

A PROBABILISTIC REPRESENTATION FOR THE VORTICITY OF A 3D VISCOUS FLUID AND FOR GENERAL SYSTEMS OF PARABOLIC EQUATIONS

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ABSTRACT. A probabilistic representation formula for general systems of linear parabolic equations, coupled only through the zero-order term, is given. On this basis, an implicit probabilistic representation for the vorticity in a 3D viscous fluid (described by the Navier-Stokes equations) is carefully analysed, and a theorem of local existence and uniqueness is proved.

1. INTRODUCTION

Consider the Navier-Stokes equation in $[0, T] \times \mathbf{R}^3$

$$(1.1) \quad \begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p = \nu \Delta u + f \\ \operatorname{div} u = 0 \\ u(0, x) = u_0(x) \end{cases}$$

This equation describes, in *Eulerian coordinates*, the evolution of a viscous incompressible Newtonian fluid, where u is the velocity field, p the pressure, f the body force and $\nu > 0$ the kinematic viscosity. The vorticity field $\xi = \operatorname{curl} u$ satisfies the equation

$$(1.2) \quad \partial_t \xi + (u \cdot \nabla)\xi = \nu \Delta \xi + (\xi \cdot \nabla)u + g$$

with $g = \operatorname{curl} f$. As we shall remark later on, the *stretching* term $(\xi \cdot \nabla)u$ can be written in the form

$$(\xi \cdot \nabla)u = \mathcal{D}_u \xi,$$

where $\mathcal{D}_u = \frac{1}{2}(\nabla u + \nabla u^T)$, which better describes the action of the deformation tensor \mathcal{D}_u on ξ . The analysis of the vorticity field is a fundamental issue related to questions like the possible emergence of singularities (see for instance Beale, Kato and Majda [3], Constantin [8]), or the description of 3D structures (see for instance Chorin [7]).

The *Lagrangian* formulation of the fluid dynamics may be important to analyse the vorticity field. Strictly speaking, the fluid particles (we mean infinitesimal portions of fluid, not the single molecules) move according to the deterministic law

$$\dot{X}(t) = u(t, X(t)).$$

However, a *virtual* Lagrangian dynamic of the particles of the form

$$(1.3) \quad dX(t) = u(t, X(t)) dt + \sqrt{2\nu} dW_t$$

(where W_t is an auxiliary 3D Brownian motion) allows us to describe the evolution of quantities which are not only transported by the fluid, but have a diffusive character. The vorticity has this

2000 *Mathematics Subject Classification.* Primary 76D05; Secondary 35A20.

Key words and phrases. Navier-Stokes equations, Feynman-Kac formula.

property, as many scalars or fields possibly spreading into the fluid. Roughly speaking, we prove the representation formula

$$\xi(t, x) = \mathbf{E}[V(t, 0)\xi_0(X(0))] + \int_0^t \mathbf{E}[V(t, s)g(s, X(s))] ds$$

where $\mathbf{E}[\cdot]$ denotes the mean value with respect to the Wiener measure, ξ_0 is the vorticity at time zero, $X(s)$ is the solution of equation (1.3) with final condition $X(t) = x$ and $V(r, s)$ is the solution of the 3×3 matrix equation

$$\begin{cases} \frac{d}{dr}V(r, s) = \mathcal{D}_u(r, X(r))V(r, s), & r \in [s, t] \\ V(s, s) = I. \end{cases}$$

The present paper is devoted to explain the formula in detail, and use it to prove a local-in-time existence and uniqueness result. This paper is in a sense the continuation of a paper of one of the authors (see Busnello [5]), where the 2D case has been considered. In the 2D case the stretching term $\mathcal{D}_u\xi$ is zero, so $V(r, s) = I$. The vorticity is purely transported and diffused, allowing for a global-in-time control which yields global existence and uniqueness results. In Busnello [5], the probabilistic formula is used to prove such a result, related to the deterministic work of Ben-Artzi [4], following a suggestion of M. Friedlin.

Girsanov transformation is used in a basic part of the work, and the Bismut-Elworthy formula is used to treat by probabilistic methods also the Biot-Savart law, which reconstructs u from ξ (necessary to solve (1.3)). In the 3D case the Biot-Savart law and its probabilistic representation are

$$u(t, x) = -\frac{1}{4\pi} \int_{\mathbf{R}^3} \frac{(x-y) \times \xi(y)}{|x-y|^3} dy = \frac{1}{2} \int_0^\infty \frac{1}{s} \mathbf{E}[\xi(t, x + W_s) \times W_s] ds.$$

In the present paper we extend as much as possible the probabilistic approach of Busnello [5] to the 3D case. Now, a priori the stretching mechanism could produce singularities and blow-up in ξ , so we can only work on a time interval $[0, \tau]$ depending on the size of the data. This is the only possible result that also the analytic approaches to equation (1.1) can reach at present. Global existence for (1.1) is known only at the level of weak solutions, but we have to work at a higher level of regularity to deal with the vorticity. In certain function spaces, global existence (and uniqueness) are known for sufficiently small data; in principle the probabilistic formulation could lead to such results, but we have found some obstacles, so a probabilistic proof of such a result remains an open problem (except for the completely different approach of Le Jan and Sznitman [19]).

The plan of the paper is the following. In Section 2 we state the precise representation formula and the local existence and uniqueness result for the Navier-Stokes equation, with the main lines of its proof. However, the full proof of the representation formula and the local result are based on three main items that we postpone to the next three sections:

- (i) a general representation formula for linear systems of parabolic equations, given in Section 3;
- (ii) the probabilistic representation of Biot-Savart law and a number of estimates on it, given in Section 4;
- (iii) a series of estimates for the expected values appearing in the formula for the vorticity, given in Section 5.

We have chosen this ordering to hi-light the results for the Navier-Stokes equation at the beginning, for the reader who is not interested in the long list of estimates and preliminaries necessary to prove the main theorem. About item (i) above, we remark that we use a method due to Krylov (in the scalar case) that introduces new variables in order to eliminate the zero order terms of the parabolic equation. Such method in the case of systems coupled through the zero order part is particularly interesting because it reduces the original system to a decoupled one. The representation proved in Section 3 can be applied, in principle, to several other systems of equations appearing in fluid dynamics, like the equation for u itself (but the term ∇p appears in the right-hand-side), the equation for the magnetisation variable (see for example [7]), the equation for the transport of passive scalars.

Concerning the literature on the subject, at an advanced state of the work we became aware of the interesting papers by Esposito, Marra, Pulvirenti and Sciarretta [10] and by Esposito and Pulvirenti [11] where a partially similar representation formula was introduced; differently with respect to these papers we treat probabilistically also the Biot-Savart law, we use different probabilistic tools, we analyse in detail the general case of systems of probabilistic equations to understand rigorously the equivalence with the probabilistic representation and we prove the local existence and uniqueness result in different function spaces (in particular, for a class of less smooth initial conditions).

There is also a paper by Rapoport [21] dealing with a general class of equations on manifolds which in particular throw light on the differential geometric structure of the formula. Also the probabilistic representation of systems of parabolic equations has been treated in the literature under certain assumptions (our work seems to be more general); see Kahane [16] and Freidlin [12].

Finally, among the literature on probabilistic analysis of PDEs there are possible connections with the geometric approach of Gliklikh [15], with recent investigations on vortex method in 3D by Meleard and co-workers, by Giet [13], and more closely with a work in preparation by Albeverio and Belopolskaya [1] where a probabilistic representation for the velocity u is employed. Concerning the huge literature on the deterministic analysis of the Navier-Stokes equations, results of local existence and uniqueness have been proved in a great amount of function spaces, see for instance collections of results in Cannone [6] and von Wahl [22], or in many works of Kato, Solonnikov and many others. We have not found a theorem exactly with the spaces used in the present paper, but it may exist somewhere or it may be proved with an adaptation of the existing techniques.

1.1. A physical interpretation of the probabilistic formula for the vorticity.

1.1.1. *Evolution of the vorticity in the non-viscous case.* Let us first recall the well-known evolution of the vorticity field of an incompressible *non-viscous* fluid (therefore described by the Euler equation). Let $\xi(t, x)$ be the value of the vorticity at time t and point $x \in \mathbf{R}^3$. The material point x moves according to the law

$$\begin{cases} \dot{X}(t) = u(t, X(t)) \\ X(0) = x, \end{cases}$$

where u is the velocity field of the fluid. From the Eulerian description of the evolution of ξ

$$\partial_t \xi + (u \cdot \nabla) \xi = \mathcal{D}_u \xi + g,$$

we deduce the Lagrangian formulation

$$(1.4) \quad \frac{d}{dt}\xi(t, X(t)) = \mathcal{D}_u(t, X(t))\xi(t, X(t)) + g(t, X(t))$$

which gives us

$$(1.5) \quad \xi(t, X(t)) = V(t, 0)\xi_0(x) + \int_0^t V(t, s)g(s, X(s)) ds$$

where

$$(1.6) \quad \begin{cases} \frac{d}{dr}V(r, s) = \mathcal{D}_u(r, X(r))V(r, s) & r \in [s, t] \\ V(s, s) = I. \end{cases}$$

Take $g = 0$ for simplicity (the general case is similar); equations (1.4) and (1.5) say that the initial vorticity $\xi_0(x)$ at point x is transported along the path $X(t)$, and during this motion it is modified by the deformation tensor. For instance, the vorticity is stretched when it is sufficiently aligned with the expanding directions of \mathcal{D}_u ; of course the relative position of ξ with respect to the expanding and contracting (remember that $\text{Trace } \mathcal{D}_u = 0$) directions of \mathcal{D}_u changes in time, so $\xi(t, X(t))$ may undergo a complicate evolution with stretching, rotations, contractions. Heuristic reasoning and numerical experiments show a predominance of the stretching mechanism, and seem to indicate even a blow-up of $\xi(t, X(t))$ in finite time, for certain initial point x .

If we want to know $\xi(\bar{t}, \bar{x})$ at a certain time \bar{t} and point \bar{x} , we have to solve the backward equation

$$(1.7) \quad \begin{cases} \dot{X}(t) = u(t, X(t)) & t \in [0, \bar{t}] \\ X(\bar{t}) = \bar{x}, \end{cases}$$

to find the initial position $x = X(0)$ which moves to \bar{x} at time \bar{t} ; then

$$(1.8) \quad \xi(\bar{t}, \bar{x}) = V^{\bar{t}, \bar{x}}(\bar{t}, 0)\xi_0(X^{\bar{t}, \bar{x}}(0)) + \int_0^{\bar{t}} V^{\bar{t}, \bar{x}}(\bar{t}, s)g(s, X^{\bar{t}, \bar{x}}(s)) ds$$

where we have denoted by $X^{\bar{t}, \bar{x}}(\cdot)$ the solution of (1.7), to stress the dependence of the final condition \bar{x} at time \bar{t} , and by $V^{\bar{t}, \bar{x}}(r, s)$ the corresponding solution of equation (1.6).

