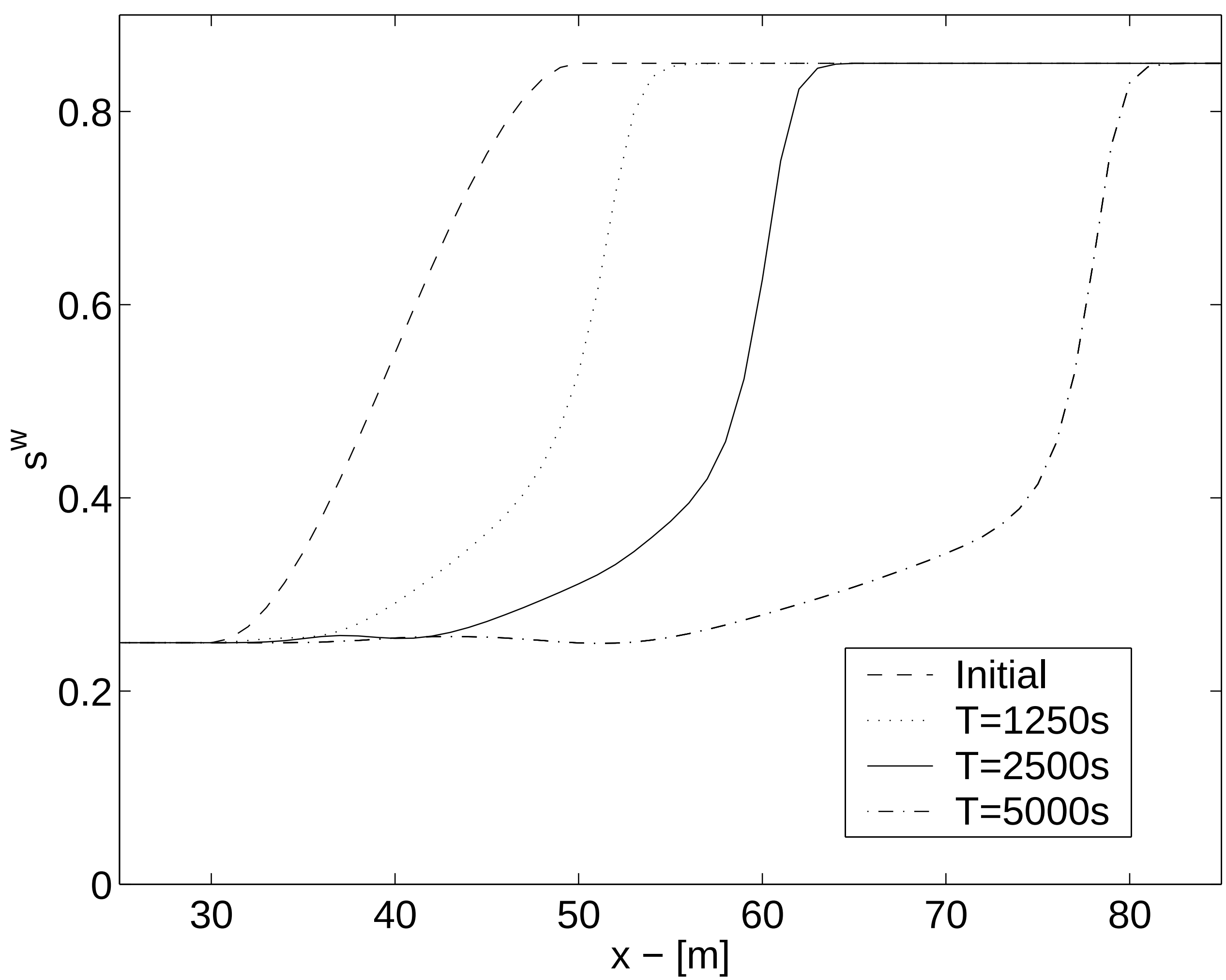
$\mathbf{D}_w$:



