Self-similar jet breakup for a generalized PTT model

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Abstract

We discuss self-similar solutions for jet breakup for a family of models of non-Newtonian fluids which includes the Phan-Thien Tanner model. The constitutive law is given by

$$\begin{aligned}\mathbf{T}_t + (\mathbf{v}\cdot\nabla)\mathbf{T} - (\nabla\mathbf{v})\mathbf{T} - \mathbf{T}(\nabla\mathbf{v})^T + \kappa\mathbf{T} \\ +\nu(\operatorname{tr}\mathbf{T})^{a-1}\mathbf{T} = \mu(\nabla\mathbf{v} + (\nabla\mathbf{v})^T),\end{aligned} \tag{1}$$

where $a > 1$. For $a = 2$, a recent paper establishes the existence of a similarity solution for surface tension driven jet breakup. In this paper, we show that analogous similarity solutions exist for $a < 7/3$. If, on the other hand $a > 7/3$, then surface tension plays no role in the breakup, and a different type of similarity solution exists.

1 Introduction

It is well known that the addition of polymers acts to suppress the surface-tension driven breakup of liquid jets. This effect is important, for example, in controlling the size of droplets into which fluids break up in such applications as fire fighting or spraying liquids from the air. The simplest problem to study is the surface-tension driven instability of a uniform jet. Linear stability analysis predicts that disturbances grow faster on a viscoelastic jet than on a Newtonian jet with the same viscosity. Numerical studies [1] confirm this. However, in the highly nonlinear regime, the viscoelastic jets become stabilized. Instead of breaking up, they tend to form a "beads on a string" structure. Numerical simulations for viscoelastic fluids have reproduced these phenomena. We refer to [14] for a review of this problem.

Several recent studies have analyzed one-dimensional models for the evolution of fluid jets, based on a thin jet approximation [8,3,7,4,2]. Such models predict breakup of Newtonian jets in finite time, and the approach to breakup can be

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described by a similarity solution [8,3]. Viscoelasticity delays the breakup, and for the Oldroyd B fluid without inertia, it can be shown that breakup will not occur at all [9]. On the other hand, the Giesekus fluid does exhibit breakup in finite time [10]. Since the Giesekus and PTT fluid behave in the same fashion at high elongation rates, the same behavior applies to the PTT model. It was recently shown [11] that this breakup can also be described by a similarity solution. In this paper, we address the question whether analogous solutions exist for a broader class of models in which the term $(\mathrm{tr}\,\mathbf{T})\mathbf{T}$ term in the PTT model is replaced by a power $(\mathrm{tr}\,\mathbf{T})^{a-1}\mathbf{T}$ with $a > 1$. Rather surprisingly, we shall find that similarity solutions which are analogous to those for $a = 2$ exist only if $a < 7/3$. If $a > 7/3$, it turns out that surface tension plays no role in the breakup and a different type of similarity solution applies. Similarity solutions for breakup without surface tension actually exist for $a > 2$. However, if $2 < a < 7/3$, surface tension will alter the characteristics of the breakup if it is present. We remark that the analysis in [11] also addressed the effect of a retardation time. In the present paper, we shall not do so, since the models we consider are elongation thinning, and the inclusion of a retardation time would simply make the breakup asymptotics Newtonian.

As we shall see in the next section, the fluid model considered here is elongation thinning at high elongation rates. Although polymers show strongly strain hardening behavior over a range of elongation rates, elongation thinning at high deformation rates is not uncommon. Elongation thinning behavior has been linked to the possibility of breakup without surface tension. In particular, the "Considère effect," i.e. a decrease in the force in a filament with increasing elongational strain, has been suggested as a factor which might lead to such a breakup [6]. Indeed, recent results of Fontelos [5] on the Johnson-Segalman model show the possibility of breakup without surface tension in such a situation. Neither the Giesekus model nor the model considered here, however, have the Considère effect. Nevertheless, breakup without surface tension is possible if $a > 2$. Indeed, McKinley [13] observed what appeared to be breakup without surface tension in numerical studies of the Giesekus model ($a = 2$). As shown in [12], no breakup actually occurs, but the radius thins extremely quickly (like an exponential of an exponential). The present results show that the Giesekus model is actually "on the edge" of allowing for breakup without surface tension.

