RAY THEORY FOR THE ELASTIC WAVE EQUATION

Three Lectures

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CONTENTS

solution of the ray and dynamic ray equations
3.5 The solution of the transport equation using the results of dynamic ray theory.

4. The elastic wave equation.

4.1 Isotropy.
4.2 P and S waves.
4.3 The transport of the S polarization.
4.4 The anisotropic elastodynamic equation.
4.5 Anisotropic ray theory.

5. Ray theory for a symmetric hyperbolic system.

5.1 Group velocity: various properties.
5.2 ...

RAY THEORY FOR THE 3-D SCALAR WAVE EQUATION

Let $\boldsymbol{x} = (x_1, x_2, x_3)$ be the position vector, x_1, x_2, x_3 cartesian coordinates, and t be time.

When necessary we will regard $\boldsymbol{x}$ and other vectors as columns unless they are explicitly transposed.

We shall consider the scalar wave equation in the form

$$u_{,jj} = \frac{1}{v^2}\ddot{u}\,; \quad \nabla^2 u = \frac{1}{v^2}\ddot{u}\,. \tag{4.1}$$

v is a smooth function of $\boldsymbol{x}$ but independent of t.

Taking a Fourier transform in t leads to

$$u_{,jj} - \frac{(\mathrm{i}\omega)^2}{v^2}\,u = 0, \tag{4.2}$$

where ω is the angular frequency.

THE RAY ANSATZ

Based on comparison with asymptotic expansions of exact solutions where they are available we seek a solution in the form

$$u(\mathbf{x}, \omega) = \sum_{\nu=0}^{\infty} \frac{U_\nu(\mathbf{x})}{(\mathrm{i}\omega)^\nu} e^{\mathrm{i}\omega T(\mathbf{x})}. \tag{5.1}$$

Here $T(\boldsymbol{x})$ is **phase function** and has the interpretation of **travel time**. The series is assumed to be asymptotic for large ω (and not necessarily convergent). That is, we are concerned with what happens as $\omega \to \infty$ rather than what happens as the number of terms $\to \infty$. The remainder after n terms is assumed to be $o\Big((\mathrm{i}\omega)^{-n}\Big)$. Typically ω is large enough that one term is a sufficient approximation.

SIDE CALCULATIONS

We shall substitute this expression into the reduced wave equation and assume that on differentiating term by term the resulting series are asymptotic. For this we shall need

$$u_{,j} = \sum_{\nu=0}^{\infty} \frac{1}{(\mathrm{i}\omega)^{\nu}} (\mathrm{i}\omega U_{\nu} T_{,j} + U_{\nu,j}) e^{\mathrm{i}\omega T(\mathbf{x})}, \tag{6.1}$$

and then

$$
\begin{aligned}
u_{,jj} &= \textstyle\sum_{\nu=0}^{\infty} \frac{1}{(\mathrm{i}\omega)^{\nu}} \Big[(\mathrm{i}\omega)^2 U_\nu T_{,j} T_{,j} + \\
&\qquad \mathrm{i}\omega(2U_{\nu,j}T_{,j} + U_\nu T_{,jj}) + U_{\nu,jj}\Big] e^{\mathrm{i}\omega T(\mathbf{x})} \\
&= \textstyle\sum_{\nu=0}^{\infty} \frac{1}{(\mathrm{i}\omega)^{\nu-2}} \Big[U_\nu T_{,j} T_{,j} + (2U_{\nu-1,j}T_{,j} + \\
&\qquad U_{\nu-1}T_{,jj}) + U_{\nu-2,jj}\Big] e^{\mathrm{i}\omega T(\mathbf{x})}.
\end{aligned}
\tag{7.1}
$$

Here U_{-1} and U_{-2} are interpreted as zero.

So, finally, substituting the ray ansatz into the reduced wave equation we get the

BASIC EQUATION OF RAY THEORY

$$\sum_{\nu=0}^{\infty}\Big[U_\nu(T_{,j}T_{,j}-\frac{1}{v^2})+ \\ +(2U_{\nu-1,j}T_{,j}+U_{\nu-1}T_{,jj})+ \\ +U_{\nu-2,jj}\Big]\frac{1}{(\mathrm{i}\omega)^{\nu-2}} = 0. \tag{8.1}$$

We now **equate to zero the coefficients of the individual powers of** $\mathrm{i}\omega$ **starting with the highest power** $(\mathrm{i}\omega)^2$**.**

$\underline{\nu = 0}$. Of the three terms in brackets in (8.1) only the first survives because $U_{-1} = U_{-2} = 0$. Then, assuming U_0 does not vanish identically, we may cancel it to get the important

Eikonal Equation

$$T_{,j}T_{,j} - \frac{1}{v^2} = 0. \tag{8.2}$$

Continuing to equate coefficients to zero in the basic equation. Assuming that a suitable $T(\boldsymbol{x})$ has been found satisfying the eikonal equation, we proceed for $\nu = 1$.

$$U_1(T_{,j}T_{,j} - \frac{1}{v^2}) + (2U_{0,j}T_{,j} + U_0T_{,jj}) + U_{-1,jj} = 0. \tag{9.1}$$

But the first term vanishes because T satisfies (8.2). Also, $U_{-1} = 0$ by definition, so that (9.1) reduces to the

First Transport Equation.

$$2U_{0,j}T_{,j} + U_0T_{,jj} = 0. \tag{9.2}$$

The first term is a directional derivative of U_0 in the direction of ∇T and so (9.2) is an O.D.E. along a curve, the **ray** everywhere tangent to ∇T. It determines how the leading amplitude U_0 is **transported** along a **ray**.