1.1.2. Path integral modification in the viscous case. In the viscous case the position $X(t)$ of a material point still evolves under the deterministic equation $\dot{X}(t) = u(t, X(t))$. However, the vorticity carried by the fluid particle at time $t = 0$ is not simply transported along its motion and modified by the action of the tensor \mathcal{D}_u ; a diffusion of ξ takes place. Let us introduce a *virtual* evolution of fluid particles, subject to u and a random diffusion:

$$(1.9) \quad dX(t) = u(t, X(t)) dt + \sqrt{2\nu} dW_t$$

where W_t is 3D Brownian motion. Whether such a motion has a physical meaning or not seems to be a similar question to the case of Feynman paths in Feynman integrals approach to quantum physics. The initial vorticity $\xi_0(x)$ decomposes, in a sense, in infinitesimal components along the different random solutions of (1.9), proportional to the probability of each evolution (strictly speaking such probabilities are zero). If $X(t, \omega)$ is a path given by (1.9), let us denote by $P(\omega)$

its probability, ignoring for a moment that $P(\omega) = 0$; then an amount of vorticity equal to $\xi_0(x)P(\omega)$ travels along $X(t, \omega)$ and is subject to the action $V(t, s)$ of \mathcal{D}_u along the path:

$$\xi(t, X(t, \omega))P(\omega) = V(t, 0)\xi_0(x)P(\omega) + \int_0^t V(t, s)g(s, X(s, \omega)) ds P(\omega)$$

(the reasoning for the integral effect of g is similar, and we omit it). Now $\xi(t, X(t, \omega))P(\omega)$ is not the total value of the vorticity field at time t and point $\bar{x} = X(t, \omega)$, but it is only the contribution due to the ω -evolution started from position x : other initial positions and other evolutions will reach the point \bar{x} at time t , and we have to add all these contributions. Therefore to compute $\xi(\bar{t}, \bar{x})$ at a certain time \bar{t} and point \bar{x} we have to solve the backward stochastic equation

$$\begin{cases} dX(t) = u(t, X(t)) dt + \sqrt{2\nu} dW_t & t \in [0, \bar{t}] \\ X(\bar{t}) = \bar{x} \end{cases}$$

to find the various positions $X(0, \omega)$ which move to \bar{x} at time \bar{t} under different noise paths $W(t, \omega)$; at the heuristic level each ω gives a contribution $\xi(\bar{t}, \bar{x}; \omega)P(\omega)$ to $\xi(\bar{t}, \bar{x})$ given by

$$\xi(\bar{t}, \bar{x}; \omega)P(\omega) = V^{\bar{t}, \bar{x}}(\bar{t}, 0; \omega)\xi_0(X^{\bar{t}, \bar{x}}(0; \omega))P(\omega) + \int_0^{\bar{t}} V^{\bar{t}, \bar{x}}(\bar{t}, s)g(s, X^{\bar{t}, \bar{x}}(s; \omega)) ds P(\omega)$$

(see (1.8) and (1.4)), so the total $\xi(\bar{t}, \bar{x})$ is given by

$$\xi(\bar{t}, \bar{x}) = \mathbf{E}[V^{\bar{t}, \bar{x}}(\bar{t}, 0)\xi_0(X^{\bar{t}, \bar{x}}(0))] + \int_0^{\bar{t}} \mathbf{E}[V^{\bar{t}, \bar{x}}(\bar{t}, s)g(s, X^{\bar{t}, \bar{x}}(s))] ds.$$

This is the heuristic derivation and the physical explanation of the formula.

2. MAIN RESULT ON THE PROBABILISTIC REPRESENTATION FOR THE VORTICITY

2.1. Some definitions and notations. First we recall some classical spaces, like the space $L^p(\mathbf{R}^3, \mathbf{R}^3)$ of 3D vector fields whose p -power is summable, with norm

$$\|f\|_p = \left(\int_{\mathbf{R}^3} |f(x)|^p dx \right)^{\frac{1}{p}},$$

the space $C_b^k(\mathbf{R}^3, \mathbf{R}^3)$ of k -times differentiable vector fields, with norm

$$\|g\|_{C_b^k} = \sum_{|\beta| \leq k} \|D^\beta g\|_\infty$$

and finally the space $C_b^{k, \alpha}(\mathbf{R}^3, \mathbf{R}^3)$ of vector fields whose k^{th} -order derivatives are Hölder-continuous with exponent α , with norm

$$\|g\|_{C_b^{k, \alpha}} = \|g\|_{C_b^k} + [g]_{k+\alpha},$$

where

$$[g]_{k+\alpha} = \sum_{|\beta|=k} \sup_{x, y \in \mathbf{R}^3} \frac{|D^\beta g(x) - D^\beta g(y)|}{\|x - y\|^\alpha}.$$

Next we define the spaces where our problem will be set. The velocity field of Navier-Stokes equations will be in the space

$$(2.1) \quad \mathcal{U}^\alpha(T) = \left\{ u \in C([0, T]; C_b^1(\mathbf{R}^3, \mathbf{R}^3)) \cap L^\infty(0, T; C_b^{1, \alpha}(\mathbf{R}^3, \mathbf{R}^3)) \mid \operatorname{div} u(t) = 0 \right\},$$

endowed with the norm

$$\sup \operatorname{ess}_{0 \leq t \leq T} \|u(t)\|_{C_b^{1,\alpha}},$$

while the vorticity will be in the space

$$(2.2) \quad \mathcal{V}^{\alpha,p}(T) = C([0, T]; C_b(\mathbf{R}^3, \mathbf{R}^3) \cap L^p(\mathbf{R}^3, \mathbf{R}^3)) \cap L^\infty(0, T; C_b^\alpha(\mathbf{R}^3, \mathbf{R}^3)),$$

endowed with the norm

$$\sup \operatorname{ess}_{0 \leq t \leq T} \|v(t)\|_{L^p \cap C_b^\alpha},$$

where $\|\psi\|_{C_b^\alpha \cap L^p} = \|\psi\|_p + \|\psi\|_{C_b^\alpha}$. We will use also the space

$$(2.3) \quad \mathcal{U}_M^\alpha(T) = \left\{ u \in \mathcal{U}^\alpha(T) \mid \sup \operatorname{ess}_{t \leq T} \|u(t)\|_{C_b^{1,\alpha}} \leq M \right\},$$

and the space

$$(2.4) \quad \mathcal{V}_L^{\alpha,p}(T) = \left\{ \psi \in \mathcal{V}^{\alpha,p}(T) \mid \sup \operatorname{ess}_{t \leq T} \|\psi(t)\|_{L^p \cap C_b^\alpha} \leq L \right\}.$$

2.2. Probabilistic representation for the vorticity. The formulation of the three dimensional Navier-Stokes equations

$$(2.5) \quad \begin{aligned} \partial_t u - \nu \Delta u + (u \cdot \nabla)u + \nabla P &= f, \\ \operatorname{div} u &= 0, \\ u(0, x) &= u_0(x), \\ \lim_{|x| \rightarrow \infty} u(t, x) &= 0, \end{aligned}$$

can be given in terms of the vorticity field $\xi = \operatorname{curl} u$ as

$$\begin{aligned} \partial_t \xi - \nu \Delta \xi + (u \cdot \nabla)\xi - (\xi \cdot \nabla)u &= g, \\ \xi(0, x) &= \xi_0(x), \\ \xi &= \operatorname{curl} u, \\ \operatorname{div} u &= 0, \\ \lim_{|x| \rightarrow \infty} u(t, x) &= 0, \end{aligned}$$

where $g = \operatorname{curl} f$. We shall write the term $(\xi \cdot \nabla)u$ as $(\nabla u)\xi$. Moreover, the same term can be written as $\mathcal{D}_u \xi$, where \mathcal{D}_u is the *deformation tensor*, the symmetric part of ∇u ,

$$\mathcal{D}_u = \frac{1}{2}(\nabla u + \nabla u^T),$$

since

$$(\nabla u)\xi - \mathcal{D}_u \xi = \frac{1}{2}(\nabla u - \nabla u^T)\xi = \xi \times \xi = 0.$$

As we explained intuitively in the introduction (see Section 1.1) and we shall describe rigorously in the sequel, using the representation formula of Theorem 4.4 and the generalised Feynman-Kac formula of Theorem 3.12, the formulation of Navier-Stokes equations can be given in the following way:

$$(2.6) \quad \begin{aligned} \xi(t, x) &= \mathbf{E}[U_t^{x,t} \xi_0(X_t^{x,t})] + \int_0^t \mathbf{E}[U_s^{x,t} g(t-s, X_s^{x,t})] ds, \\ u(t, x) &= \frac{1}{2} \int_0^\infty \frac{1}{s} \mathbf{E}[\xi(t, x + W_s) \times W_s] ds, \end{aligned}$$

where the *Lagrangian paths* $(X_s^{x,t})_{0 \leq s \leq t}$ are processes solutions of the following stochastic differential equations

$$\begin{cases} dX_s^{x,t} = -u(t-s, X_s^{x,t}) ds + \sqrt{2\nu} dW_s, & s \leq t, \\ X_0^{x,t} = x, \end{cases}$$

and the *deformation matrices* $(U_s^{x,t})_{0 \leq s \leq t}$ are the solutions to the following differential equations with random coefficients

$$\begin{cases} dU_s^{x,t} = U_s^{x,t} \mathcal{D}_u(t-s, X_s^{x,t}) ds, & s \leq t, \\ U_0^{x,t} = I, \end{cases}$$

Here \mathcal{D}_u is either ∇u or the deformation tensor (the name deformation matrices of $U_s^{x,t}$ refers to the latter case). Notice that, with respect to the introduction, we have made a time-reversion which simplifies the mathematical analysis.

A sufficiently regular solution of the classical formulation (2.5) is a solution of (2.6) and vice-versa. The main aim of this section is to show that, under suitable conditions, problem (2.6) has a unique local in time solution. The claim is proved in the following theorem.

Theorem 2.1. *Given $p \in [1, \frac{3}{2})$, $\alpha \in (0, 1)$ and $T > 0$, let $\xi_0 \in C_b^\alpha(\mathbf{R}^3, \mathbf{R}^3) \cap L^p(\mathbf{R}^3, \mathbf{R}^3)$ and $g \in \mathcal{V}^{\alpha,p}(T)$, and set*

$$\varepsilon_0 = \|\xi_0\|_{C_b^\alpha \cap L^p} + \int_0^T \|g(s)\|_{C_b^\alpha \cap L^p} ds.$$

Then there exists $\tau \in (0, T]$, depending only on ε_0 , such that there is a unique solution $u \in \mathcal{U}^\alpha(\tau)$, with $\xi \in \mathcal{V}^{\alpha,p}(\tau)$, of problem (2.6).

Proof. We will show that there are suitable L , M and τ such that the map $\mathcal{B} \circ \mathcal{N}$, where $\mathcal{B} : \mathcal{V}_L^{\alpha,p}(\tau) \rightarrow \mathcal{U}_M^\alpha(\tau)$ is defined as

$$\mathcal{B}(\xi)(t, x) = \frac{1}{2} \int_0^\infty \frac{1}{s} \mathbf{E}[\xi(t, x + W_s) \times W_s] ds,$$

and $\mathcal{N} : \mathcal{U}_M^\alpha(\tau) \rightarrow \mathcal{V}_L^{\alpha,p}(\tau)$ is defined as

$$\mathcal{N}(u)(t, x) = \mathbf{E}[U_t^{x,t} \xi_0(X_t^{x,t})] + \int_0^t \mathbf{E}[U_t^{x,t} g(t-s, X_s^{x,t})] ds,$$

is contractive.