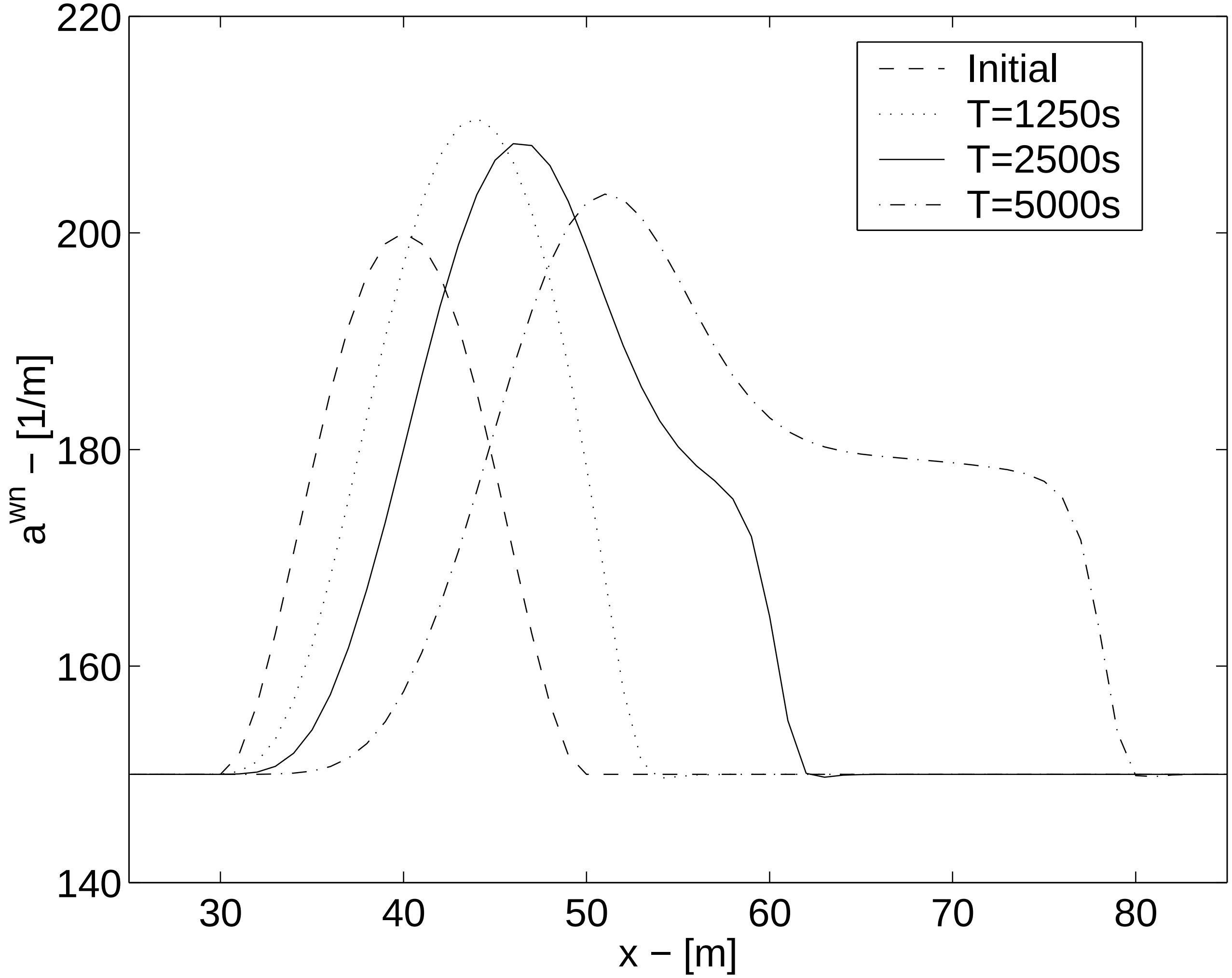
$\mathbf{D}_{wn}$:



Saturation vs. time



Interfacial area vs. time



Summary

- 2×2 system of eq.'s is derived with saturation and wn-interfacial area as primary variables
- functional dependencies of resistances and parametrization of wn-interfacial area w.r.t. capillary pressure and saturation are given
- remaining modelling difficulties:
  - momentum equation for wn-interfacial area
  - production term for wn-interfacial area
- benifits of 'complex' model:
  - do not have to worry about hysteresis
  - obtain wn-interfacial area as function of space and time

Nomenclature

The following terminology is used in the article:

ε	porosity of the medium.
s^α	saturation of the α-phase,i.e.,
	volume fraction of the void space occupied by fluid phase α.
$\varepsilon^\alpha = \varepsilon s^\alpha$	volume fraction of the α-phase.
ρ^α	density of α-phase.
$\mathbf{v}^\alpha$	velcoity of α-phase.
$\mathbf{v}^{wn}$	velocity of the wn-interface.
p^α	pressure of α-phase.
$\mathbf{R}^\alpha_{wn}$	resistance for the α-phase due to the wn-interface (fluid-fluid).
$\mathbf{R}^\alpha_{\alpha s}$	resistance for the α-pahse due to the αs-interface (fluid-solid).
a^{wn}	specicfic interfacial area of the wn-interaface (fluid-fluid).
a^s	specific area of the solid phase surface.
J^w_{wn}	average curvature of the wn-interface (calculated with $\mathbf{n}^\alpha$ positive).
$\gamma^{\alpha\beta}$	surface tension of the $\alpha\beta$-interface.
Φ^w	average contact angle between the w- and s-phase.
$\mathbf{G}^{wn}$	geometric tensor.
x^{ws}_s	fraction of the solid surface covered by the wetting phase.
L_p	coefficient for the dynamic capillary pressure equation.
L_x	coefficient for the surface area fraction equation.

In addition we have introduced:

$\mathbf{u}^\alpha$		Darcy velocity for the α-phase.
$\mathbf{u}$	$=$	total Darcy velocity.
$\mathbf{u}^w + \mathbf{u}^n$		
μ^α		viscosity for the α-phase.
$\mathbf{K}$		permeability for the medium.
$a^{\alpha s}$		specicfic interfacial area of the αs-interaface (fluid-solid).