Notice that in this context the ray is normal to the surfaces of constant phase $T =$ constant. More basically the factor $2T_{,j}$ in (9.2) is the gradient of $T_{,j}T_{,j} - \dfrac{1}{v^2}$ with respect to $T_{,j}$. Continuing for

$\nu > 1$**:**

Writing the equation for $\nu + 1$, and assuming that $U_{\nu-1}$ has already been determined, we get the

Higher Transport Equations.

$$(2U_{\nu,j}T_{,j} + U_\nu T_{,jj}) + U_{\nu-1,jj} = 0. \qquad (10.1)$$

These are linear O.D.E.'s with the same left sides as the first transport equation but with a nonhomogeneous term depending on previously determined amplitudes. **We thus have a scheme to determine the amplitudes U_ν, $\nu = 0, 1, 2 \ldots$ in turn by solving a recursive system of O.D.E.'s.**

Solution of the eikonal and transport equations.
The eikonal equation is

$$T_{,j}T_{,j} - \frac{1}{v^2} = 0. \tag{11.1}$$

Assuming sufficient differentiability, we take the gradient of this equation to get

$$2T_{,ij}T_{,j} + 2\frac{1}{v^3}v_i = 0. \tag{11.2}$$

This equation is an O.D.E. for $T_{,i}$ along the same rays. Let us parameterize points on a ray by arclength s. Then $\boldsymbol{x}(s)$ and ∇T satisfy

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}s} = v\nabla T, \quad \frac{\mathrm{d}}{\mathrm{d}s}\nabla T = -\frac{1}{v^2}\nabla v. \tag{11.3}$$

The factor v ensures that $\mathrm{d}\boldsymbol{x}/\,\mathrm{d}s$ is a unit vector.

THE RAY EQUATIONS

Or, writing $\boldsymbol{p}$ for ∇T, we may rewrite (11.3) & (**??**) as

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}s} = v\,\boldsymbol{p}, \quad \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}s} = -\frac{1}{v^2}\nabla v. \qquad (12.1)$$

These are the **ray equations.**

They can be thought of as a Hamiltonian system with Hamiltonian H given by

$$H = \frac{1}{2}\Big(v(\boldsymbol{x})\boldsymbol{p}\cdot\boldsymbol{p} - \frac{1}{v(\boldsymbol{x})}\Big). \qquad (12.2)$$

Different forms of the these equations, also called ray equations, are obtained by for instance using the unit normal $\boldsymbol{t} = v\boldsymbol{p}$ instead of $\boldsymbol{p}$:

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}s} = \boldsymbol{t}, \quad \frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}s} = -\Big[\nabla \log v - (\boldsymbol{t}\boldsymbol{\cdot}\nabla \log v)\,\boldsymbol{t}\Big], \qquad (12.3)$$

or by using T instead of s as a parameter along the ray. Each form has advantages in certain contexts. In (12.3) the quantity in brackets is the **transverse gradient** of $\log v$. On each ray initial values for $\boldsymbol{x}$ and $\boldsymbol{t}$ (or $\boldsymbol{p}$) are required. These are an **initial point** on the ray and an **initial direction.**

Simultaneously T may be found since

$$\frac{\mathrm{d}T}{\mathrm{d}s} = \frac{\partial T}{\partial \boldsymbol{x}} \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}s} = \boldsymbol{p} \cdot v\boldsymbol{p} = \frac{1}{v}. \tag{13.1}$$

Thus $\mathrm{d}s/\,\mathrm{d}T = v$ reinforcing the interpretation of T as travel time. It may be verified that the Hamiltonian is constant (equal to zero) along rays. If initial values of $\boldsymbol{x}$, T, and $\boldsymbol{p} = \nabla T$ are given on a family of rays they must be compatible with the eikonal equation. If they are compatible initially they will remain compatible.

RAY TUBES.

The first transport equation (9.2)

$$2U_{0,j}T_{,j} + U_0 T_{,jj} = 0 \tag{14.1}$$

when multiplied by U_0 gives rise to a conservation equation

$$(U_0^2 T_{,j})_{,j} = 0. \tag{14.2}$$

I.e.

$$\nabla \cdot \Big(\frac{U_0^2 \boldsymbol{t}}{v}\Big) = 0. \tag{14.3}$$

The vector in parentheses is (the leading term in) the energy flux, and so (14.3) represents conservation of energy. So energy flows along the rays.

AMPLITUDES AND RAYTUBE AREAS.

Consider a narrow tube of rays originating at the boundary of a small patch on a wave front $T = T_0$ and terminating in the the boundary of a patch on the wavefront $T = T_1$. Let us integrate (13.1) over the volume enclosed by this curvilinear cylinder and the patches at its ends. On applying the divergence theorem we find that the cylindrical surfaces contribute nothing because the vector field is tangential to the surface.

Thus the integrals over the ends must cancel each other. Since there the vector field is normal to the surface we find that the contribution at the two ends are approximately $-\sigma U_0^2/v\Big|_0$ evaluated on $T = T_0$ and $\sigma U_0^2/v\Big|_1$ on $T = T_1$ respectively, where σ is the cross-sectional area of the narrow tube of rays. Since these add to zero we find

$$\sigma U_0^2/v \tag{16.1}$$

is conserved along a narrow tube of rays.

Hence

$$U_0 = C\sqrt{\frac{v}{\sigma}}\,, \tag{16.2}$$

where C is a constant, v is the wave speed, and σ is the raytube area.

CAUSTICS.

It is clear from this formula that when the ray tube area becomes small the amplitude increases in inverse proportion. But a ray tube may flatten to zero if neighboring rays cross. This happens if the rays envelope a surface (curve in the plane) and may arise because of an inhomogeneous wave speed or even in a uniform medium if there has been reflection at a concave surface or refraction through a convex lens. But in a uniform medium the wave field is analytic. What is happening?

It turns out that the wavefield, though large, is indeed finite at a caustic, but the ray ansatz breaks down there in the form we have used. The more sophisticated ansatz, developed by Ludwig (1967), should be used.

DYNAMIC RAY THEORY FOR THE 3-D SCALAR WAVE EQUATION

If we try to calculate the amplitude by means of the ray tube area formula we are at first sight faced with taking the differences of the positions on neighboring rays which normally leads to loss of accuracy. This may be avoided if further quantities are carried along in the ray calculation.

Notice that in the transport equations (9.2) and (9.1) we need the Laplacian $\nabla\nabla T = T_{,jj}$ of T. To calculate this we shall first find a transport equation for the Hessian $H = \nabla\nabla T$.

TRANSPORT EQUATION FOR THE HESSIAN $H = \nabla\nabla T$.

We recall (11.2), which we rewrite as

$$\nabla T \cdot \nabla\nabla T + \frac{1}{v^3}\nabla v \ = \ 0. \tag{19.1}$$

Let us take the gradient of this to get

$$\nabla T \cdot \nabla\nabla\nabla T + \nabla\nabla T \cdot \nabla\nabla T = -\frac{1}{v^3}\nabla\nabla v + \frac{3}{v^4}\nabla v\nabla v. \tag{19.2}$$

If we multiply by v and interpret $v\nabla T \cdot \nabla$ as d/ds and write H for $\nabla\nabla T$ we get

$$H' + vH^2 = -\frac{1}{v^2}\nabla\nabla v + \frac{3}{v^3}\nabla v\nabla v, \tag{19.3}$$

which is a transport equation for H, since we shall regard v and its derivatives as immediately available.