First, in view of Corollary 4.5, $M \geq \tilde{C}L$. Using Proposition 5.5, we see that \mathcal{N} maps $\mathcal{U}_M^\alpha(\tau)$ to $\mathcal{V}_L^{\alpha,p}(\tau)$ if

$$(2.7) \quad e^{3\tau M} (1 + \tau M) \varepsilon_0 \leq L.$$

By means of Corollary 4.5 and Proposition 5.6, $\mathcal{B} \circ \mathcal{N}$ is contractive if

$$(2.8) \quad \tilde{C}C(\nu, p)C_M(\tau)\varepsilon_0 < 1,$$

where $C(\nu, p)$ is a constant depending only on p and ν , and $\lim_{\tau \rightarrow 0} C_M(\tau) = 0$. Hence, it is sufficient to choose τ small enough in order to have both conditions (2.7) and (2.8) verified. \square

Remark 2.2. As usual, the statement of the above theorem can be read in terms of small initial data. More precisely, for each fixed time T , there is a constant ϵ such that if $\varepsilon_0 \leq \epsilon$, there exists a unique solution $u \in \mathcal{U}^\alpha(T)$, with $\xi \in \mathcal{V}^{\alpha,p}(T)$, of problem (2.6)

3. THE FEYNMAN-KAC FORMULA FOR A DETERMINISTIC SYSTEM OF PARABOLIC EQUATIONS

This section is devoted to the development of a probabilistic representation formula for the following system of parabolic equations with final condition:

$$(3.1) \quad \begin{cases} \partial_t v_k + \frac{1}{2} \sum_{i,j} a_{ij} \partial_{x_i x_j}^2 v_k + \sum_{i=1}^d b_i \partial_{x_i} v_k + (\mathcal{D}v)_k + f_k = 0, & k = 1, \dots, l, \\ v_k(T, x) = \varphi_k(x), & x \in \mathbf{R}^d, \end{cases}$$

for $(t, x) \in [0, T] \times \mathbf{R}^d$, or the following system of parabolic equations with initial condition

$$(3.2) \quad \begin{cases} \partial_t v_k = \frac{1}{2} \sum_{i,j} a_{ij} \partial_{x_i x_j}^2 v_k + \sum_{i=1}^d b_i \partial_{x_i} v_k + (\mathcal{D}v)_k + f_k, & k = 1, \dots, l, \\ v_k(0, x) = \varphi_k(x), & x \in \mathbf{R}^d, \end{cases}$$

where $a = \sigma \sigma^*$ and

$$(3.3) \quad \begin{aligned} \sigma &: [0, T] \times \mathbf{R}^d \longrightarrow \mathbf{R}^{d \times d}, & b &: [0, T] \times \mathbf{R}^d \longrightarrow \mathbf{R}^d, \\ \mathcal{D} &: [0, T] \times \mathbf{R}^d \longrightarrow \mathbf{R}^{l \times l}, & \varphi &: \mathbf{R}^d \longrightarrow \mathbf{R}^l \\ f &: [0, T] \times \mathbf{R}^d \longrightarrow \mathbf{R}^l \end{aligned}$$

are Borel measurable functions. Additional assumptions will be stated in the sequel.

At first, for simplicity, assume that $f \equiv 0$ and all the data are regular. If $l = 1$, the equation (3.1), with final condition, has a unique solution given by the Feynman-Kac formula

$$v(t, x) = \mathbf{E}[\varphi(X_T^{t,x}) e^{\int_t^T \mathcal{D}(r, X_r^{t,x}) dr}]$$

where $X_s^{t,x}$ is the solution of the SDE

$$(3.4) \quad \begin{cases} dX_s^{t,x} = b(s, X_s^{t,x}) ds + \sigma(s, X_s^{t,x}) dW_s, & s \in [t, T], \\ X_t^{t,x} = x, \end{cases}$$

where $(W_t)_{t \geq 0}$ is a d -dimensional Brownian motion on some filtered probability space. Our aim is to extend such formula to the case $l > 1$.

Notice that in the case $l = 1$, for each ω , the function

$$u_r^{t,x} = e^{\int_t^r \mathcal{D}(s, X_s^{t,x}) ds},$$

is the solution of the following equation (now \mathcal{D} is a scalar)

$$(3.5) \quad \begin{cases} du_r^{t,x} = u_r^{t,x} \mathcal{D}(r, X_r^{t,x}) dr, & r \in [t, T], \\ u_t^{t,x} = 1. \end{cases}$$

So, in the same way, in the case $l > 1$, we will consider the process $U^{t,(x,Y)}$, solution of the equation

$$(3.6) \quad \begin{cases} dU_r^{t,(x,Y)} = U_r^{t,(x,Y)} \mathcal{D}(r, X_r^{t,x}) dr & r \in [t, T], \\ U_t^{t,(x,Y)} = Y, \end{cases}$$

where now both \mathcal{D} and $U^{t,(x,Y)}$ are $l \times l$ matrices. If $Y \equiv I$ we will write $U^{t,x}$ in place of $U^{t,(x,I)}$. Now, the natural conjecture is that, under suitable regularity conditions, the solution of (3.1) is

$$(3.7) \quad v(t, x) = \mathbf{E}[U_T^{t,x} \varphi(X_T^{t,x})].$$

In Section 3.1 we shall prove (3.7), under suitable regularity conditions on the coefficients. Such formula needs to be modified in order to handle the case $f \not\equiv 0$, as we show in Section 3.2. In Section 3.3 we shall provide sufficient conditions for the uniqueness of strong solutions to system (3.1). Finally in Section 3.4 we shall give a Feynman-Kac representation for the solutions of the system, with initial condition, (3.2).

Remark 3.1. When $l = 1$, we can write without distinction in formula (3.5) both $u_r^{t,x} \mathcal{D}$ and $\mathcal{D} u_r^{t,x}$, since they are both scalars. If $l > 1$, the lack of commutativity for the matrix products gives that $U_r^{t,x} \mathcal{D}$ and $\mathcal{D} U_r^{t,x}$ are different. The choice in the order of the matrix product in equation (3.6), and in formula (3.7) as well, derives from the form of the term $\mathcal{D} \cdot v$ in system (3.1). To have an intuitive idea of this fact, the reader can see the computations in the proof of the uniqueness in Proposition 3.9 (it is convenient to take $f \equiv 0$ for simplicity). However, when one uses backward stochastic equations to represent solutions, the order of U_r and \mathcal{D} in equation (3.6) changes, see Section 3.4.

3.1. The homogeneous case. Throughout this section, we will assume

$$f \equiv 0$$

and that the functions b , σ and \mathcal{D} , given in (3.3), are Borel measurable functions such that

- (A₁) b , σ are sub-linear with respect to x , uniformly in t ,
- (A₂) b , σ are locally Lipschitz-continuous in x , uniformly in t ,
- (A₃) a is differentiable in x and $\partial_{x_i} a$ are locally Lipschitz-continuous in x , uniformly in t ,
- (A₄) \mathcal{D} is bounded and locally Lipschitz-continuous in x , uniformly in t ,
- (A₅) φ is bounded and continuous.

In particular, assumptions (A₁), (A₂) and (A₄) ensure the existence of strong solutions, unique in law, for the equations (3.4) and (3.6). Moreover, from assumption (A₄), it easily follows that

$$(3.8) \quad \|U_T^{t,x}\|_{\mathbf{R}^{l \times l}} \leq e^{T\|\mathcal{D}\|_\infty},$$

where $\|\mathcal{D}\|_\infty$ is the sup-norm. Finally, the previous formula and assumption (A₅), imply that the function v given by formula (3.7) is well defined and bounded.

We can now state the main result of this section:

Theorem 3.2. *Assume (A₁)-(A₅) and $\varphi \in C_b(\mathbf{R}^d, \mathbf{R}^l)$. Then the function*

$$v(t, x) = \mathbf{E}[U_T^{t,x} \varphi(X_T^{t,x})]$$

is continuous and bounded and solves the Kolmogorov equation (3.1) in the sense of distributions, that is

$$(3.9) \quad \int_0^T \int_{\mathbf{R}^d} v M^* \eta \, dx \, dt = 0, \quad \text{for all } \eta \in C_c^\infty((0, T) \times \mathbf{R}^d, \mathbf{R}^l),$$

where

$$(3.10) \quad M^* \eta = -\partial_t \eta + \frac{1}{2} \sum_{ij} \partial_{x_i x_j}^2 (a_{ij} \eta) - \sum_i \partial_{x_i} (b_i \eta) + \mathcal{D}^* \eta.$$

Remark 3.3. The operator M^* makes sense since, by assumptions (A₂), (A₃) and Rademacher theorem, the functions $\partial_{ij} a_{ij}$ and $\partial_i b_i$ are well defined a.e. and essentially bounded in compact sets. Moreover, $M^* \eta$ is bounded in compact sets.

To prove Theorem 3.2, we shall use the *method of new variables* given by Krylov in [17]. Krylov used such method in order to transform a parabolic equation on $\mathbf{R}^d \times [0, T]$ with potential term, into a parabolic equation on $\mathbf{R}^{d+2} \times [0, T]$ without potential term. As observed in the introduction, we extend this method to systems of parabolic equations. In our case, the elimination of the potential term has the additional advantage that the coupling between the equations in (3.1) disappears. In other words, we turn system (3.1) into a system of l *independent* parabolic equations on $\mathbf{R}^{d+l \times l} \times [0, T]$ without the potential term.

We define the new variables $\bar{x} = (x, Y) \in \mathbf{R}^{d+l \times l}$, and, for each function $\psi : \mathbf{R}^d \rightarrow \mathbf{R}^l$, we define the function $\bar{\psi} : \mathbf{R}^{d+l \times l} \rightarrow \mathbf{R}^l$ as $\bar{\psi}(\bar{x}) = Y\psi(x)$. Finally, if $u(t, x) : [0, T] \times \mathbf{R}^d \rightarrow \mathbf{R}^l$, we set $\bar{u}(t, \bar{x}) = Yu(t, x)$.

Prior to the computation of the derivatives of \bar{v} , we give some notations. We denote by $\mathbf{0}_{m \times n}$ the $m \times n$ matrix with all entries equal to zero. Given a column vector $\alpha \in \mathbf{R}^d$ and a $l \times l$ matrix A , we define the $(d+l)$ (*exotic*) column vector $\begin{bmatrix} \alpha \\ A \end{bmatrix}$, where the first d rows are given by the components of α and the other l rows are the rows of A (the apparent inconsistency is inessential, since we shall only use the scalar product defined below). The scalar product between two such vectors is defined as

$$\left\langle \begin{bmatrix} \alpha \\ A \end{bmatrix}, \begin{bmatrix} \beta \\ B \end{bmatrix} \right\rangle = \alpha \cdot \beta + \langle A : B \rangle,$$

where, as usual, $\langle A : B \rangle = \text{Trace}(A \cdot B) = \sum_{i,j=1}^l A_{ij}B_{ij}$.