2 Steady flow behavior of the model

In this section, we shall briefly review the properties of the model under study in steady flows. For shear flow with shear rate $\dot{\gamma}$, the constitutive relation reduces to

$$
\begin{aligned}
-2\dot{\gamma}T_{11} + \kappa T_{11} + \nu T_{11}^a &= 0, \\
\kappa T_{12} + \nu T_{11}^{a-1} T_{12} &= \mu\dot{\gamma}.
\end{aligned} \tag{2}
$$

Like for the PTT model, there is no second normal stress difference. The equations can be recast in the form

$$
\dot{\gamma} = \frac{\kappa}{\mu} T_{12} + \frac{2^{a-1}\nu}{\mu^a} T_{12}^{2a-1}, \tag{3}
$$

and

$$
T_{11} = \frac{2T_{12}^2}{\mu}. \tag{4}
$$

This yields a monotone relation between shear stress and shear rate. At high shear rate, we have the power law $T_{12} \sim \dot{\gamma}^{1/(2a-1)}$, $T_{11} \sim \dot{\gamma}^{2/(2a-1)}$.

In steady elongational flow with stretch rate $\dot{\epsilon}$, we have the relations

$$
\begin{aligned}
-2\dot{\epsilon}T_{11} + \kappa T_{11} + \nu (T_{11} + 2T_{rr})^{a-1} T_{11} &= 2\mu\dot{\epsilon}, \\
\dot{\epsilon}T_{11} + \kappa T_{rr} + \nu (T_{11} + 2T_{rr})^{a-1} T_{rr} &= -\mu\dot{\epsilon}.
\end{aligned} \tag{5}
$$

With $\tau = T_{11} - T_{rr}$, these equations can be shown to yield

$$
\dot{\epsilon} = \frac{\kappa}{4\tau} Y + \frac{\nu}{2^{a+1}\tau} Y^a, \tag{6}
$$

where

$$
Y = -3\mu - \tau + \sqrt{8\tau^2 + (3\mu + \tau)^2}. \tag{7}
$$

This reduces to

$$
\dot{\epsilon} = \frac{\kappa\tau}{3\mu} \tag{8}
$$

at low elongation rates, and

$$
\dot{\epsilon} = \frac{3^a \nu}{2^{a+1}} \tau^{a-1} \tag{9}
$$

at high elongation rates. If $a > 2$, this means that τ increases more slowly than $\dot{\epsilon}$, i.e. the fluid is elongation thinning.

3 Governing equations

The situation considered is that of a free jet, with an undisturbed uniform configuration. The description is Lagrangian, i.e. all variables are regarded as functions of the position X of a particle in the undisturbed uniform jet and time t. Let s denote the amount of stretch, δ the unperturbed jet radius, σ the surface tension coefficient, T_{11} and T_{rr} the axial and radial stress components, and u the axial velocity. If inertial terms are neglected, the force balance in the jet leads to

$$\frac{T_{11} - T_{rr}}{s} + \frac{\sigma}{\delta\sqrt{s}} = \lambda(t), \tag{10}$$

where $\lambda(t)$ is an undetermined function representing the stress per unit undeformed area of the jet (this is proportional to the force in the jet). Moreover, the equality of mixed partial derivatives implies that

$$s_t = u_X. \tag{11}$$

To complete the description, we need to relate $T_{11} - T_{rr}$ to s. This depends on the constitutive law for the fluid. For the generalized PTT model mentioned above, we have the ordinary differential equations

$$\begin{aligned} (T_{11})_t - 2T_{11}s_t/s + \kappa T_{11} + \nu(T_{11} + T_{rr})^{a-1}T_{11} &= 2\mu s_t/s, \\ (T_{rr})_t + T_{rr}s_t/s + \kappa T_{rr} + \nu(T_{11} + T_{rr})^{a-1}T_{rr} &= -\mu s_t/s. \end{aligned} \tag{12}$$

We note that in the Lagrangian description, the time derivative represents the material time derivative, and the velocity gradient has the form s_t/s. Here $\mu > 0$, $\kappa > 0$, $\nu > 0$.