‘SIMPLIFICATION’: THE TRANSVERSE COMPONENTS

But H has six independent components and we are really only interested in one scalar $\nabla^2 T$. We cannot derive an equation just for $\nabla^2 T$, but we can reduce the calculation by obtaining a closed system of **transport equations for just the three independent transverse components.** Unfortunately this reduction requires an excursion into the differential geometry of the rays and the definition of certain orthonormal vectors transverse to the ray by means of which we can define what we mean by transverse components.

THE RAY-CENTERED FRAME OF REFERENCE

Although we shall not follow Cerveny in using ray-centered coordinates, to develop dynamic ray tracing, we shall define the orthonormal frame of coordinate vectors used in that system.

Consider a orthonormal right-handed triad $\{\boldsymbol{e}_1(s), \boldsymbol{e}_2(s), \boldsymbol{t}(s)$ attached t the point $\boldsymbol{x}(s)$ of the ray. For definiteness we shall assume that each the three vectors are columns of their cartesian components in the (x_1, x_2, x_3) system. We shall derive transport equations for these vectors so that they have zero component of angular velocity around $\boldsymbol{t}$.

Let

$$E = (\boldsymbol{e}_1 \vdots \boldsymbol{e}_2 \vdots \boldsymbol{t}). \tag{22.1}$$

Then

$$\frac{\mathrm{d}E}{\mathrm{d}s} = E\,\Omega_E \tag{22.2}$$

for some skew matrix Ω_E.

Let $\boldsymbol{a}(s)$ be any vector function of s and suppose

$$\boldsymbol{a}(s) = E(s)\boldsymbol{a}^E(s), \tag{22.3}$$

so that $\boldsymbol{a}^E(s)$ are the components of $\boldsymbol{a}(s)$ relative to the frame E. Then

$$\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}s} = E\frac{\mathrm{d}\boldsymbol{a}^E}{\mathrm{d}s} + E\Omega_E\boldsymbol{a}^E = E\Big(\frac{\mathrm{d}\boldsymbol{a}^E}{\mathrm{d}s} + \Omega_E\boldsymbol{a}^E\Big). \tag{22.4}$$

It is easy to verify that if

$$\Omega_E = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\omega}_E = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}, \tag{23.1}$$

then

$$\Omega_E\, \boldsymbol{a} = \boldsymbol{\omega}_E \times \boldsymbol{a}, \tag{23.2}$$

where $\times$ is the vector product. We may now write (22.4) as

$$\frac{\mathrm{d}\boldsymbol{a}}{\mathrm{d}s} = E\frac{\mathrm{d}\boldsymbol{a}^E}{\mathrm{d}s} + E\Omega_E\boldsymbol{a}^E = E\Big(\frac{\mathrm{d}\boldsymbol{a}^E}{\mathrm{d}s} + \boldsymbol{\omega}_E \times \boldsymbol{a}^E\Big). \tag{23.3}$$

We may think of $\boldsymbol{\omega}_E$ as the components of the angular velocity of the frame E referred to E as basis.

E DOES NOT ROTATE AROUND THE RAY

We define E so that it has no angular velocity component about $\boldsymbol{t}$. It follows that $\omega_3 = 0$. ω_1 and ω_2 may then be easily be found. Using (12.3), from

$$\begin{aligned} \frac{\mathrm{d}\boldsymbol{t}}{\mathrm{d}s} &= \omega_2 \boldsymbol{e}_1 - \omega_1 \boldsymbol{e}_2 \\ &= -(\boldsymbol{e}_1 \cdot \nabla \log v)\boldsymbol{e}_1 - (\boldsymbol{e}_2 \cdot \nabla \log v)\boldsymbol{e}_2 \qquad (24.1) \\ &= -\frac{v_1}{v}\boldsymbol{e}_1 - \frac{v_2}{v}\boldsymbol{e}_2, \end{aligned}$$

where $v_\alpha = \boldsymbol{e}_\alpha \cdot \nabla v$. Thus

$$\Omega_E = \frac{1}{v}\begin{pmatrix} 0 & 0 & -v_1 \\ 0 & 0 & -v_2 \\ v_1 & v_2 & 0 \end{pmatrix}, \qquad (24.2)$$

Then, writing the other columns of $\mathrm{d}E/\mathrm{d}s$ we have

$$\frac{\mathrm{d}\boldsymbol{e}_\alpha}{\mathrm{d}s} = (\boldsymbol{e}_\alpha \cdot \nabla \log v)\boldsymbol{t} = \frac{v_\alpha}{v}\boldsymbol{t}, \quad \alpha = 1, 2. \tag{25.1}$$

LONGITUDINAL COMPONENTS OF H

We repeat yet again

$$\nabla T \cdot \nabla\nabla T + \frac{1}{v^3}\nabla v \;=\; 0. \tag{25.2}$$

On multiplying by v this becomes

$$\boldsymbol{t} \cdot H = -\frac{1}{v^2}\nabla v. \tag{25.3}$$

This already gives the longitudinal components of H in frame E.

Resolving H in the frame E we get

$$H = E\, H_E\, E^T. \tag{25.4}$$

Then, using primes for derivatives with respect to s,

$$H' = E\,(H'_E + \Omega_E\,H + H\Omega_E^T)\,E^T. \tag{26.1}$$

Hence, using (19.3),

$$\begin{aligned} H'_E + \Omega_E\,H + H\Omega_E^T &= E^T\,H'\,E \\ &= -vH_E^2 - \frac{1}{v^2}E^T\nabla\nabla vE + \frac{3}{v^3}E^T\nabla v\nabla vE, \end{aligned} \tag{26.2}$$

PARTITIONING THE MATRICES

Let us partition these 3×3 matrices into the upper left 2×2 block (the transverse components) and the border. We set

$$H_E = \begin{pmatrix} H_{\boldsymbol{ee}} & -\dfrac{v_{\boldsymbol{e}}}{v^2} \\ -\dfrac{v_{\boldsymbol{e}}^T}{v^2} & -\dfrac{v'}{v^2} \end{pmatrix}, \tag{26.3}$$

where we have used (25.3).

