# FUNCTORIAL REMARKS ON THE GENERAL CONCEPT OF CHAOS

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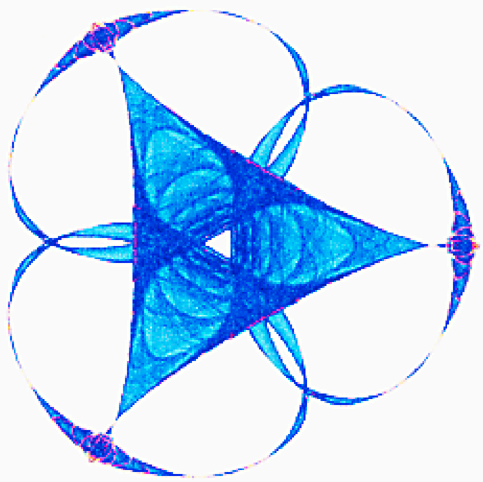

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# Functorial Remarks on the General Concept of Chaos

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Although the term "chaos" is employed in various ways in current dynamics literature, several instances [1], [2] of a precise usage have in common two features of a categorical nature: surjectivity of an induced map and right adjointness in the inducing process. Abstracting these features we propose a general definition and then note that there are examples which are rather different from the usual ones but which nonetheless have an intuitively "chaotic" quality.

Recall that a functor

$$\mathcal{X} \xrightarrow{U} \mathcal{Y}$$

between two categories is said to have a right adjoint

$$\mathcal{Y} \xrightarrow{H} \mathcal{X}$$

if there are natural transformations $1_{\mathcal{X}} \xrightarrow{\eta} HU$, $UH \xrightarrow{\varepsilon} 1_{\mathcal{Y}}$ which induce bijections

$$\frac{UX \longrightarrow Y}{X \longrightarrow HY}$$

between the indicated sets of $\mathcal{Y}$-morphisms and $\mathcal{X}$-morphisms,

for each $X$ in $\mathfrak{X}$ and $Y \varepsilon \mathcal{Y}$. An important class of examples may be constructed as follows. Let $\mathcal{Y}$ be a suitable category of topological or differentiable spaces, and let $T$ be a monoid in $\mathcal{Y}$. For example $T$ could be the additive monoid of nonnegative reals or of nonnegative integers. Define $\mathfrak{X} = \mathcal{Y}^T$, the category whose objects are objects of $\mathcal{Y}$ equipped with actions of $T$, and whose morphisms are $T$-equivariant maps $\varphi$:

$$
\begin{tikzcd}
X \times T \arrow[r, "\varphi \times T"] \arrow[d, "\text{action}"'] & Y \times T \arrow[d, "\text{action}"] \\
X \arrow[r, "\varphi"'] & Y
\end{tikzcd}
$$

commutative in $\mathcal{Y}$

We will write the actions of $T$ multiplicatively and on the right. Define $U$ to be the "underlying space" functor which forgets the action

$$
\begin{tikzcd}
\mathfrak{X} \arrow[r, "U"] \arrow[d, equal] & \mathcal{Y} \\
\mathcal{Y}^T \arrow[ur, "\text{forget}"'] &
\end{tikzcd}
$$

As usual we will omit the symbol $U$ from the notation when it causes no confusion. Then $U$ has a right adjoint given by

$$H(Y) = Y^T$$

the internal function space (presumed to exist in $\mathcal{Y}$ ) for each space $Y$ in $\mathcal{Y}$. The adjointness is demonstrated

in essence by the bijection $\dfrac{X \xrightarrow{\bar{\varphi}} Y^T}{X \xrightarrow[\varphi]{} Y}$

between arbitrary maps $\varphi$ and equivariant maps $\bar{\varphi}$ defined by

$$\varphi(x) = \bar{\varphi}(x)(u)$$ where $u$ is the unit element of $T$

$$\bar{\varphi}(x)(t) = \varphi(xt)$$ where $xt$ is the action given on $X$

Here $\bar{\varphi}$ is equivariant if $\varphi$ is given and we have equipped $Y^T$ with the usual action by "translation":

$$(yt)(s) = y(ts) \qquad \text{for all } s \in T,\ y \in Y^T$$

Thus if $X$ is a space equipped with a dynamical action of $T$ and $Y$ is a given space (without action) and if we consider a map $X \xrightarrow{\varphi} Y$ as an "observable" of state, then $\bar{\varphi}(x)$ is the function of time giving the progression of observed values of $\varphi$ if we start in state $x$ at $u =$ time $0$. This $\bar{\varphi}$ is sometimes referred to as "symbolic dynamics", the points of $Y$ being considered as symbols for the blocks into which $\varphi$ divides the state space $X$; then starting in state $x$, $\bar{\varphi}(x)$ is the $T$-sequence of blocks through which the dynamics takes the system.