We can simplify the equations if we set

$$\begin{aligned} T_{11}(X,t) &= p(X,t)s(X,t)^2 - \mu, \\ T_{rr}(X,t) &= q(X,t)/s(X,t) - \mu. \end{aligned} \tag{13}$$

The advantage of this substitution is to transform away the terms involving s_t/s in the constitutive equation, and we obtain

$$\begin{aligned} p_t + \kappa p + \nu p(ps^2 + \frac{q}{s} - 2\mu)^{a-1} &= \kappa\mu s^{-2}, \\ q_t + \kappa q + \nu q(ps^2 + \frac{q}{s} - 2\mu)^{a-1} &= \kappa\mu s. \end{aligned} \tag{14}$$

4 Similarity solutions

We look for similarity solutions analogous to those of [11]. With $t = 0$ denoting the breakup time, such solutions have the form

$$s(X,t) = (-t)^{-\alpha}\tilde{s}(\frac{X}{(-t)^{\beta}}), \quad p(X,t) = (-t)^{3\alpha/2}\tilde{p}(\frac{X}{(-t)^{\beta}}), \quad q(X,t) = 0,$$
$$u(X,t) = (-t)^{\beta-\alpha-1}\tilde{u}(\frac{X}{(-t)^{\beta}}), \tag{15}$$

and the force $\lambda(t)$ is of the form $\lambda = k(-t)^{\alpha/2}$. Here

$$\alpha = \frac{2}{a-1}. \tag{16}$$

We insert this ansatz into the equations above, retaining only the leading order terms for $t \to 0$. With the similarity variable $\xi = X/(-t)^{\beta}$, the resulting system of equations is

$$\tilde{p}\tilde{s}^3 + \frac{\sigma}{\delta}\tilde{s}^{3/2} = k\tilde{s}^2,$$
$$-\frac{3\alpha}{2}\tilde{p} + \beta\xi\tilde{p}'(\xi) + \nu\tilde{p}^a\tilde{s}^{2a-2} = 0,$$
$$\alpha\tilde{s} + \beta\xi\tilde{s}'(\xi) = \tilde{u}'(\xi). \tag{17}$$

As in [8] and [11], the system (17) is invariant under a scaling of ξ and therefore has no characteristic length scale. The length scale over which the self-similar solution is valid is therefore determined by the global dynamics of the jet. We can rescale s by $(\sigma/\delta)^{-2}\nu^{-2/(a-1)}$, p by $\nu^{3/(a-1)}(\sigma/\delta)^4$, k by $(\sigma/\delta)^2\nu^{1/(a-1)}$. After this rescaling of variables, we recover the same equations with the parameters ν and σ/δ set to one, so we shall henceforth set $\nu = \sigma/\delta = 1$. We can eliminate $\tilde{p}$ from the first equation of (17) and insert into the second equation. In addition, we substitute $\tilde{s}(\xi) = \psi(\xi)^2$. This results in the equation

$$2(-\psi + k\psi^2)^a + 3\alpha\psi(1 - k\psi) + 6\beta\xi\psi' - 4\beta k\xi\psi\psi' = 0. \tag{18}$$

We seek solutions of (18) which have the behavior

$$\psi(\xi) \sim a_0 + a_2\xi^2 + O(\xi^4) \tag{19}$$

near $\xi = 0$. After some calculation, this leads to the conditions

$$k = \frac{3(-1 + 2\beta)}{2a_0(-3 + 2\beta)}, \tag{20}$$

and

$$\Big(\frac{a_0(3+2\beta)}{4\beta-6}\Big)^{a-1} = \frac{3}{a-1}. \tag{21}$$

We now look for solutions to (18) which start out from a value $a_0 = \psi(0) > 3/(2k)$. With increasing ξ, ψ decreases. According to (18), ψ' then becomes infinite when $\psi = 3/(2k)$. At this point, ψ jumps to zero, see [11]. We note that equation (10) has the form

$$ps - \frac{q}{s^2} + s^{-1/2} = \lambda(t). \tag{22}$$

For given p and q and λ sufficiently large, this equation has three solutions for s; let us refer to these three solutions as the upper, middle and lower branch. The similarity solutions for breakup follow the upper branch until it merges with the middle branch; at this point there is a jump to the lower branch. The solution on the lower branch, however, is not governed by the same asymptotic scalings as the similarity solution. With these scalings, the term q/s^2 in (22) becomes negligible, and the lower branch collapses to $s = 0$.