$$E^T \nabla\nabla v E = \begin{pmatrix} V_{\boldsymbol{ee}} & V_{\boldsymbol{et}} \\ V_{\boldsymbol{te}} & V_{\boldsymbol{tt}} \end{pmatrix},$$

$$E^T \nabla v = \begin{pmatrix} v_{\boldsymbol{e}} \\ v' \end{pmatrix} \tag{27.1}$$

$$E^T \nabla v \nabla v^T E = \begin{pmatrix} v_{\boldsymbol{e}} v_{\boldsymbol{e}}^T & v_{\boldsymbol{e}} v' \\ v' v_{\boldsymbol{e}}^T & v' v' \end{pmatrix}$$

$$\Omega_E = \begin{pmatrix} \mathbf{0} & -\dfrac{v_{\boldsymbol{e}}}{v} \\ \dfrac{v_{\boldsymbol{e}}^T}{v} & 0 \end{pmatrix}. \tag{27.2}$$

BASIC EQUATION OF DYNAMIC RAY TRACING

Notice that $H_{\boldsymbol{ee}}$ is the only block that is not explicitly available in terms of v. We now substitute the partitioned expressions into (26.2) and take the leading 2×2 block. This is trivial except for the products $\Omega_E H$, $H\Omega_E^T$, and vH_E^2. Using the notation $[A]_{2\times2}$ to represent this leading block of a matrix A, we have

$$[\Omega_E H]_{2\times2} = [H\Omega_E^T]_{2\times2} = \frac{v_{\boldsymbol{e}} v_{\boldsymbol{e}}^T}{v^3}, \tag{28.1}$$

$$[vH_E^2]_{2\times2} = H_{\boldsymbol{ee}}^2 + \frac{v_{\boldsymbol{e}} v_{\boldsymbol{e}}^T}{v^3}, \tag{28.2}$$

It is very covenient that the three terms $v_{\boldsymbol{e}} v_{\boldsymbol{e}}^T / v^3$ cancel the term $[3E^T \nabla v \nabla v E / v^3] = 3 v_{\boldsymbol{e}} v_{\boldsymbol{e}}^T / v^3$ in (26.2) to leave the remarkably simple

DYNAMIC RAY EQUATION.

$$H'_{\boldsymbol{ee}} + v H^2_{\boldsymbol{ee}} + \frac{V_{\boldsymbol{ee}}}{v^2} = 0, \tag{29.1}$$

where

$$V_{\boldsymbol{ee}} = [E^T \nabla\nabla v E]_{2\times 2}. \tag{29.2}$$

$V_{\boldsymbol{ee}}$ consists of the components of $\nabla\nabla v$ in the directions $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$.

SOME SIMPLE SPECIAL CASES

We repeat

$$H'_{\boldsymbol{ee}} + vH^2_{\boldsymbol{ee}} + \frac{V_{\boldsymbol{ee}}}{v^2} = 0. \qquad (30.1)$$

In order to calculate $V_{\boldsymbol{ee}}$ we need to know $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ as well as the second derivatives of v. In some special cases this either is not necessary or poses no difficulty.

- If $v(\boldsymbol{x})$ is linear in $\boldsymbol{x}$ then $\nabla\nabla v$ is zero.

- If the initial point and initial direction of the ray lie in a plane of symmetry then the ray lies wholly in this plane. Then one special solution for $\boldsymbol{e}_1$ is the constant unit vector perpendicular to that plane. The general solution is any vector perpendicular to $\boldsymbol{t}$ and making a fixed angle with the special constant solution.

- If the medium is plane- or spherically-stratified the above symmetry case applies.

- If the principal normal $\boldsymbol{n}$, binormal $\boldsymbol{b}$, and torsion τ of the ray are known as a functions of s, the angle ϕ that $\boldsymbol{e}_1$ makes with $\boldsymbol{n}$ (towards $\boldsymbol{b}$) satisfies $\mathrm{d}\phi/\,\mathrm{d}s = -\tau$.

THE SOLUTION OF THE LEADING TRANSPORT EQUATION

The leading transport equation is

$$2U_{0,j}T_{,j} + U_0 T_{,jj} = 0. \tag{32.1}$$

Thus, to calculate U_0 we need $T_{,jj} = \nabla^2 T$. This is the trace of $H = \nabla\nabla T$. The trace is invariant under rotations of coordinate system and so we have

$$\nabla^2 T = \mathrm{tr}\{H\} = \mathrm{tr}\{H\boldsymbol{ee}\} - \frac{v'}{v^2}. \tag{32.2}$$

Multiplying (32.1) by v and dividing by 2 we have

$$U_0' + \tfrac{1}{2}(v\,\mathrm{tr}\{H\boldsymbol{ee}\} - \frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}s})U_0 = 0, \tag{32.3}$$

which may be solved by quadratures to get

$$U_0(s) = U_0(s_0)\sqrt{\frac{v}{v_0}}\,\exp\Big[-\tfrac{1}{2}\int_{s_0}^{s} v\,\mathrm{tr}\{H_{\boldsymbol{ee}}\}\,\mathrm{d}s\Big]. \quad (33.1)$$

The quantity $\frac{1}{2}\,v\,H_{\boldsymbol{ee}}$ is the curvature tensor of the wave front $T =$ constant.

THE FULL SYSTEM OF DYNAMIC RAY TRACING EQUATIONS

$$
\begin{aligned}
T' &= \frac{1}{v}, \\
\boldsymbol{x}' &= \boldsymbol{t}, \\
\boldsymbol{t}' &= -\Big[\nabla \log v - (\boldsymbol{t} \cdot \nabla \log v)\, \boldsymbol{t}\Big], \\
\boldsymbol{e}' &= \boldsymbol{t}\, [\boldsymbol{e} \cdot \nabla(\log v)], \\
H'_{\boldsymbol{ee}} &= -vH^2_{\boldsymbol{ee}} - \frac{V_{\boldsymbol{ee}}}{v^2}, \\
U'_0 &= -\tfrac{1}{2}(v\, \mathrm{tr}\{H_{\boldsymbol{ee}}\} - \frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}s})U_0 = 0.
\end{aligned}
\tag{34.1}
$$

Here $\boldsymbol{e}$ stands for the matrix $(\boldsymbol{e}_1 \vdots \boldsymbol{e}_2)$. (Actually, when one of $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ is known the other may be obtained by completing the orthonornal triple with $\boldsymbol{t}$.)