Given $u \in C^1([0, T] \times \mathbf{R}^d; \mathbf{R}^l)$, since

$$(3.11) \quad \frac{\partial \bar{u}_h}{\partial Y_{ij}} = \frac{\partial (Yu)_h}{\partial Y_{ij}} = \frac{\partial}{\partial Y_{ij}} \sum_k Y_{hk} u_k = \delta_{ih} u_j, \quad h = 1, \dots, l$$

it follows that, for each $h = 1, \dots, l$, the gradient $\nabla_{\bar{x}} \bar{u}_h$ of \bar{u}_h with respect to all its variables is given by the following (*exotic*) column vector

$$\nabla_{\bar{x}} \bar{u}_h = \begin{bmatrix} \nabla_x (Yu)_h \\ \nabla_Y \bar{u}_h \end{bmatrix} = \begin{bmatrix} \nabla_x (Yu)_h \\ \mathbf{0}_{1 \times l} \\ \dots \\ u \\ \dots \\ \mathbf{0}_{1 \times l} \end{bmatrix},$$

where the d -column vector is the gradient with respect to x and the $l \times l$ matrix has its rows all equal to the l -dimensional vector $\mathbf{0}_{1 \times l} = (0, \dots, 0)^T$ except for the h^{th} , which is the vector u .

We want to evaluate next the scalar product $\langle \begin{bmatrix} b \\ Y\mathcal{D} \end{bmatrix}, \nabla_{\bar{x}} \bar{u}_h \rangle$. Since

$$(Y\mathcal{D})_{ij} \partial_{Y_{ij}} (Yu)_h = (Y\mathcal{D})_{ij} \delta_{ih} u_j = (Y\mathcal{D})_{hj} u_j \delta_{ih},$$

it follows that

$$\left\langle \begin{bmatrix} b \\ Y\mathcal{D} \end{bmatrix}, \nabla_{\bar{x}} \bar{u}_h \right\rangle = b \cdot \nabla_x (Yu)_h + (Y\mathcal{D}u)_h.$$

In particular, if $Y = I$, the above quantity is equal to $b \cdot \nabla_x u_h + (\mathcal{D}u)_h$.

Let

$$(3.12) \quad \alpha(t, \bar{x}) = \begin{pmatrix} a(t, x) & \mathbf{0}_{d \times l^2} \\ \mathbf{0}_{l^2 \times d} & \mathbf{0}_{l^2 \times l^2} \end{pmatrix}, \quad \beta(t, \bar{x}) = \begin{bmatrix} b(t, x) \\ Y\mathcal{D}(t, x) \end{bmatrix},$$

where we understand that α is defined in blocks, where each entry is a matrix itself (notice that also $D_{\bar{x}}^2 \bar{u}_h$ is defined in blocks, and the product $\langle \alpha : D_{\bar{x}}^2 \bar{u}_h \rangle$ is defined as the sum of the four

$\langle \cdot \rangle$ -products of the corresponding blocks). With these notations, if $u \in C^{1,2}([0, T] \times \mathbf{R}^d; \mathbf{R}^l)$, we have for each $h = 1, \dots, l$,

$$\begin{aligned} \partial_t \bar{u}_h + \frac{1}{2} \langle \alpha : D_{\bar{x}}^2 \bar{u}_h \rangle + \langle \beta, \nabla_{\bar{x}} \bar{u}_h \rangle &= \\ &= (Y \partial_t u)_h + \frac{1}{2} \sum_{i,j} a_{ij} \partial_{x_i x_j}^2 (Y u)_h + \sum_i b_i \partial_{x_i} (Y u)_h + (Y \mathcal{D} u)_h \\ &= \left[Y \left(\partial_t u + \frac{1}{2} \sum_{i,j} a_{ij} \partial_{x_i x_j}^2 u + \sum_i b_i \partial_{x_i} u + \mathcal{D} u \right) \right]_h. \end{aligned}$$

From this identity it is straightforward to prove that a field u is a strong solution of (3.1) if and only if \bar{u} is a strong solution of system (3.13), where by strong solution we mean a continuous function having continuous first derivatives in time and second derivatives in space, and satisfying the corresponding equation point-wise. In the same way, applying the same ideas used above on the adjoint operator, we have the following equivalence.

Proposition 3.4. *A function u is a weak solution of system (3.1), with final condition, if and only if \bar{u} is a weak solution of*

$$(3.13) \quad \partial_t \bar{u}_h + \frac{1}{2} \langle \alpha : D_{\bar{x}}^2 \bar{u}_h \rangle + \langle \beta, \nabla_{\bar{x}} \bar{u}_h \rangle = 0, \quad h = 1, \dots, l,$$

with final condition $\bar{u}(T, \bar{x}) = Y \varphi(x)$.

In the sequel we prove that, under suitable conditions, the vector field $\bar{v}(t, \bar{x}) = Y v(t, x)$, where v is given by (3.7), is a weak solution of (3.13). In view of the above lemma, this implies that the function given by (3.7) solves system (3.1) in the weak sense.

The main part is contained in the following proposition, where we relax some regularity assumptions on the coefficients of a theorem of Krylov [17]. Indeed, the drift and the diffusion defined in formulae (3.12) are neither bounded nor globally Lipschitz-continuous, in contrast to the assumptions of [17]. The same problem occurs for the final condition. On the other hand, both the drift and the diffusion are locally Lipschitz-continuous and with linear growth (in all variables, including Y).

Proposition 3.5. *Let $m \in \mathbf{N}$ and consider the scalar parabolic equation*

$$(3.14) \quad \partial_t u + \frac{1}{2} \langle \alpha : D^2 u \rangle + \langle \beta, \nabla u \rangle = 0 \quad (t, x) \in [0, T] \times \mathbf{R}^m$$

with final condition $u(T, x) = \psi(x)$, where $\alpha = \gamma \gamma^*$ and

$$\beta : [0, T] \times \mathbf{R}^m \rightarrow \mathbf{R}^m, \quad \gamma : [0, T] \times \mathbf{R}^m \rightarrow \mathbf{R}^{m \times m}, \quad \psi : [0, T] \times \mathbf{R}^m \rightarrow \mathbf{R},$$

and assume that

- (i) β, γ are Borel measurable, sub-linear and locally Lipschitz continuous in x , uniformly in t ,
- (ii) ψ is continuous and with polynomial growth,
- (iii) $\gamma(t, \cdot)$ is continuously differentiable for each t and $\partial_{x_i} \gamma$ are locally Lipschitz continuous in x , uniformly in t .

Set $u(t, x) = \mathbf{E}[\psi(Z_T^{t,x})]$, where $Z_r^{t,x}$ is the solution of the SDE

$$\begin{cases} dZ_r^{t,x} = \beta(r, Z_r^{t,x}) dr + \gamma(r, Z_r^{t,x}) dW_r, & r \in [t, T], \\ Z_t^{t,x} = x, \end{cases}$$

where $(W_t)_{t \geq 0}$ is an m -dimensional standard Brownian motion. Then u is a weak solution of (3.14): for each $\eta \in C_c^\infty((0, T) \times \mathbf{R}^m)$, we have

$$\int_0^T \int_{\mathbf{R}^m} u N^* \eta dx dt = 0,$$

where

$$N^* \eta = -\partial_t \eta + \frac{1}{2} \sum_{i,j} \partial_{x_i x_j}^2 (\alpha_{ij} \eta) - \sum_i \partial_{x_i} (\beta_i \eta).$$

Proof. If everywhere in the assumptions of the proposition we have global Lipschitz-continuity (instead of local Lipschitz-continuity), the proposition follows from Theorem 5.13 of Krylov [17]. In the general case, we proceed by truncation. Let $\Psi_n \in C^\infty(\mathbf{R}^m)$ be such that

$$\Psi_n(x) = \begin{cases} 1 & |x| \leq n \\ 0 & |x| \geq n+1 \end{cases}$$

and set $\beta^{(n)} = \Psi_n \beta$ and $\gamma^{(n)} = \Psi_n \gamma$. Fix a Brownian motion $(\Omega, \mathcal{F}, \mathcal{F}_t, W_t, \mathbf{P})$ and denote by $Z_t^{s,x,n}$ the solutions to the corresponding SDEs. The sequence $Z_t^{s,x,n}$ converges to $Z_t^{s,x}$ in probability uniformly on compact subsets of $[0, T] \times \mathbf{R}^m$.

Suppose first that ψ is bounded. Then $u_n(t, x) = \mathbf{E}[\psi(Z_t^{s,x,n})]$ converges to $u(t, x)$ and $\beta_{x_i}^{(n)}$ converges to β_{x_i} , $\partial_{x_i} \alpha^{(n)}$ to $\partial_{x_i} \alpha$ and $\partial_{x_i x_j} \alpha^{(n)}$ to $\partial_{x_i x_j} \alpha$ uniformly on compact subsets of $[0, T] \times \mathbf{R}^m$. Let $\eta \in C_c^\infty$, since $N_n^* \eta$ is a bounded sequence (see Remark 3.3), by the dominated convergence theorem, $\int u_n N_n^* \eta$ converges to $\int u N^* \eta$, where N_n^* is the operator corresponding to the approximate coefficients. Since u_n are weak solutions, it follows that u is also a weak solution.

If ψ is not bounded, we take a sequence of bounded continuous functions $\psi_n \rightarrow \psi$ such that $|\psi_n(x)| \leq |\psi(x)|$. From Theorem 4.6 of Krylov [17], we have $\mathbf{E}[|Z_T^{t,x}|^k] \leq c(1 + |x|^k)$, so that $u_n(t, x) \leq c(1 + |x|^k)$ by assumption (ii), and again we conclude by the dominated convergence theorem. \square

We are now ready to prove the main theorem.

Proof of Theorem 3.2. First we show that v is bounded and continuous. The boundedness comes from (3.8) and the assumptions on φ . In order to show the continuity, we take a sequence (x_n, t_n) converging to (x, t) . From Lemma 2.9 of Krylov [17], the function $(t, x) \rightarrow (X_s^{t,x}, U_s^{t,(x,I)}) \in C([0, T], \mathbf{R}^{d+l \times l})$ (where by convention $(X_s^{t,x}, U_s^{t,(x,I)}) = (x, I)$ if $s < t$) is continuous in probability. Hence, there is a subsequence such that convergence is almost sure. Finally, the conclusion follows from the bound (3.8), the assumptions on φ and the dominated convergence theorem.

We show then that v is a weak solution. We have the following two ingredients:

- (i) the two systems of SDEs (3.4) and (3.6) can be thought as a unique system where the solution $(X_r^{t,x}, U_r^{t,(x,Y)})$ takes values in $\mathbf{R}^{d+l \times l}$ and drift and diffusion are given by formulae (3.12).

(ii) Since by uniqueness for equation (3.6) it follows that $U_T^{t,(x,Y)} = YU_T^{t,x}$, for the function v defined in (3.7), we have

$$\bar{v}(t, \bar{x}) = Yv(t, x) = \mathbf{E}[YU_T^{t,x} \varphi(X_T^{t,x})] = \mathbf{E}[U_T^{t,(x,Y)} \varphi(X_T^{t,x})] = \mathbf{E}[\bar{\varphi}(X_T^{t,x}, U_T^{t,(x,Y)})].$$

From these two facts, by Proposition 3.5, \bar{v} is a weak solution to system (3.13). By Proposition 3.4, v is a weak solution to system (3.1). \square

The regularity assumption (A_4) on the term \mathcal{D} can be relaxed with the following condition

(A'_4) \mathcal{D} is bounded and uniformly continuous.