Following [8] and [11], we shall assume that large velocities are confined to the self-similar region. If $\beta < \alpha + 1$, then this requirement, together with the last equation of (17) implies that

$$\int_{-\infty}^{\infty} \alpha\tilde{s} + \beta\xi\tilde{s}'\,d\xi = 0. \tag{23}$$

If $\tilde{s}$ jumps to zero at a finite ξ_0, this condition reads

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} \alpha\tilde{s} + \beta\xi\tilde{s}'\,d\xi = 2\int_{0}^{\infty} \alpha\tilde{s} + \beta\xi\tilde{s}'\,d\xi \\ &= 2\int_{0}^{\xi_0+} \alpha\tilde{s} + \beta\xi\tilde{s}'\,d\xi = 2(\alpha-\beta)\int_{0}^{\xi_0} \tilde{s}\,d\xi. \end{aligned} \tag{24}$$

Consequently, $\beta = \alpha = 2/(a-1)$. In (20) and (21), this leads to

$$k = \frac{3(5-a)}{(7-3a)a_0}, \tag{25}$$

and

$$\Big(\frac{a_0(1+3a)}{14-6a}\Big)^{1/a-1} = \frac{3}{a-1}. \tag{26}$$

In order for a_0 to be positive, we need to restrict a to be less than 7/3. In the limit $a \longrightarrow 7/3$, a_0 tends to zero, and k tends to infinity like $1/a_0^2$. This means that, at the breakup point, the term $k\tilde{s}^2$ is proportional to a_0^2, while the term $\tilde{s}^{3/2}$ (which represents the surface tension force) is proportional to $a_0^3 << a_0^2$. We conclude that in the limit $a \longrightarrow 7/3$, surface tension ceases to play a role in the breakup. In the next section, we shall look for similarity solutions for breakup without surface tension (i.e., breakup occurs because fluid is pulled out of the filament by elastic forces).

Since $\beta = \alpha$, the Eulerian position

$$x(X,t) = \int_0^X s(Y,t)\,dY = \int_0^\xi \tilde{s}(\zeta)\,d\zeta \tag{27}$$

is a time-independent function of the similarity variable. Hence the jet profile remains fixed in space as breakup is approached, and the breakup occurs simultaneously over a finite length of the jet.

We remark that, as in the Newtonian case, the assumption of neglecting inertia ultimately becomes inconsistent as breakup is approached. We can see that the stress ps^2 is proportional to $(-t)^{-alpha/2}$ as breakup is approached while the Reynolds stress u^2 is proportional to $(-t)^{-2}$. Hence inertial terms will eventually change the dominant balance if $a > 3/2$.

5 Breakup without surface tension

We now drop the term involving surface tension in the equations of motion. If we do so, there is no longer a reason why $\lambda(t)$ should behave like $(-t)^{\alpha/2}$, and our similarity ansatz becomes

$$\begin{aligned} s(X,t) &= (-t)^{-\alpha}\tilde{s}\Big(\frac{X}{(-t)^\beta}\Big), \quad p(X,t) = (-t)^{\alpha+\gamma}\tilde{p}\Big(\frac{X}{(-t)^\beta}\Big), \quad q(X,t) = 0, \\ u(X,t) &= (-t)^{\beta-\alpha-1}\tilde{u}\Big(\frac{X}{(-t)^\beta}\Big), \end{aligned} \tag{28}$$

and $\lambda(t) = k(-t)^\gamma$. We obtain the equations

$$\begin{aligned}\tilde{p}\tilde{s} &= k,\\ -(\alpha+\gamma)\tilde{p}+\beta\xi\tilde{p}'(\xi)+\tilde{p}^a\tilde{s}^{2a-2} &= 0,\\ \alpha\tilde{s}+\beta\xi\tilde{s}'(\xi) &= \tilde{u}'(\xi).\end{aligned} \tag{29}$$