In special cases the equations for $\boldsymbol{e}$ do not need to be solved and $\boldsymbol{e}$ can be found more directly, in keeping with earlier remarks. $\boldsymbol{e}$ is needed to calculate $V_{\boldsymbol{ee}}$.

THE ELASTIC WAVE EQUATION

The new factor introduced by the elastic wave equation is that **the wavefield is a vector.** Also anisotropy arises more naturally.

To set it up briefly let $\boldsymbol{u}(\boldsymbol{x}, t)$ be the displacement from static equilibrium, assumed small, of the material particle at position $\boldsymbol{x}$ and time t. Then the strain tensor, which describes the local small change in shape and size from static equilibrium, is the symmetric part of the displacement gradient

$$\boldsymbol{\epsilon} = \tfrac{1}{2}[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T]. \tag{36.1}$$

The skew part of $\nabla \boldsymbol{u}$ represents pure rotation and does change the state of stress apart from an infinitesimal rotation which does not enter the linear theory.

The **momentum equation** is

$$\rho\,\ddot{\boldsymbol{u}} = \nabla\cdot\boldsymbol{\tau} + \boldsymbol{F}, \tag{37.1}$$

where $\boldsymbol{F}$ is the body-force density and $\boldsymbol{\tau}$ is the stress tensor.

The constitutive relation

$\boldsymbol{\tau}$ is linearly related to $\boldsymbol{\epsilon}$ by the constitutive law

$$\boldsymbol{\tau} = \boldsymbol{\mathcal{C}}\boldsymbol{:}\boldsymbol{\epsilon}. \tag{37.2}$$

Here $\boldsymbol{\mathcal{C}}$ is the fourth-rank **stiffness tensor**, which in general depends upon position. Because $\boldsymbol{\tau}$ does not depend on the skew part of $\nabla\boldsymbol{u}$ it is assumed without loss of generality to be symmetric in its last two subscripts and we may write

$$\boldsymbol{\tau} = \tfrac{1}{2}\boldsymbol{\mathcal{C}}\boldsymbol{:}[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T] = \boldsymbol{\mathcal{C}}\boldsymbol{:}\nabla\boldsymbol{u}. \tag{37.3}$$

It is more explicit, though less attractive to look at, if we write these equations in subscript notation.

$$\rho\,\ddot{u}_i = \tau_{ij,j}, \;\; \tau_{ij} = c_{ijkl}u_{kl}. \tag{38.1}$$

We note the symmetries following symmetries of $\mathcal{C}$

$$c_{jikl} = c_{ijlk} = c_{klij} = c_{ijkl}, \tag{38.2}$$

and the fact that it is positive in the sense that

$$c_{ijkl}e_{ij}e_{kl} > 0 \tag{38.3}$$

for any non-zero symmetric second rank tensor e_{ij}.

From (38.1) we get **the elastic wave equation.**

$$\rho\,\ddot{u}_i = (c_{ijkl}u_{kl})_{,j}, \tag{38.4}$$

where we have taken $\boldsymbol{F}$ to be zero. Equation (38.3) implies that (38.4) is (symmetric) hyperbolic.

GENERALIZED PROGRESSING WAVES

Since in seismology we are concerned with transient signals and relatively short pulses we will stay in the time domain. Instead of a power series in inverse powers of iω we shall consider a suite of functions $f^{(\nu)}$ (ν an integer) satisfying

$$\frac{\partial f^{(\nu)}}{\partial t} = f^{(\nu-1)}. \tag{39.1}$$

so that each one is one (integration) step smoother than the one before. It is worth noting that $(\mathrm{i}\omega)^{-\nu}\mathrm{e}^{\mathrm{i}\omega t}$ satisfy this relationship.

We write **the time-dependent form of the ray ansatz** as

$$\boldsymbol{u}(\mathrm{x},t) = \sum_{\nu=0}^{\infty} \boldsymbol{U}^{(\nu)}(\mathrm{x}) f^{(\nu)}[t - T(\mathrm{x})]. \tag{40.1}$$

$\boldsymbol{U}^{(\nu)}$ are the **polarization vectors**, $\boldsymbol{U}^{(\nu)} = 0$ for $\nu < 0$.

T is the **travel time**, and

$f^{(0)}$ is the **pulse shape**.

The surfaces $T(\boldsymbol{x}) = t$ are the (moving) **wavefronts.**

This time-dependent form is referred to as a **generalized progressing wave**.

THE USUAL SUBSTITUTION

We rewrite (38.4) as

$$(c_{ijkl}u_{k,l})_{,j} - \rho\ddot{u}_i = 0, \qquad (41.1)$$

and then substitute (40.1).

Consider

$$\begin{aligned}
&\left\{c_{ijkl}[U_k^{(\nu)}f^{(\nu)}(t-T)]_{,l}\right\}_{,j} \\
&= \left\{c_{ijkl}[-T_{,l}U_k^{(\nu)}f^{(\nu-1)} + U_{k,l}^{(\nu)}f^{(\nu)}]_{,l}\right\}_{,j} \\
&= c_{ijkl}T_{,j}T_{,l}U_k^{(\nu)}f^{(\nu-2)} - (c_{ijkl}T_{,l}U_k^{(\nu)})_{,j}f^{(\nu-1)} \\
&\quad -c_{ijkl}T_{,j}U_{k,l}^{(\nu)}f^{(\nu-1)} + c_{ijkl}U_{k,jl}^{(\nu)}f^{(\nu)} = 0,
\end{aligned} \qquad (41.2)$$

Then the full substitution is

$$\sum_{\nu=0}^{\infty}(c_{ijkl}T_{,j}T_{,l}U_k^{(\nu)}-\rho U_i^{(\nu)})f^{(\nu-2)}-(c_{ijkl}T_{,l}U_k^{(\nu)})_{,j}f^{(\nu-1)}$$

$$+c_{ijkl}T_{,j}U_{k,l}^{(\nu)}f^{(\nu-1)}+c_{ijkl}U_{k,jl}^{(\nu)}f^{(\nu)}$$

$$=$$

$$\sum_{\nu=0}^{\infty}\Big[(c_{ijkl}T_{,j}T_{,l}U_k^{(\nu)}-\rho U_i^{(\nu)})-(c_{ijkl}T_{,l}U_k^{(\nu-1)})_{,j}$$

$$-c_{ijkl}T_{,j}U_{k,l}^{(\nu-1)}+c_{ijkl}U_{k,jl}^{(\nu-2)}\Big]f^{(\nu-2)}=0. \tag{42.1}$$

THE EIKONAL EQUATION

Equating the coefficient of the most singular term to zero we get

$$\left(c_{ijkl}T_{,l}T_{,j} - \rho\delta_{ik}\right) U_k^{(0)} = 0. \qquad (43.1)$$

It makes no sense for $U_k^{(0)}$ to be zero and so the symmetric matrix $\boldsymbol{Q}$ with entries

$$Q_{ik}(\nabla T) = c_{ijkl}T_{,l}T_{,j} \qquad (43.2)$$

has ρ as an eigenvalue and $\boldsymbol{U}^{(0)}$ as corresponding eigenvector.