In fact we can deduce the following corollary.

Corollary 3.6. *Assume (A_1) - (A_3) , (A'_4) and (A_5) . Then the function*

$$v(t, x) = \mathbf{E}[U_T^{t,x} \varphi(X_T^{t,x})]$$

is continuous and bounded and solves the Kolmogorov equation (3.1) in the sense of distributions:

$$\int_0^T \int_{\mathbf{R}^d} v M^* \eta \, dx \, dt = 0 \quad \text{for all } \eta \in C_c^\infty((0, T) \times \mathbf{R}^d, \mathbf{R}^d).$$

Proof. Let ρ_n be a sequence of mollifiers and set $\mathcal{D}_n = \mathcal{D} * \rho_n$ and $v_n(t, x) = \mathbf{E}[U_{T,n}^{t,x} \varphi(X_T^{t,x})]$, where $U_{r,n}^{t,x}$ is the solution of (3.6) corresponding to \mathcal{D}_n .

Since $\mathcal{D}_n \rightarrow \mathcal{D}$ uniformly in $[0, T] \times \mathbf{R}^d$, we have $U_{t,n}^{t,x} \rightarrow U_t^{t,x}$ in $L^1(\Omega)$, uniformly in $[0, T] \times \mathbf{R}^d$. Consequently, $v_n(t, x) \rightarrow v(t, x)$ and $\mathcal{D}_n v_n \rightarrow \mathcal{D}v$ uniformly $[0, T] \times \mathbf{R}^d$. Since v_n are weak solutions of the corresponding approximate problem, in the limit v is a weak solution of $Mv = 0$. \square

3.2. The inhomogeneous case. In this section, Theorem 3.2 will be extended to the inhomogeneous case. We will show a Feynman-Kac representation formula for the complete system (3.1), that is with $f \not\equiv 0$, with final condition. Throughout this section we will assume (A_1) - (A_3) , (A'_4) , (A_5) and the following

(A_6) $f : [0, T] \times \mathbf{R}^d \rightarrow \mathbf{R}^l$ is bounded and uniformly continuous.

Theorem 3.7. *Assume (A_1) - (A_3) , (A'_4) , (A_5) - (A_6) . Then the function*

$$(3.15) \quad v(t, x) = \mathbf{E}[U_T^{t,x} \varphi(X_T^{t,x})] + \int_t^T \mathbf{E}[U_r^{t,x} f(r, X_r^{t,x})] \, dr$$

is a weak solution of (3.2), that is,

$$\int_0^T \int_{\mathbf{R}^d} (u M^* \eta + f \eta) \, dt \, dx = 0, \quad \eta \in C_c^\infty((0, T) \times \mathbf{R}^d, \mathbf{R}^l).$$

The main idea to prove the theorem is to introduce a new component (we apply again the *method of new variables* of Krylov [17]) and prove that v is a solution of system (3.1) if and only if $\tilde{v} = (v_1, \dots, v_l, 1)$ solves the system

$$(3.16) \quad \partial_t \tilde{v} + \frac{1}{2} \sum_{i,j} a_{ij} \partial_{x_i x_j} \tilde{v} + \sum_i b_i \partial_{x_i} \tilde{v} + (\tilde{\mathcal{D}} \tilde{v})_k = 0,$$

with final condition $\tilde{v}(T, \cdot) = (\varphi_1, \dots, \varphi_l, 1)$, where $\tilde{\mathcal{D}} = \begin{pmatrix} \mathcal{D} & f \\ 0 & 0 \end{pmatrix}$. Notice that $\tilde{\mathcal{D}} \tilde{v} = \begin{pmatrix} \mathcal{D}v + f \\ 0 \end{pmatrix}$, so that the component \tilde{v}_{l+1} is obviously a solution.

The key lemma is the following.

Lemma 3.8. *The function $\tilde{v} = (v_1, v_2, \dots, v_l, 1)$ is a weak solution of (3.16) if and only if $v = (v_1, v_2, \dots, v_l)$ is a weak solution of (3.1).*

Proof. A weak solution of (3.1) is a function v such that $\iint (vM^*\eta + f\eta) = 0$ for each test function η , or equivalently $\iint (vL^*\eta + v\mathcal{D}^*\eta + f\eta) = 0$, where the operator M^* has been defined in (3.10) and L^* is defined as

$$L^*\eta = -\partial_t\eta + \frac{1}{2} \sum_{i,j} \partial_{x_i x_j}^2 (a_{ij}\eta) - \sum_i \partial_{x_i} (b_i\eta).$$

Let $\tilde{\eta} = (\eta, \eta_{l+1})$ be a \mathbf{R}^{l+1} -valued test function. Since $\mathcal{D}^* = \begin{pmatrix} \mathcal{D}^* & 0 \\ f^* & 0 \end{pmatrix}$, we have

$$\tilde{v}\tilde{\mathcal{D}}^*\tilde{\eta} = \begin{pmatrix} v\mathcal{D}^*\eta + f\eta \\ 0 \end{pmatrix}.$$

It comes out that v is a solution of the inhomogeneous equation if and only if \tilde{v} solves $\iint (\tilde{v}L^*\tilde{\eta} + f\tilde{\eta}) = 0$, that is, if and only if \tilde{v} is a weak solution of system (3.16). \square

We can now prove the main theorem of this section.

Proof of Theorem 3.7. Let $\tilde{\varphi}$ be the function $(\varphi_1, \dots, \varphi_l, 1)$ and $\tilde{U}_s^{t,x}$ be the solution of

$$(3.17) \quad \begin{cases} d\tilde{U}_s^{t,x} = \tilde{U}_s^{t,x}\tilde{\mathcal{D}}(s, X_s^{t,x}) ds, & s \in [t, T], \\ \tilde{U}_t^{t,x} = I_{l+1}. \end{cases}$$

Since φ , \mathcal{D} and f satisfy assumptions (A_4) , (A_5) and (A_6) , the functions $\tilde{\varphi}$ and $\tilde{\mathcal{D}}$ satisfy assumptions (A'_4) and (A'_5) . Hence, by Corollary 3.6, the function

$$(x, t) \rightarrow \mathbf{E}[\tilde{U}_T^{t,x}\tilde{\varphi}(X_T^{t,x})]$$

is a weak solution of system (3.16).

We write $\tilde{U}_s^{t,x}$ in blocks:

$$\tilde{U}_s^{t,x} = \begin{pmatrix} A_s^{t,x} & b_s^{t,x} \\ c_s^{t,x} & d_s^{t,x} \end{pmatrix},$$

where A_s is a $l \times l$ matrix, $b_s \in \mathbf{R}^d$ is a column vector, $c_s \in \mathbf{R}^d$ is a row vector and d_s is a scalar. With this position, the Cauchy problem (3.17) is equivalent to

$$\begin{cases} dA_s^{t,x} = A_s^{t,x}\mathcal{D}(s, X_s^{t,x}) ds, & A_t^{t,x} = I_l, \\ db_s^{t,x} = A_s^{t,x}f(s, X_s^{t,x}) ds, & b_t^{t,x} = 0, \\ dc_s^{t,x} = c_s^{t,x}\mathcal{D}(s, X_s^{t,x}) ds, & c_t^{t,x} = 0, \\ dd_s^{t,x} = c_s^{t,x}f(s, X_s^{t,x}) ds, & d_t^{t,x} = 1, \end{cases}$$

and it is easy to see that

$$\begin{aligned} A_s^{t,x} &= U_s^{t,x} & b_s^{t,x} &= \int_t^s U_r^{t,x} f(r, X_r^{t,x}) dr \\ c_s^{t,x} &= 0 & d_s^{t,x} &= 1. \end{aligned}$$

Consequently,

$$\begin{aligned} \mathbf{E}[\tilde{U}_T^{t,x} \tilde{\varphi}(X_T^{t,x})] &= \mathbf{E}\left[\left(\begin{matrix} U_T^{t,x} & b_T^{t,x} \\ 0 & 1 \end{matrix}\right) \left(\varphi\left(X_T^{t,x}\right)\right)\right] = \mathbf{E}\left[\left(U_T^{t,x} \varphi\left(X_T^{t,x} + b_T^{t,x}\right)\right)\right] \\ &= \mathbf{E}\left[\left(U_T^{t,x} \varphi\left(X_T^{t,x} + b_T^{t,x}\right) + \int_t^s U_r^{t,x} f(r, X_r^{t,x}) dr\right)\right]. \end{aligned}$$

□

3.3. A uniqueness result. In the preceding sections, we were concerned with the existence of a weak solution of the parabolic system (3.1) having a nice probabilistic representation. The aim of the present section is to provide sufficient conditions for the uniqueness of solutions. In Proposition 3.9 we shall see that the strong solution, if exists, is given by our probabilistic representation, hence is unique. In Theorem 3.10 we will show, under some special conditions on the coefficients, that weak solutions are also unique and are given by the probabilistic representation. Such special conditions on the coefficients are satisfied in the application of the probabilistic representation to the Navier-Stokes system: if the velocity field is regular enough, the coefficients in the equations for the vorticity satisfy the special conditions. Hence, for each fixed regular velocity, there exists a unique weak solution of the vorticity equation given by the Feynman-Kac formula.

Let $C_b^{1,2}([0, T] \times \mathbf{R}^d, \mathbf{R}^l)$ be the space of continuous functions having first and second derivatives in x and first derivative in t continuous and bounded. We start by showing that, if the solution of the parabolic system is regular, then it is given by formula (3.15).

Proposition 3.9. *Let $v \in C_b^{1,2}([0, T] \times \mathbf{R}^d, \mathbf{R}^l)$ be a strong solution of system (3.1), with final condition. Then v is given by formula (3.15).*

Proof. It is sufficient to show that the process

$$U_r^{t,x} v(r, X_r^{t,x}) + \int_t^r U_s^{t,x} f(s, X_s^{t,x}) ds, \quad r \in [t, T],$$

is a martingale. Indeed, if $h \in \{1, \dots, l\}$, by Itô formula (we omit for simplicity $(r, X_r^{t,x})$ from the term $v(r, X_r^{t,x})$ and from the coefficients, and the subscript r from the term $U_r^{t,x}$)

$$\begin{aligned} d_r(U^{t,x} v)_h &= \sum_k d(U_{hk}^{t,x} v_k) = \sum_k (U_{hk}^{t,x} dv_k + v_k dU_{hk}^{t,x}) \\ &= \sum_k U_{hk}^{t,x} \left[(\partial_r v_k + \sum_i b_i \partial_{x_i} v_k + \frac{1}{2} \sum_{i,j} a_{ij} \partial_{x_i x_j}^2 v_k) dr \right. \\ &\quad \left. + \sum_{i,j} \partial_{x_i} v_k \sigma_{ij} dW_r^j \right] + \sum_{k,i} v_k U_{hi}^{t,x} \mathcal{D}_{ik} dr \\ &= - \sum_k U_{hk}^{t,x} (f_k + \sum_i \mathcal{D}_{ki} v_i) dr + (dM_r)_h + \sum_{k,i} v_k U_{hi}^{t,x} \mathcal{D}_{ik} dr \\ &= -d_r \left(\int_t^r (U_s^{t,x} f)_h ds \right) + (dM_r)_h \end{aligned}$$

since v is a solution of system (3.1); $(M_r)_{r \in [t, T]}$ is the d -dimensional martingale, vanishing at $r = t$, given by

$$(dM_r)_h = \sum_k U_{hk}^{t,x} \sum_{i,j} \partial_{x_i} v_k \sigma_{ij} dW_r^j.$$

Moreover, M_r is square-integrable, since $v \in C_b^{1,2}$, $U_T^{t,x}$ is bounded by (3.8) and

$$\sup_{t \leq r \leq T} \mathbf{E}[|X_r^{t,x}|^2]$$

is bounded. □

Theorem 3.10. *Let φ be bounded and continuous, f and \mathcal{D} bounded and uniformly continuous. Suppose that σ is constant and b is a Borel measurable and Lipschitz-continuous in x function such that $\operatorname{div} b = 0$. Then the function*

$$v(t, x) = \mathbf{E}[U_T^{t,x} \varphi(X_T^{t,x})] + \int_t^T \mathbf{E}[U_r^{t,x} f(r, X_r^{t,x})] dr$$

is the unique weak solution of the parabolic system (3.1).