Moreover, we find the relationship

$$\gamma = \alpha - \frac{1}{a-1}. \tag{30}$$

Combining the first two equations of (29) results in the equation

$$(k\tilde{s})^a - k((2\alpha - \frac{1}{a-1})\tilde{s} + \beta\xi\tilde{s}') = 0. \tag{31}$$

We set $k\tilde{s} = \phi$, and we can solve the differential equation in the form

$$\phi^{1-a}(1+2\alpha-2a\alpha) + a - 1 = C\xi^{(-1-2\alpha+2a\alpha)/\beta}. \tag{32}$$

We expect $2a\alpha - 2\alpha - 1$ to be positive, and ϕ to be a decreasing function of ξ. In this case C must be negative, and we may rescale ξ such that $C = -1$. We obtain quadratic behavior near $\xi = 0$ if

$$\beta = (a-1)\alpha - \frac{1}{2}. \tag{33}$$

In this case, the solution becomes

$$\phi(\xi) = \Big(\frac{2(a-1)\alpha - 1}{a-1+\xi^2}\Big)^{1/(a-1)}. \tag{34}$$

Moreover, α is determined by the condition that

$$\int_0^\infty \alpha\phi(\xi) + \beta\xi\phi'(\xi)\, d\xi = 0. \tag{35}$$

This condition leads to

$$\alpha = \frac{1}{2a-4}. \tag{36}$$

Self-similar solutions for breakup without surface tension therefore exist for $a > 2$. We need to consider, however, how surface tension would affect them if it is present. We note that the term λs^2 in the momentum equation is proportional to $(-t)^{-2\alpha+\gamma}$ and

$$-2\alpha + \gamma = -\alpha - \frac{1}{a-1}. \tag{37}$$

On the other hand the surface tension term $s^{3/2}$ behaves like $(-t)^{-3\alpha/2}$. In order for surface tension terms to be of lower order and not affect the breakup asymptotics, we must have $3\alpha/2 < \alpha + 1/(a-1)$, i.e.

$$\alpha = \frac{1}{2a-4} < \frac{2}{a-1}. \tag{38}$$

This is the case if $a > 7/3$.

We note that there is a fundamental difference between the case $a < 3$ and $a > 3$ (I am grateful to an anonymous referee for pointing this out). For $a < 3$, γ is positive, and $\phi(\xi)$ is integrable. This means that the spatial length of the self-similar region is finite and the force in the jet tends to zero as breakup is approached. For $a > 3$, on the other hand, γ is negative and $\phi(\xi)$ is not integrable. Hence the spatial length of the self-similar region is infinite, and the solution cannot be connected to any "outer" solution in another part of the jet where s stays finite. Moreover, the force in the jet approaches infinity as breakup is approached.

6 Comparison with the generalized Newtonian model

One of the properties of the generalized PTT model considered above is that it is elongation thinning for $a > 2$, i.e. the elongational viscosity tends to zero at infinite elongation rates. It is a natural question whether a generalized Newtonian fluid which is thinning at high stretch rates behaves in a qualitatively similar fashion. We shall show that indeed this is not the case.

Let us consider the generalized Newtonian model

$$T_{11} = 2\eta\frac{s_t}{s}|\frac{s_t}{s}|^{a-1}, \quad T_{rr} = -\eta\frac{s_t}{s}|\frac{s_t}{s}|^{a-1}. \tag{39}$$

The force balance in the jet becomes

$$3\eta\frac{s_t}{s}|\frac{s_t}{s}|^{a-1} + \frac{\sigma}{\delta}s^{1/2} = \lambda(t)s. \tag{40}$$

We make the similarity ansatz

$$s(X,t) = (-t)^{-\alpha}\phi(\frac{X}{(-t)^{\beta}}), \quad \lambda(t) = k(-t)^{\alpha/2}. \tag{41}$$

By rescaling variables, we may assume without loss of generality that $3\eta = \sigma/\delta = 1$. The resulting equation is

$$(\alpha\phi + \beta\xi\phi')|\alpha\phi + \beta\xi\phi'|^{a-1} + \phi^{a+1/2} - k\phi^{a+1} = 0, \tag{42}$$

and $\alpha = 2a$.