THE SLOWNESS SURFACE Let us write $\boldsymbol{p} = \nabla T$. Then (43.1) may be rewritten

$$(\boldsymbol{Q}(\boldsymbol{p}) - \rho I)\boldsymbol{U}^{(0)} = 0. \tag{44.1}$$

To understand this better replace $\boldsymbol{p}$ by a unit vector $\boldsymbol{n}$ and consider the eigenvector problem

$$(\boldsymbol{Q}(\boldsymbol{n}) - \rho v^2(\boldsymbol{n})I)\boldsymbol{R} = 0. \tag{44.2}$$

For fixed $\boldsymbol{n}$, because of the positivity of $\boldsymbol{\mathcal{C}}$ there will in general be three real values of $v(\boldsymbol{n})$, $v = v^{(1)}$, $v = v^{(2)}$, $v = v^{(3)}$, for which there exist non-zero vectors $\boldsymbol{R}^{(1)}$, $\boldsymbol{R}^{(2)}$, $\boldsymbol{R}^{(3)}$. If define $\boldsymbol{p} = \boldsymbol{n}/v^{(N)}$ then, because $\boldsymbol{Q}(\boldsymbol{p})$ is homogeneous of second degree in $\boldsymbol{p}$, we see that (44.1) is satisfied with $\boldsymbol{U}^{(0)}$ a multiple of $\boldsymbol{R}^{(N)}$. Thus, for each direction $\boldsymbol{n}$ there are three vectors $\boldsymbol{p}$ having that direction, and for which $\boldsymbol{Q}(\boldsymbol{p})$ has eigenvalue ρ.

THE SLOWNESS SURFACE (CONT.)

Because the eigenvalues depend continuously on the matrix we see that the three vectors $\boldsymbol{p}^{(N)}(\boldsymbol{n})$ describe three closed ovals as $\boldsymbol{n}$ ranges over the unit sphere. The union of these three sheets is called **the slowness surface**. It is a sextic surface with equation

$$\det\{Q(\boldsymbol{p}) - \rho I\} = 0. \tag{45.1}$$

Equation (43.1) tells us that ∇T lies on the slowness surface. We shall be concerned with individual sheets of this surface. Suppose the medium is uniform and we substitute the plane wave

$$\boldsymbol{u} = \boldsymbol{U}\, g(vt - \boldsymbol{n} \cdot \boldsymbol{x}) \tag{45.2}$$

into the wave equation we will see that $v(\boldsymbol{n})$ is its wave speed. A sheet of the slowness surface then has the equation $v(\boldsymbol{p}) = 1$ and v is homogeneous of degree 1.

SLOWNESS

Why **slowness?**

Consider a moving point $\boldsymbol{x}(t)$ which stays on a wavefront $T(\boldsymbol{x}) = t$. By differentiation we see that

$$\nabla T \cdot \dot{\boldsymbol{x}} = \boldsymbol{p} \cdot \dot{\boldsymbol{x}} = 1. \tag{46.1}$$

Suppose that $\dot{\boldsymbol{x}}$ is parallel to $\boldsymbol{p} = \nabla T$. Then

$$\boldsymbol{p} \cdot \dot{\boldsymbol{x}} = |\dot{\boldsymbol{x}}|\,|\boldsymbol{p}| = 1, \tag{46.2}$$

and so we see that **the magnitude of $|p|$ is the reciprocal of the speed of the wavefront normal to itself.**

The vector $\boldsymbol{p}$ captures the direction of the wavenormal and the normal **slowness.** It is the **slowness vector.** For the elastic wave equation to be satisfied $\boldsymbol{p} = \nabla T$ must lie on the slowness surface.

AN IDENTITY DUE TO COURANT

For arbitrary 3-vector $\boldsymbol{p}$ let $\lambda(\boldsymbol{p})$ be an eigenvalue of $Q(\boldsymbol{p})$, i.e.

$$(\boldsymbol{Q}(\boldsymbol{p}) - \lambda(\boldsymbol{p})I)\boldsymbol{e}_\beta(\boldsymbol{p}) = 0 \tag{47.1}$$

for some non-zero vector $\boldsymbol{e}_\beta$. Then (43.1) implies that

$$\lambda(\boldsymbol{p}) = \rho. \tag{47.2}$$

For each $\boldsymbol{p}$ there are three eigenvalues $\lambda(\boldsymbol{p})$ and correspondingly three orthogonal eigenvectors. Equation (45.2) defines a three-sheeted surface in $\boldsymbol{p}$-space and it may be shown that each real branch of $\lambda(\boldsymbol{p})$ defines a closed surface surrounding the origin.

AN IDENTITY DUE TO COURANT (CONT.)

Let us now differentiate (45.1) i.e.

$$(\boldsymbol{Q}(\boldsymbol{p}) - \lambda(\boldsymbol{p})I)\boldsymbol{e}_\beta(\boldsymbol{p}) = 0, \tag{48.1}$$

with respect to $\boldsymbol{p}$, denoting this $\boldsymbol{p}$-gradient by $_{,\boldsymbol{p}}$.

$$(\boldsymbol{Q}_{,\boldsymbol{p}} - \lambda_{,\boldsymbol{p}}I)\boldsymbol{e}_\beta + (\boldsymbol{Q} - \lambda I)\boldsymbol{e}_{\beta,\boldsymbol{p}} = 0. \tag{48.2}$$

Let $\boldsymbol{e}_\alpha^T$ be any left eigenvector belonging to λ. Then on left multiplying (48.2) by $\boldsymbol{e}_\alpha^T$ we get

$$\boldsymbol{e}_\alpha^T(\boldsymbol{Q}_{,\boldsymbol{p}} - \lambda_{,\boldsymbol{p}}I)\boldsymbol{e}_\beta = 0. \tag{48.3}$$

That is

$$\boldsymbol{e}_\alpha^T\boldsymbol{Q}_{,\boldsymbol{p}}\boldsymbol{e}_\beta = \lambda_{,\boldsymbol{p}}\boldsymbol{e}_\alpha^T\boldsymbol{e}_\beta = 0, \tag{48.4}$$

an identity ascribed to Courant by Ludwig (1960). (The subscripts α and β allow for a multidimensional null space.)