The proof of the theorem is based on a regularisation by convolution, in order to apply the uniqueness result of the previous proposition.

Let $\rho \in C^\infty(\mathbf{R}^d, \mathbf{R})$, $0 \leq \rho \leq 1$, with support in the ball of radius one, and set $\rho_n(x) = n^d \rho(nx)$. Let J_n be the convolution operator: $J_n(u) = \rho_n * u$.

Lemma 3.11. *Let $b : \mathbf{R}^d \rightarrow \mathbf{R}^d$ be a Lipschitz-continuous function, such that $\operatorname{div} b = 0$ (in the sense of distributions). Then there is a constant C such that for each $u \in C_b(\mathbf{R}^d, \mathbf{R}^l)$,*

$$(3.18) \quad |([J_n, b \cdot \nabla]u)(x)| \leq C \sup_{y \in B_{1/n}(x)} |u(y)| \quad \text{for all } n,$$

where $[J_n, b \cdot \nabla]u = J_n((b \cdot \nabla)u) - (b \cdot \nabla)J_n u$ is the commutator. Moreover

$$(3.19) \quad [J_n, b \cdot \nabla]u \xrightarrow{n \rightarrow \infty} 0 \quad \text{uniformly on compact sets}$$

Proof. Fix $u \in C_b(\mathbf{R}^d, \mathbf{R}^l)$. Since $\operatorname{div} b = 0$, by integration by parts we have

$$\begin{aligned} & ([J_n, b \cdot \nabla]u)(x) = \\ &= (\rho_n * (b \cdot \nabla)u - (b \cdot \nabla)(\rho_n * u))(x) \\ &= \int_{\mathbf{R}^d} \rho_n(x-y) (b(y) \cdot \nabla_y)u(y) - (b(x) \cdot \nabla_x)(\rho_n(x-y))u(y) dy \\ &= \int_{\mathbf{R}^d} \nabla_y \rho_n(x-y) (b(x) - b(y))u(y) dy. \end{aligned}$$

Taking the norms in \mathbf{R}^l we get

$$\begin{aligned} |([J_n, b \cdot \nabla]u)(x)| &\leq \int_{B_{1/n}(x)} |\nabla \rho_n(x-y)| \cdot |b(y) - b(x)| \cdot |u(y)| dy \\ &\leq cL \|\nabla \rho\|_\infty \sup_{y \in B_{1/n}(x)} |u(y)| \end{aligned}$$

where L is the Lipschitz constant of b . So far, we have proved (3.18). Concerning (3.19), it is easy to see that the claim is true for $u \in C_b^\infty(\mathbf{R}^d, \mathbf{R}^l)$. If u is only C_b , the claim follows from approximation with C_b^∞ functions (in the sup-norm, on compact sets) and from the bound (3.18). □

We apply now the previous lemma to prove the main theorem.

Proof of Theorem 3.10. Let v be a bounded and continuous weak solution of system (3.1). The sequence $v_n = \rho_n * v$ belongs to $C([0, T], C_b^\infty(\mathbf{R}^d, \mathbf{R}^l))$ and $v_n \rightarrow v$ uniformly on compact sets. We want to show that v_n is a weak solution of

$$(3.20) \quad \partial_t v_n + \frac{1}{2} \sum_{i,j} a_{ij} \partial_{x_i x_j}^2 v_n + \sum_i b_i \partial_{x_i} v_n + \mathcal{D}v_n + \rho_n * f + R_n = 0,$$

with final condition $v_n(T) = \rho_n * \varphi$, where $R_n = [J_n, b \cdot \nabla]v + [J_n, \mathcal{D}]v$. Indeed, v is a weak solution of (3.1), so that we can use $\zeta_n = \check{\rho}_n * \eta$ as a test function, where η is again a test function and $\check{\rho}_n(x) = \rho_n(-x)$, to obtain with some easy computations

$$\begin{aligned} & \int_0^T \int_{\mathbf{R}^d} (v M^* \zeta + f \zeta) = \\ &= \iint v \left(-\partial_t \zeta_n + \frac{1}{2} \sum_{i,j} a_{x_i x_j} \partial_{ij}^2 \zeta_n - \sum_i b_i \partial_{x_i} \zeta_n + \mathcal{D}^* \zeta_n \right) + f \zeta_n \\ &= \iint [v_n \left(-\partial_t \eta + \frac{1}{2} \sum_{i,j} a_{ij} \partial_{x_i x_j}^2 \eta - \sum_i b_i \partial_{x_i} \eta + \mathcal{D}^* \eta \right)] \\ & \quad + \iint \eta (J_n f + [J_n, b \cdot \nabla]v + [J_n, \mathcal{D}]v) \end{aligned}$$

(notice that for each u , $\int u(\check{\rho}_n * \eta) = \int \eta(\rho_n * u)$).

Since v_n belongs to $C([0, T], C_b^\infty(\mathbf{R}^d, \mathbf{R}^l))$ and $\rho_n * f + R_n$ is bounded and continuous, we argue that the distributional derivative $\partial_t v_n$ is bounded and continuous and, therefore, a strong derivative. Hence $v_n \in C_b^{1,2}$ and it is a strong solution of (3.20). Proposition 3.9 yields

$$v_n(t, x) = \mathbf{E}[U_T^{t,x} \rho_n * \varphi(X_T^{t,x})] + \int_t^T \mathbf{E}[U_r^{t,x} (\rho_n * f + R_n)(X_r^{t,x})] dr.$$

It is easy to check that $\|[J_n, D]v\|_\infty \leq 2\|D\|_\infty \|v\|_\infty$ and $[J_n, D]v \rightarrow 0$, uniformly on compact sets. Hence, by the previous lemma, R_n is bounded, independently of n , and $R_n \rightarrow 0$ uniformly on compact sets. Using (3.8) and the dominated convergence theorem, we obtain

$$v(t, x) = \lim_{n \rightarrow \infty} v_n(t, x) = \mathbf{E}[U_T^{t,x} \varphi(X_T^{t,x})] + \int_t^T \mathbf{E}[U_r^{t,x} f(r, X_r^{t,x})] dr.$$

□

3.4. The formula for parabolic systems with initial condition. In this section we describe the probabilistic representation of weak solutions to the system (3.2), with initial condition. Indeed, in the sequel we will use the results of this sections to give a probabilistic representation for the solutions to the Navier-Stokes equations, which is a parabolic equation with initial condition.

We will obtain the representation formula for the forward parabolic system using the representation for the backward parabolic system and a time inversion of the coefficients. To this aim, we will consider the following stochastic differential equations

$$(3.21) \quad \begin{cases} dX_r^{s,x,t} = b(t-r, X_r^{s,x,t}) dr + \sigma(t-r, X_r^{s,x,t}) dW_r, & r \in [s, t], \\ X_s^{s,x,t} = x, \end{cases}$$

and

$$(3.22) \quad \begin{cases} dU_r^{s,(x,Y),t} = U_r^{s,(x,Y),t} \mathcal{D}(t-r, X_r^{s,x,t}) dr, & r \in [s, t], \\ U_s^{s,(x,Y),t} = Y, \end{cases}$$

where, as usual, $U^{s,(x,Y),t} = U^{s,x,t}$ when $Y = I$.

Theorem 3.12. *Let the data b , σ , φ , \mathcal{D} and f satisfy assumptions (A_1) - (A_3) , (A'_4) (in page 13, (A_5) (in page 9) and (A_6) (in page 13). Then the function*

$$(3.23) \quad v(t, x) = \mathbf{E}[U_t^{0,x,t} \varphi(X_t^{0,x,t})] + \int_0^t \mathbf{E}[U_r^{0,x,t} f(t-r, X_r^{0,x,t})] dr$$

is a weak solution of (3.2), with initial condition.

Moreover, if σ is constant and b is globally Lipschitz-continuous in x , then v is the unique weak solution.

Proof. Let $\tilde{v}(t, x) = v(T-t, x)$. If v is a weak solution of (3.2), by easy computations it follows that \tilde{v} is a weak solution of

$$(3.24) \quad \begin{aligned} \partial_t \tilde{v}(t, x) + \frac{1}{2} \sum_{i,j} a_{ij}(T-t, x) \partial_{x_i x_j}^2 \tilde{v}(t, x) + \sum_i b_i(T-t, x) \partial_{x_i} \tilde{v} \\ + \mathcal{D}(T-t, x) \tilde{v}(t, x) + f(T-t, x) = 0, \end{aligned}$$

for $t \in [0, T]$, with final condition $\tilde{v}(T, x) = \varphi(x)$ (and vice-versa).

By Theorem 3.7, a solution \tilde{v} of (3.24) is given by the following formula

$$\tilde{v}(t, x) = \mathbf{E}[U_T^{t,x,T} \varphi(X_T^{t,x,T})] + \int_t^T \mathbf{E}[U_r^{t,x,T} f(T-r, X_r^{t,x,T})] dr,$$

where $U_r^{t,x,T}$ and $X_r^{t,x,T}$ are given respectively in (3.22) and (3.21). We can conclude that a solution v of the forward parabolic equation (3.2) is given by

$$v(t, x) = \mathbf{E}[U_T^{T-t,x,T} \varphi(X_T^{T-t,x,T})] + \int_{T-t}^T \mathbf{E}[U_r^{T-t,x,T} f(T-r, X_r^{T-t,x,T})] dr$$

Finally, one can easily check that, for each $r \in [T-t, T]$, the joint law of the random variables $U_r^{T-t,x,T}$ and $X_r^{T-t,x,T}$ is equal to the joint law of the random variables $U_{r+t-T}^{0,x,t}$ and $X_{r+t-T}^{0,x,t}$. In conclusion, formula (3.23) holds. \square

The representation formula above appears more complicated than the formula for parabolic systems with final condition (3.15): the stochastic processes X_r in (3.15) are the solutions of a fixed SDE corresponding to different initial conditions, while the stochastic processes $X_r^{0,x,t}$ and $U_r^{0,x,t}$ in (3.23) solve for each t a different SDE. A different representation can be given, which is more appealing at the heuristic level, even if less suitable for stochastic calculus.