<u>Definition</u> If $X \varepsilon \mathcal{Y}^T$, $Y \varepsilon \mathcal{Y}$, an observable $X \xrightarrow{\varphi} Y$ is $T$-<u>chaotic</u> iff the induced map $X \xrightarrow{\bar{\varphi}} Y^T$ is an epimorphism. More generally if $\mathcal{X} \xrightarrow{U} \mathcal{Y}$ is a functor with right adjoint $H$ and $UX \xrightarrow[\varphi]{} Y$, $\varphi$ is $U$-<u>chaotic</u> iff $X \xrightarrow{\bar{\varphi}} H(Y)$ is an epimorphism.

That is (for those categories $\mathcal{Y}$ in which epimorphisms are surjective) $X \xrightarrow{\varphi} Y$ is a chaotic observable iff <u>every</u> $T$-sequence of symbols $T \xrightarrow{y} Y$ is realized as $y = \bar{\varphi}(x)$ for at least one state $x \varepsilon \underline{X}$. Refinements of this condition can be formulated by considering certain homomorphisms $T \rightarrow T'$ as periods and requiring the same condition after applying the corresponding change of action functors.

A standard example of a map $\varphi$ often considered in this context is the adjunction

$$\underline{X} \longrightarrow \pi_0(\underline{X}) = Y$$

where $\pi_0$ is the "space of components" functor left adjoint to the inclusion $\mathcal{Y}_0 \hookrightarrow \mathcal{Y}$ of the category of prodiscrete spaces. In case $Y = \pi_0(X)$ is actually finite and $T$ is countable discrete, then $H(Y) = Y^T$ is a Cantor space.

Finally, in accord with loose usage we may say that a system $\bar{\underline{X}}$ in $\mathcal{X} = \mathcal{Y}^T$ is "chaotic" in case there exists an $\mathcal{X}$-subobject $\underline{X} \hookrightarrow \bar{\underline{X}}$ (that is, a $T$-invariant subspace) and a nontrivial space $Y$ and a chaotic $\underline{X} \rightarrow Y$, or perhaps that $\pi_0(\underline{X})$ is nontrivial and $\underline{X} \rightarrow \pi_0 \underline{X}$ chaotic.

The extent to which the above general categorical definition expresses the informal notion of chaos can be illustrated by an example of a dual algebraic nature. Let $\mathcal{Y} = \mathcal{A}$ = the category of all commutative algebras over the reals; for example the function ring $C^\infty(Y)$ is an object of $\mathcal{A}$ for a manifold $Y$. Let $\mathfrak{X} = \mathcal{A}'$ = the category of all such algebras $A$ equipped moreover with derivations $A \circlearrowleft (\ )'$, that is $(\ )'$ is a real-linear map satisfying the Leibniz product rule, where the morphisms of $\mathcal{A}'$ are algebra homomorphisms which moreover commute with the given derivations. Of course any vector field supplies an example of an object in $\mathcal{A}'$. Then the forgetful functor

$$\mathcal{A}' \xrightarrow{U} \mathcal{A}$$

has the right adjoint $H$ which assigns, to any algebra $A$, the algebra

$$H(A) = A[[t]]$$

of all formal (divided) power series with coefficients in $A$, equipped with the obvious formal derivation $\frac{d}{dt}$. The adjointness is verified as follows: If $B$ is equipped with any given derivation and $B \xrightarrow[\varphi]{} A$ is an algebra homomorphism, then

$$B \xrightarrow[\bar{\varphi}]{} A[[t]]$$

is defined by $\bar{\varphi}(f) = \sum \dfrac{\varphi(f^{(n)})}{n!} t^n$ where $f^{(n)}$ is defined by iterating the given derivation on $\mathcal{B}$, and clearly $\bar{\varphi}(f') = \frac{d}{dt}\bar{\varphi}(f)$ as required for $\mathcal{a}'$-morphisms.

Proposition: If $\mathcal{B} = C^{\infty}(\mathfrak{X})$, where $\mathfrak{X}$ is a manifold, is equipped with the derivation induced by any given non-trivial vector field, then $\mathcal{B}$ has a map $\mathcal{B} \xrightarrow{\varphi} \mathbb{R}$ which is chaotic relative to the adjunction $\mathcal{a}' \leftrightarrows \mathcal{a}$.

Proof: Let $X$ be a point where the vector field is non-trivial and let $\varphi(f) = f(X)$ for all $f \in \mathcal{B}$. Then $\bar{\varphi}(f)$ is the Taylor series of $f$. Borel's theorem states that any element of $H(\mathbb{R}) = \mathbb{R}[[t]]$ is the Taylor series at $X$ of some smooth $f$. Thus $\bar{\varphi}$ is surjective, so that $\varphi$ is chaotic relative to $\mathcal{a}' \to \mathcal{a}$, according to our definition.