THE LEADING TRANSPORT EQUATION

Equating to zero the coefficient of $f^{(-1)}$ in (42.1) we find that

$$\begin{aligned}(c_{ijkl}T_{,j}T_{,l}U_k^{(1)} - \rho U_i^{(1)}) - (c_{ijkl}T_{,l}U_k^{(0)})_{,j} \\ + c_{ijkl}T_{,j}U_{k,l}^{(0)} = 0,\end{aligned} \tag{49.1}$$

The first term does not vanish even though T satisfies the eikonal equation. But we can get rid of it by contracting with $U_i^{(0)}$. For the moment suppose for simplicity that the eigenvalue is simple and so the eigenspace is one-dimensional. The term in $U_i^{(1)}$ is anihilated since, by the symmetry of $Q(\boldsymbol{p})$, $\boldsymbol{U}^{(0)T}$ is a left eigenvector.

THE TRANSPORT EQUATION (CONT.)

We are then left with terms which combine to form

$$\left(c_{ijkl}U_i^{(0)}U_k^{(0)}T_{,l}\right)_{,j} = 0. \qquad (50.1)$$

where we have used $c_{ijkl}U_{i,j}^{(0)}U_k^{(0)}T_{,l} = c_{ijkl}U_i^{(0)}U_{k,l}^{(0)}T_{,j}$, which follows from the symmetry $c_{ijkl} = c_{klij}$.

Let us simplify the expression $c_{ijkl}U_i^{(0)}U_k^{(0)}T_{,l}$ with the use of Courant's identity.

Taking both $\boldsymbol{e}_\alpha$ and $\boldsymbol{e}_\beta$ equal to $\boldsymbol{U}^{(0)}$ in (48.4) we have

$$
\begin{aligned}
c_{ijkl}U_i^{(0)}U_k^{(0)}T_{,l} &= \tfrac{1}{2}\boldsymbol{U}^{(0)T}\boldsymbol{Q}_{,p_j}\boldsymbol{U}^{(0)} \\
&= \tfrac{1}{2}\lambda_{,p_j}|\boldsymbol{U}^{(0)}|^2 = \rho v_{,p_j}|\boldsymbol{U}^{(0)}|^2,
\end{aligned}
\tag{51.1}
$$

Where we have used $v = 1$. To see this in more detail

$$
\begin{aligned}
(\boldsymbol{Q}_{,p_q})_{ik} &= \frac{\partial}{\partial p_q} c_{ijkl}p_jp_l \\
&= c_{ijkl}\delta_{jq}p_l + c_{ijkl}p_j\delta_{lq}.
\end{aligned}
\tag{51.2}
$$

Hence

$$
\begin{aligned}
\boldsymbol{U}^{(0)T}\boldsymbol{Q}_{,p_q}\boldsymbol{U}^{(0)} &= c_{ijkl}(\delta_{jq}p_l + p_j\delta_{lq})U_i^{(0)}U_k^{(0)} \\
&= (c_{iqkl}p_l + c_{ijkq}p_j)U_i^{(0)}U_k^{(0)} \\
&= 2\,c_{iqkl}\,p_l U_i^{(0)}U_k^{(0)}
\end{aligned}
\tag{52.1}
$$

by the symmetry $c_{ijkl} = c_{klij}$.

So the transport equation (50.1) becomes

$$
\left\{\rho v_{,p_j}|\boldsymbol{U}^{(0)}|^2\right\}_{,j} = 0. \tag{52.2}
$$

We can now use the raytube-area argument to calculate the amplitudes. The vector $\rho v_{,p_j}|\boldsymbol{U}^{(0)}|^2$ is **the energy flux vector. It is normal to the slowness surface $v(\boldsymbol{p}) = 1$.** By Euler's theorem $\boldsymbol{p}\cdot v_{,\boldsymbol{p}} = v = 1$. Hence $v_{,\boldsymbol{p}}$ **is the ray velocity.**

ISOTROPY

The elastic material is isotropic if the fourth rank tensor $\boldsymbol{\mathcal{C}}$ is isotropic. The general form of an isotropic fourth rank tensor is

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk}. \tag{53.1}$$

Because of the symmetries of $\boldsymbol{\mathcal{C}}$, $\mu = \nu$, and we write

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \tag{53.2}$$

Like $\boldsymbol{\mathcal{C}}$, λ and μ are functions of position $\boldsymbol{x}$. They are the Lamé constants. μ is the rigidity or **shear modulus.** λ does not have an independent name but $\kappa = \lambda + \frac{2}{3}\mu$ is the bulk modulus.

P AND S WAVES Then

$$Q_{ik}(\boldsymbol{p}) = c_{ijkl}p_jp_k = \lambda p_ip_k + \mu(\delta_{ik}p_jp_j + p_ip_k) = (\lambda+\mu)p_ip_k + \mu|\boldsymbol{p}|^2\delta_{ik}. \tag{54.1}$$

Then the eikonal equation becomes

$$(\mu|\boldsymbol{p}|^2 - \rho)\boldsymbol{U}^{(0)} + (\lambda+\mu)\boldsymbol{p}\cdot\boldsymbol{U}^{(0)}\boldsymbol{p} = 0. \tag{54.2}$$

It is obvious that **neither term vanishes** or **both terms vanish simultaneously.**

We shall consider these two alternatives.

P AND S WAVES (CONT.)