Consider the following backward SDE

$$(3.25) \quad Y_r^{t,x} = x + \int_r^t b(s, Y_s^{t,x}) ds + \int_r^t \sigma(s, Y_s^{t,x}) \hat{d}W_s, \quad r \in [0, t],$$

where $\hat{d}W_s$ denotes the backward stochastic integral with respect to the Brownian motion W_s (see Kunita [18] for the definition of the backward integral). Notice that the final condition

$Y_t^{t,x} = x$ has been imposed here. Let $V_r^{s,t,x}$, $0 \leq s \leq r \leq t$ be the solution of

$$(3.26) \quad \begin{cases} dV_r^{s,t,x} = \mathcal{D}(r, Y_r^{t,x}) V_r^{s,t,x} dr, & r \in [s, t], \\ V_s^{s,t,x} = I \end{cases}$$

Theorem 3.13. *Under the same assumptions of the previous theorem, a weak solution of the parabolic system (3.2), with initial condition, is given by the following formula*

$$(3.27) \quad v(t, x) = \mathbf{E}[V_t^{0,t,x} \varphi(Y_0^{t,x})] + \int_0^t \mathbf{E}[V_t^{r,t,x} f(r, Y_r^{t,x})] dr,$$

where $Y^{t,x}$ and $V^{r,t,x}$ are given respectively by (3.25) and (3.26).

Remark 3.14. We want to give an interpretation of the representation formula given above. Suppose for clarity that $f \equiv 0$. Consider the trajectory $Y_r^{t,x}(\omega)$ of a virtual particle which is in x at time t , transported by a velocity field and subject to a diffusion, and evaluate $v(0, Y_0^{t,x}(\omega)) = \varphi(Y_0^{t,x}(\omega))$. Then we take into account, through the vector field $V_t^{0,t,x}$, the effects of the tensor \mathcal{D} along the given trajectory in the time interval $[0, t]$. Finally, by taking the expectation, we consider the mean effect of all virtual particles.

Before giving the proof of the theorem, we need the following simple lemma for the time inversion of a stochastic integral.

Lemma 3.15. *Let $(W_s)_{s \geq 0}$ be a Brownian motion. Fix $t > 0$ and set*

$$B_s = W_t - W_{t-s} \quad s \in [0, t].$$

Let $\mathcal{F}_s^W = \sigma(W_r \mid r \in [0, s])$ and $\mathcal{F}_{s,t}^B = \sigma(B_u - B_v \mid s \leq v \leq u \leq t)$ and let $g(s)$ be a continuous and bounded process adapted to the filtration \mathcal{F}_s^W . Then the process $f(s) = g(t-s)$, $s \in [0, t]$ is $\mathcal{F}_{s,t}^B$ -adapted and for all a, b such that $0 \leq a \leq b \leq t$,

$$\int_a^b g(s) dW_s = \int_{t-b}^{t-a} f(s) \hat{d}B_s.$$

Proof. Since $B_u - B_v = W_{t-v} - W_{t-u}$, we have $\mathcal{F}_{t-s}^W = \mathcal{F}_{s,t}^B$ and this gives the first statement.

Take now a sequence of partitions of the interval $[a, b]$:

$$\pi_n : \{a = s_0^n \leq s_1^n \leq \dots \leq s_{k_n}^n = b\}$$

such that $|\pi_n| \rightarrow 0$. We have

$$\begin{aligned} \int_a^b g(s) dW_s &= \lim_{n \rightarrow \infty} \sum g(s_i^n) (W_{s_{i+1}^n} - W_{s_i^n}) \\ &= \lim_{n \rightarrow \infty} \sum g(t - r_i^n) (W_{t-r_{i+1}^n} - W_{t-r_i^n}) \\ &= \lim_{n \rightarrow \infty} \sum f(r_i^n) (B_{r_i^n} - B_{r_{i+1}^n}) \\ &= \int_{t-b}^{t-a} f(s) \hat{d}B_s, \end{aligned}$$

where $r_i^n = t - s_i^n$, $i = 1 \dots k_n$. □

Proof of Theorem 3.13. We need only to show that

$$X_{t-r}^{0,x,t} = Y_r^{t,x} \quad \text{and} \quad U_{t-r}^{0,x,t} = V_t^{r,x,t}, \quad \mathbf{P} - \text{a.s.}$$

since such formulas, formula (3.23) and a change of variables give us (3.27).

We prove the first equality. Fix a Brownian motion $(W_r)_{r \geq 0}$ and consider the solution $X_r^{0,x,t}$ of equation (3.21). By Lemma (3.15) above, it follows that $X_{t-r}^{0,x,t}$ satisfies the backward SDE (3.25) with respect to the Brownian motion B_s defined in Lemma 3.15. Since equation (3.25) has a unique strong solution, we have the first equality.

We proceed to prove the second equality. Fix ω so that $r \rightarrow Y_r^{t,x}(\omega)$ is continuous. The key observation is that

$$V_r^{s,t,x}(\omega) = V_r^{0,t,x}(\omega)(V_s^{0,t,x}(\omega))^{-1}, \quad 0 \leq s \leq r \leq t,$$

and it is true since

$$d(V_r^{0,t,x}(\omega))^{-1} = -(V_r^{0,t,x}(\omega))^{-1} \mathcal{D}(r, Y_r^{t,x}(\omega)),$$

with initial condition $(V_0^{0,t,x}(\omega))^{-1} = I$, so that it easy to check that $V_r^{0,t,x}(\omega)(V_s^{0,t,x}(\omega))^{-1}$ satisfy equation (3.26). Finally, by evaluating

$$d_r V_t^{r,t,x}(\omega) = d_r [V_t^{0,t,x}(\omega)(V_r^{0,t,x}(\omega))^{-1}],$$

we see that both $V_t^{r,t,x}(\omega)$ and $r \rightarrow U_{t-r}^{0,x,t}(\omega)$ solves the ODE:

$$dU_r = -U_r \mathcal{D}(r, Y_r^{t,x}(\omega)) dr \quad r \in [0, t],$$

with final condition $U_t = I$. □

4. A PROBABILISTIC REPRESENTATION FOR THE NEWTONIAN POTENTIAL AND THE BIOT-SAVART LAW

In the present section we aim to give a probabilistic representation for the velocity field of an incompressible fluid in terms of the vorticity field $\xi = \text{curl } u$.

Under suitable assumptions on ξ , the Poisson equation $-\Delta \psi = \xi$ has a solution, given by

$$\psi(x) = \frac{1}{4\pi} \int_{\mathbf{R}^3} \frac{\xi(y)}{|x-y|} dy$$

(ψ is a vector field and the equation is interpreted component-wise). Let $u(x)$ be defined as $u(x) = \text{curl } \psi(x)$, i.e.

$$(4.1) \quad u(x) = \frac{1}{4\pi} \int_{\mathbf{R}^3} \frac{\xi(y) \times (x-y)}{|x-y|^3} dy.$$

If $\text{div } \xi = 0$, then also $\text{div } \psi = 0$ and $\text{div } u = 0$, and this implies also $\text{curl } \text{curl } \psi = -\Delta \psi$. Therefore $\text{curl } u = \xi$, i.e. u is the divergence-free velocity field associated to ξ . The equality (4.1) is the *Biot-Savart law*.

In order to give a probabilistic representation of this formula, it is necessary to give a representation of the solution of the Poisson equation and of its derivatives.

4.1. A probabilistic representation for the Newtonian potential. In this section we study a probabilistic representation of the solution of the Poisson equation. The deterministic regularity results are classical (see for example Gilbarg and Trudinger [14] and Ziemer [23]), so we will focus on the probabilistic formula.

Let $f : \mathbf{R}^3 \rightarrow \mathbf{R}$ be an integrable function. We define the Newtonian potential with density f as

$$Nf(x) = \frac{1}{4\pi} \int_{\mathbf{R}^3} \frac{1}{|x-y|} f(y) dy.$$

If f is regular and with compact support, Nf is a solution of the Poisson equation.

Let $A = \frac{1}{2}\Delta$, it is well known that A generates, on the space $C_0(\mathbf{R}^3)$ of all continuous functions vanishing at infinity, the strongly continuous semigroup

$$P_t f(x) = \mathbf{E}[f(x + W_t)] \quad x \in \mathbf{R}^3, t \geq 0, \quad f \in C_0(\mathbf{R}^3),$$

where $(W_t)_{t \geq 0}$ is a 3D-standard Brownian motion. The resolvent of A can be written as

$$((A - \lambda I)^{-1} f)(x) = \int_0^\infty e^{-\lambda t} \mathbf{E}[f(x + W_t)] dt, \quad f \in C_0(\mathbf{R}^3),$$

so that we can argue that the integral

$$(4.2) \quad \int_0^\infty \mathbf{E}[f(x + W_t)] dt.$$

converges to $A^{-1}f(x) = 2Nf(x)$ (indeed, at this stage, we do not know if A is invertible).

As a first step, we find some conditions on f in such a way that formula (4.2) produces a solution of the Poisson equation.

Proposition 4.1. *Let $f \in L^p(\mathbf{R}^3) \cap L^q(\mathbf{R}^3)$, with $1 \leq p < \frac{3}{2} < q < \infty$. Then the integral in (4.2) is convergent for all $x \in \mathbf{R}^3$ and is equal to $2Nf(x)$. Moreover $Nf \in C_0(\mathbf{R}^3)$ and*

$$\|Nf\|_\infty \leq C_{p,q}(\|f\|_p + \|f\|_q).$$

Proof. For every $r > 1$, by Hölder inequality,

$$(4.3) \quad \mathbf{E}|f(x + W_t)| = \frac{1}{(2\pi t)^{3/2}} \int_{\mathbf{R}^3} |f(x + y)| e^{-\frac{1}{2t}|y|^2} dy \leq C_r t^{-\frac{3}{2r}} \|f\|_r,$$

so that, by using the above inequality with $r = p$ and $r = q$ and by integrating by time,

$$\int_0^\infty \mathbf{E}|f(x + W_t)| dt \leq \int_0^1 \mathbf{E}|f(x + W_t)| dt + \int_1^\infty \mathbf{E}|f(x + W_t)| dt \leq C(\|f\|_p + \|f\|_q).$$

This will prove also the final inequality, once the other properties are verified. The integral in (4.2) is equal to $2Nf(x)$ since (we can use Fubini theorem because of the previous inequality)

$$\begin{aligned} \int_0^\infty \mathbf{E}[f(x + W_t)] dt &= \int_{\mathbf{R}^3} f(x + y) \int_0^\infty \frac{1}{(2\pi t)^{3/2}} e^{-\frac{1}{2t}|y|^2} dt dy \\ &= \int_{\mathbf{R}^3} \frac{1}{2\pi|y|} f(x + y) dy = 2Nf(x). \end{aligned}$$

We know from Gilbarg and Trudinger [14], that $f \in L^q(\mathbf{R}^3)$ implies, by Sobolev embeddings, $Nf \in C(\mathbf{R}^3)$. The behaviour at infinity is less standard, so we give a probabilistic proof of it. Thus let us show that $Nf \in C_0(\mathbf{R}^3)$. Indeed, for each $R > 0$,

$$\int_0^\infty \mathbf{E}[f(x + W_t)] dt = \int_0^\infty \mathbf{E}f(x + W_t)I_{\{|W_t|>R\}} dt + \int_0^\infty \mathbf{E}f(x + W_t)I_{\{|W_t|\leq R\}} dt$$

and, in order to show that $Nf(x)$ converges to 0 as $|x| \rightarrow \infty$, we will prove that the first term converges to 0, uniformly in x , as $R \rightarrow \infty$, and the second term converges to 0 as $|x| \rightarrow \infty$ for each $R > 0$.