We first note that there cannot be any similarity solutions for breakup without surface tension. If we ignore surface tension, the term $\phi^{a+1/2}$ is absent from (42), and it follows that $\alpha\phi+\beta\xi\phi'$ has the same sign throughout. This, however, is inconsistent with the condition that

$$\int_{-\infty}^{\infty} \alpha\phi + \beta\xi\phi' \, d\xi = 0. \tag{43}$$

We shall now show that, in fact, a solution of (42) exists for any positive a. We begin by analyzing the behavior near $\xi = 0$. At $\xi = 0$, $\alpha\phi + \beta\xi\phi' = \alpha\phi$ is positive, and we can omit the absolute value signs in (42). We now assume the behavior

$$\phi(\xi) = y_0 + y_2\xi^2 + O(\xi^4), \tag{44}$$

near $\xi = 0$. By inserting this into (42), we can derive the relations

$$k = \frac{(2a)^a}{y_0} + \frac{1}{\sqrt{y_0}}, \quad \beta = 1 + 2^{-1-a}y_0^{1/2+a}. \tag{45}$$

For any choice of $y_0 > 0$, we can determine k and β according to (45), and then find a solution of (42) which has $\phi(0) = y_0$, decreases with increasing ξ, and has quadratic behavior near 0. We next consider the asymptotic behavior of this solution near infinity. As $\xi \to \infty$, we have $\phi \to 0$, and the terms of order ϕ^a in (42) dominate. It follows that ϕ tends to zero like $\xi^{-\alpha/\beta}$: $\phi(\xi) \sim C\xi^{-\alpha/\beta}$, $C > 0$. The next order correction results from the term $\phi^{1/2+a}$ in the equation. If we make the asymptotic ansatz

$$\phi(\xi) = C\xi^{-\alpha/\beta} + A\xi^{-\gamma} + ..., \tag{46}$$

$\gamma > \alpha/\beta$, we obtain

$$\gamma = \frac{\alpha}{\beta}(1 + \frac{1}{2a}) = \frac{2a+1}{\beta}, \tag{47}$$

and

$$A(\alpha - \beta\gamma) = -A = -C^{1+1/(2a)}. \tag{48}$$

Consequently, A is positive and $\gamma > 1$ as long as $\beta < 2a + 1$. If y_0 is small, then , according, to (45), β is close to 1 and this last inequality is satisfied. As y_0 is increased, there is a critical value where β reaches $2a + 1$. At this point, the integral

$$\int_{-\infty}^{\infty} \alpha\phi(\xi) + \beta\xi\phi'(\xi)\, d\xi \tag{49}$$

becomes divergent and tends to $-\infty$. Consequently, for y_0 close to this critical value,

$$\int_{-\infty}^{\infty} \alpha\phi(\xi) + \beta\xi\phi'(\xi)\, d\xi \tag{50}$$

must be negative.

On the other hand, we may consider the limit $y_0 \to 0$. In this case, we set $\phi(\xi) = y_0\psi(\xi)$ in (42), use the formulas (45) for k and β and retain only those terms in the equation which are at the leading order for $y_0 \to 0$. The result is the equation

$$(2a\psi(\xi) + \xi\psi'(\xi))^a = (2a)^a\psi(\xi)^{1+a}. \tag{51}$$

It is evident from this equation that

$$\int_{-\infty}^{\infty} 2a\psi(\xi) + \xi\psi'(\xi)\, d\xi \tag{52}$$

is positive.

Consequently, the integral in (43) is positive when y_0 is small, but becomes negative when y_0 becomes large enough. There must thus be an intermediate value of y_0 where it is equal to zero.

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