If both terms are non-zero, $\boldsymbol{U}^{(0)}$ is parallel to $\boldsymbol{p}$ and

$$|\boldsymbol{p}|^2 = \frac{\rho}{\lambda + 2\mu}. \tag{55.1}$$

This describes a longitudinal wave called a P **wave.**

If both terms vanish, $\boldsymbol{U}^{(0)}$ is perpendicular to $\boldsymbol{p}$ and

$$|\boldsymbol{p}|^2 = \frac{\rho}{\mu}. \tag{55.2}$$

This is an S wave. P and S stand for "primary" and "secondary", the P wave traveling faster.

THE ISOTROPIC SLOWNESS SURFACE If we choose axes so that the x_1 axis is parallel to $\boldsymbol{p}$, so that $\boldsymbol{p} = (|\boldsymbol{p}|, 0, 0)$, we find that the determinant of $\boldsymbol{Q}(\boldsymbol{p})$ becomes

$$\begin{vmatrix} \frac{\lambda+2\mu}{\rho}|\boldsymbol{p}|^2 - 1 & 0 & 0 \\ 0 & \frac{\mu}{\rho}|\boldsymbol{p}|^2 - 1 & 0 \\ 0 & 0 & \frac{\mu}{\rho}|\boldsymbol{p}|^2 - 1 \end{vmatrix} = 0, \qquad (56.1)$$

or

$$\left(\frac{\lambda + 2\mu}{\rho}|\boldsymbol{p}|^2 - 1\right)\left(\frac{\mu}{\rho}|\boldsymbol{p}|^2 - 1\right)^2 = 0. \qquad (56.2)$$

THE ISOTROPIC SLOWNESS SURFACE (CONT.)

So, the slowness surface consists of

- **a sphere of radius $\sqrt{\rho/(\lambda+2\mu)}$ corresponding to the P wave with speed $v_P = \sqrt{(\lambda+2\mu)/\rho}$, and**

- **a repeated sphere of radius $\sqrt{\rho/\mu}$ corresponding to the S wave with speed $v_S = \sqrt{\mu/\rho}$.**

P WAVES

P waves are longitudinal (polarization parallel to $\boldsymbol{p}$). The function $v(\boldsymbol{p})$ for the P wave is given by

$$v(\boldsymbol{p}) = \sqrt{\frac{\lambda + 2\mu}{\rho}}\boldsymbol{p}, \tag{58.1}$$

and $v_{,\boldsymbol{p}} = v_P \boldsymbol{t} = v_P^2 \boldsymbol{p}$ and the transport equation for P waves is $\nabla \cdot (\rho v_P |\boldsymbol{U}^{(0)}|^2 \boldsymbol{t}) = 0$, which can be solved as for the scalar wave equation.

S WAVES

S **waves are transverse** (the polarization is perpendicular to $\boldsymbol{p}$, $\boldsymbol{p} \cdot \boldsymbol{U}^{(0)} = 0$).

We repeat the leading transport equation (49.1)

$$
\begin{aligned}
&(c_{ijkl} p_j p_l U_k^{(1)} - \rho U_i^{(1)}) - (c_{ijkl} p_l U_k^{(0)})_{,j} \\
&\qquad + c_{ijkl} p_j U_{k,l}^{(0)} = 0,
\end{aligned}
\tag{59.1}
$$

and substitute the expression (53.2) for c_{ijkl}.

$$
\begin{aligned}
&(\rho - \mu p_j p_j) U_i^{(1)} - (\lambda + \mu) p_j U_j^{(1)} p_i \\
&+ \left(\lambda U_l^{(0)} p_l\right)_{,i} + \left(\mu U_i^{(0)} p_j\right)_{,j} + \left(\mu U_j^{(0)} p_i\right)_{,j} \\
&+ \lambda U_{l,l}^{(0)} p_i + \mu U_{i,j}^{(0)} p_j + \mu U_{j,i}^{(0)} p_j = 0.
\end{aligned}
\tag{59.2}
$$

S WAVES (CONTD.)

The first term is zero because T satisfies the eikonal equation (55.2). The third term is zero because it involves the scalar product of $\boldsymbol{U}^{(0)}$ and $\boldsymbol{p}$, which are perpendicular to each other. The second and sixth are in the direction of $\boldsymbol{p}$. Let us examine the others. The fourth and seventh combine to give

$$2\mu p_j U_{i,j}^{(0)} + (\mu p_j)_{,j} U_i^{(0)}. \tag{60.1}$$

The fifth and eighth give

$$\begin{aligned} &\mu p_{i,j} U_j^{(0)} + (\mu U_j^{(0)})_{,j} p_i + \mu p_j U_{j,i}^{(0)} \\ &= \mu (p_j U_j^{(0)})_{,i} + (\mu U_j^{(0)})_{,j} p_i = (\mu U_j^{(0)})_{,j} p_i, \end{aligned} \tag{60.2}$$

where we have again used $\mathrm{U}^{(0)} \cdot \boldsymbol{p} = 0$.

S WAVES (CONTD.)

Notice that the terms are of three types: vectors parallel to ∇T, vectors parallel to $\boldsymbol{U}^{(0)}$, and the directional derivative $\boldsymbol{p}\cdot\nabla\boldsymbol{U}^{(0)}$ of $\boldsymbol{U}^{(0)}$ in the direction of $\boldsymbol{p}$, i.e. $p_j U_{i,j}$. This implies that the directional derivative of $\boldsymbol{U}^{(0)}$ in the direction of $\boldsymbol{p}$ is a linear combination of $\boldsymbol{p}$ and $\boldsymbol{U}^{(0)}$ and so has no component in the direction of $\boldsymbol{p}\times\boldsymbol{U}^{(0)}$. So it has **no component of angular velocity around $\boldsymbol{p}$** as the field point advances along the ray. Thus the polarization is transported along the ray just like one of the vectors $\mathbf{e}_1$, $\mathbf{e}_2$ of (34.1).

S WAVES (CONTD.)

Let us denote the unit polarization vector of S by $\mathbf{e}$, so that $\boldsymbol{U}^{(0)} = U^{(0)}\boldsymbol{e}$. Then,

$$\frac{\mathrm{d}\mathbf{e}}{\mathrm{d}s} = (\mathbf{e} \cdot \nabla \log v_S)\, \mathbf{t}. \qquad (62.1)$$

and the transport equation for the $U^{(0)}$ is

$$\nabla \cdot [\rho v_S (U^{(0)})^2 \boldsymbol{t}] = 0. \qquad (62.2)$$

Of course the rays must be computed using the appropriate velocity field $v = v_S(\boldsymbol{x})$.

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