Some other simple examples of right adjoints which may be of interest in this connection are determined by starting with a homomorphism $T' \longrightarrow T$ of monoids in a special category and considering the induced-action functor

$$\mathcal{S}^T \xrightarrow{\;U\;} \mathcal{S}^{T'}$$

whose right adjoint (generalizing ( ) when =1) is

$$H(Y) = \mathrm{Hom}_{T'}(T, Y)$$

the space of all $T'$-equivariant maps $T \longrightarrow Y$, where $Y$ has some given $T'$-action and $T'$ acts on $T$ via translation and the given $T' \to T$. For example if we consider the inclusion $\mathbb{N} \hookrightarrow \mathbb{R}^+$ of additive monoids, $Y$ is essentially equipped with a single $\mathcal{S}$-endomorphism $\tau$ (the action of $1 \in \mathbb{N}$) and so is a "discrete time dynamical system", and the associated continuous-time dynamical system $H(Y)$ has as its states all $\mathcal{S}$-maps $\mathbb{R}^+ \xrightarrow{f} Y$ for which

$$f(t+1) = f(t)\cdot\tau \quad \text{for all} \quad t \geq 0$$

A morphism $\bar{X} \xrightarrow[\varphi]{} Y$, where $\bar{X} \in \mathcal{S}^{\mathbb{R}^+}$ is now required to satisfy the condition

$$\varphi(X\cdot t) = \varphi(X)\,\tau^n$$

in case $t = n$ is a whole number. [We remark that, as in many examples, our $U$ has also a <u>left</u> adjoint $Y \mapsto \mathbb{R}^+ \times_{\mathbb{N}} Y$, a tensor-like quotient space of $\mathbb{R} \times Y$, which gives a <u>different</u> notion of "continuous-time system associated to a discrete-time system".] Such a morphism $\varphi$ is chaotic (in the present relative sense) if and only if every $f$ of the kind described above are, for some $x \in \bar{X}$ of the form

$$f(t) = \varphi(xt) \quad \text{for all } t.$$

Of course there are standard examples of such $\varphi$, namely the evaluation at $0 : \bar{X} = H(Y) \longrightarrow Y$, but the novelty would be to find $\bar{X}$ constructed by other means.

In the case of a surjective $T' \longrightarrow T$, such as $\mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$ induced by a given period,

$$H(Y) \hookrightarrow Y$$

is simply the subspace of the $T'$-system $Y$ consisting of points having that period, so for $X \xrightarrow[\varphi]{} Y$ to be "chaotic" relative to $T' \longrightarrow T$ simply means that all those points of $Y$ having the period in question are values of $\varphi$.

As a final example, consider the problem of solving differential equations. From the point of view of "Synthetic Differential Geometry" [3] this can be considered as a special case of the special class of adjoints arising from a change of monoid $\langle D \rangle \longrightarrow \mathbb{R}$, but in any case, if $\mathcal{S}$ is a reasonable category of differentiable spaces, we can consider the category $\mathcal{Y}$ whose typical object $Y$ is a space in $\mathcal{S}$ equipped with a vector field, and whose morphisms are smooth maps $Y_1 \to Y_2$ which commute with the designated vector fields. Then if $\mathfrak{X} = \mathcal{S}^{\mathbb{R}}$ is the category of reversible continuous-time flows, there is a functor $\mathfrak{X} \xrightarrow{U} \mathcal{Y}$ given by differentiating each flow at $t = 0$. The right adjoint to $U$ assigns to each such differential equation $Y$ the space $H(Y)$ of all solution curves which are defined for all time. Thus for the adjunction map $H(Y) \to Y$ to be "chaotic" in this context merely means that through each

point of $Y$ there is a solution curve which extends for all time. More generally if $\overline{\underline{X}}$ is a flow and $Y$ a vector field, a compatible map $X \xrightarrow{\varphi} Y$ is "chaotic" relative to the differentiation functor $U$ iff every solution curve for $Y$ is for some $x$ the $\varphi$-image of the flow through $x$.

That the last few examples sound more like normal, reasonable behavior than like pathological chaos is explained in terms of the extent to which the functor $U$, or more particularly the monoid homomorphism $T' \longrightarrow T$, is an isomorphism: In the basic example $1 \to T$, $Y \mapsto H(Y) = Y^T$ is a big jump, so surjectivity of $\bar{\varphi}$ is harder to come by. In the case of $\langle D \rangle \hookrightarrow \mathbb{R}$, the functor $U$ is not quite an equivalence but reasonable $Y$ "think" it is in that $H(Y) \xrightarrow{\sim} Y$ due to the strong existence and uniqueness property; thus for such $Y$, the surjectivity of $\varphi$ itself is sufficient for $\varphi$ to be "chaotic".

[1] Saari, D.G., and Urenko, J.B, "Newton's method, Circle Maps, and Chaotic Motion", American Mathematical Monthly, Jan. 1984

[2] Coppel, W.A., "Maps on an Interval", IMA Preprint Series # 26, June 1983, Institute for Mathematics and its Applications, University of Minnesota.

[3] Kock, A. Synthetic Differential Geometry, LMS Lecture Note Series # 51, Cambridge University Press 1981.