For the first term the claim is true since, as in (4.3),

$$\sup_{x \in \mathbf{R}^3} \mathbf{E}|f(x + W_t)|I_{\{|W_t|>R\}} \leq C(\|f\|_p + \|f\|_q)(t^{-3/2p}I_{[1,\infty)}(t) + t^{-3/2q}I_{[0,1)}(t))$$

and

$$\sup_{x \in \mathbf{R}^3} \mathbf{E}|f(x + W_t)|I_{\{|W_t|>R\}} \leq Ct^{-3/2}\|f\|_p \left(\int_{|y|>R} e^{-\frac{1}{2t}|y|^2} \right)^{1/p'} \rightarrow 0$$

as $R \rightarrow \infty$. As regards the second term, we can proceed as in (4.3) and bound the term $\mathbf{E}|f(x + W_t)|I_{\{|W_t|\leq R\}}$ with

$$C(t^{-\frac{3}{2p}}\|f(y)I_{\{|y-x|\leq R\}}\|_p I_{[1,\infty)}(t) + t^{-\frac{3}{2q}}\|f(y)I_{\{|y-x|\leq R\}}\|_q I_{[0,1)}(t)),$$

so that, after the integration in time, the above term converges to 0, since $f \in L^p(\mathbf{R}^3) \cap L^q(\mathbf{R}^3)$. \square

In the second step, we study the derivatives of Nf . Notice that, for a regular f , Bismut-Elworthy formula (see for example [9]) gives

$$D_{x_i} \mathbf{E}[f(x + W_t)] = \frac{1}{t} \mathbf{E}[f(x + W_t)W_t^i].$$

In this simple case, with the Brownian motion, such formula can be easily checked by means of the Gaussian density.

As in the previous proposition, one could expect that, under suitable conditions, it is possible to write the derivatives of Nf with the probabilistic representation suggested by the formula above. Indeed, this is the case, as the following proposition shows.

Proposition 4.2. *Let $f \in L^p(\mathbf{R}^3) \cap L^q(\mathbf{R}^3)$ for some $1 \leq p < \frac{3}{2} < 3 < q < +\infty$. Then $\nabla Nf \in C_0(\mathbf{R}^3)$ and for each $x \in \mathbf{R}^3$,*

$$(4.4) \quad 2D_{x_i} Nf(x) = \int_0^\infty \frac{1}{t} \mathbf{E}[f(x + W_t)W_t^i], \quad i = 1, 2, 3.$$

Moreover

$$(4.5) \quad \|\nabla Nf\|_\infty \leq C_{p,q}(\|f\|_p + \|f\|_q)$$

Proof. By Hölder inequality,

$$(4.6) \quad \begin{aligned} \frac{1}{t} \mathbf{E}|f(x + W_t)W_t^i| &= \frac{C}{t^{5/2}} \int_{\mathbf{R}^3} f(x + y)y^i e^{-\frac{1}{2t}|y|^2} \\ &\leq \frac{C}{t^{5/2}} \|f\|_p \sqrt{tt^{\frac{3}{2p}}} \\ &\leq C\|f\|_p t^{-\frac{1}{2} - \frac{3}{2p}} \end{aligned}$$

and, as in the proof of the previous proposition, the time integral is finite and bounded with respect to x , by the assumptions on p and q . Moreover it can be easily seen, by the same arguments used in the previous proposition, that formula (4.4) and inequality (4.5) hold and that $\nabla Nf \in C_0(\mathbf{R}^3)$. \square

In the last step, we study the second derivatives of the Newtonian potential. The regularity of the following theorem is based on the classical Schauder estimates.

Proposition 4.3. *Let $f \in L^p(\mathbf{R}^3) \cap C_b^\alpha(\mathbf{R}^3)$, with $1 \leq p < \frac{3}{2}$. Then $Nf \in C_b^{2,\alpha}(\mathbf{R}^3) \cap C_0(\mathbf{R}^3)$,*

$$\|Nf\|_{C_b^{2,\alpha}(\mathbf{R}^3)} \leq \tilde{C}(\|f\|_{L^p(\mathbf{R}^3)} + \|f\|_{C_b^\alpha(\mathbf{R}^3)})$$

and Nf is the unique solution of the Poisson equation in $C_0(\mathbf{R}^3) \cap C^2(\mathbf{R}^3)$.

Proof. From the previous proposition, we know that $Nf \in C_b^1(\mathbf{R}^3)$. Bismut-Elworthy formula gives us

$$(4.7) \quad D_{x_i x_j} \mathbf{E}f(x + W_t) = \frac{2}{t} \mathbf{E}[(D_{x_i} \psi)(x + W_{\frac{t}{2}}) W_{\frac{t}{2}}^j],$$

where $\psi(x) = \mathbf{E}f(x + W_{\frac{t}{2}})$. Hence, in order to show that

$$D_{x_i x_j} Nf(x) = \int_0^\infty \frac{1}{t} \mathbf{E}[(D_{x_i} \psi)(x + W_{\frac{t}{2}}) W_{\frac{t}{2}}^j] dt$$

holds, it is sufficient to show that (4.7) is integrable in time in the interval $[0, \infty)$.

First, by the Bismut-Elworthy formula, we see that

$$D_{x_i} \psi(x) = \frac{2}{t} \mathbf{E}[f(x + W_{\frac{t}{2}}) W_{\frac{t}{2}}^i]$$

and, by (4.6), that

$$(4.8) \quad \|D_{x_i} \psi\|_\infty \leq Ct^{-\frac{1}{2} - \frac{3}{2p}} \|f\|_p.$$

Moreover, since $f \in C_b^\alpha(\mathbf{R}^3)$,

$$(4.9) \quad \begin{aligned} |D_{x_i} \psi(y) - D_{x_i} \psi(x)| &= \frac{2}{t} \mathbf{E}|f(y + W_{\frac{t}{2}}) - f(x + W_{\frac{t}{2}})| \cdot |W_{\frac{t}{2}}^i| \\ &\leq Ct^{-\frac{1}{2}} [f]_\alpha |x - y|^\alpha. \end{aligned}$$

Now we show that (4.7) is integrable in time. By (4.8)

$$\frac{2}{t} \mathbf{E}|(D_{x_i} \psi)(x + W_{\frac{t}{2}}) W_{\frac{t}{2}}^j| \leq Ct^{-\frac{3}{2} - \frac{3}{2p}} \|f\|_p \mathbf{E}|W_{\frac{t}{2}}^j| \leq Ct^{-1 - \frac{3}{2p}} \|f\|_p$$

and (4.7) is integrable in $[1, \infty)$. By (4.9) it follows that

$$\begin{aligned} \frac{2}{t} \mathbf{E}|(D_{x_i} \psi)(x + W_{\frac{t}{2}}) W_{\frac{t}{2}}^j| &= \frac{2}{t} \mathbf{E}|[(D_{x_i} \psi)(x + W_{\frac{t}{2}}) - (D_{x_i} \psi)(x)] W_{\frac{t}{2}}^j| \\ &\leq Ct^{-\frac{3}{2}} \mathbf{E}|W_{\frac{t}{2}}|^\alpha |W_{\frac{t}{2}}^j| \\ &\leq Ct^{-1 + \frac{\alpha}{2}} [f]_\alpha \end{aligned}$$

and (4.7) is integrable in $[0, 1)$.

In conclusion, the probabilistic representation formula for the second derivatives holds and

$$\|D_{x_i x_j} Nf\|_\infty \leq C(\|f\|_p + [f]_\alpha).$$

By Schauder's theory, since $Nf \in C_b^2(\mathbf{R}^3)$ and $f \in C_b^\alpha(\mathbf{R}^3)$, it follows that $Nf \in C_b^{2,\alpha}(\mathbf{R}^3)$ and

$$\|Nf\|_{C_b^{2,\alpha}(\mathbf{R}^3)} \leq C(\|f\|_p + \|f\|_{C_b^\alpha})$$

(see for example Lunardi [20]). Moreover, Nf solves the Poisson equation (Lemma 4.2 of Gilbarg and Trudinger [14]) and the solution is unique by the maximum principle. \square

4.2. A probabilistic representation for the Biot-Savart law. We apply now the theory developed in the previous section. The following theorem, which is actually a mere corollary of the above results, is nothing but the well known *Biot-Savart law*.

Theorem 4.4. *Let $\xi \in L^p(\mathbf{R}^3, \mathbf{R}^3) \cap C_b^\alpha(\mathbf{R}^3, \mathbf{R}^3)$, with $1 \leq p < \frac{3}{2}$ and $0 < \alpha < 1$. There is a unique $u \in C_b^{1,\alpha}(\mathbf{R}^3, \mathbf{R}^3) \cap C_0(\mathbf{R}^3, \mathbf{R}^3)$ such that*

$$\operatorname{curl} u = \xi, \quad \operatorname{div} u = 0$$

and such solution is given by the following formula

$$u(x) = \frac{1}{2} \int_0^\infty \frac{1}{t} \mathbf{E}[\xi(x + W_t) \times W_t] dt, \quad x \in \mathbf{R}^3,$$

where $(W_t)_{t \geq 0}$ is a standard 3D-Brownian motion.

Proof. The probabilistic formula derives from Proposition 4.2 and the regularity of u from Propositions 4.2 and 4.3. We prove the uniqueness of the representation: since $\operatorname{div} u = 0$, we have $u = \operatorname{curl} \psi$, where ψ is the stream function. Now, by the maximum principle, the unique solution of the problem

$$\Delta u = 0, \quad u \rightarrow 0 \quad \text{as } |x| \rightarrow \infty$$

is $u \equiv 0$. \square

Since we are interested in the time evolution of the vector fields, it is appropriate to give a time-dependent version of the previous theorem. We recall that the spaces $\mathcal{U}^\alpha(T)$ and $\mathcal{U}_M^\alpha(T)$ have been defined in (2.1) and (2.3), the spaces $\mathcal{V}^{\alpha,p}(T)$ and $\mathcal{V}_L^{\alpha,p}(T)$ have been defined in (2.2) and (2.4).

Corollary 4.5. *Let $\alpha \in (0, 1)$ and $1 \leq p < \frac{3}{2}$. The map $\mathcal{B}\mathcal{S} : \mathcal{V}^{\alpha,p}(T) \rightarrow \mathcal{U}^\alpha(T)$, defined as*

$$\mathcal{B}\mathcal{S}(\xi)(t, x) = \frac{1}{2} \int_0^\infty \frac{1}{s} \mathbf{E}[\xi(t, x + W_s) \times W_s] ds,$$

is linear bounded and $\|\mathcal{B}\mathcal{S}\| \leq \tilde{C}$, where \tilde{C} is the constant, independent of T , appearing in Proposition 4.3.

Moreover, if $L, M > 0$ are constant such that $M \geq \tilde{C}L$, then the map $\mathcal{B}\mathcal{S} : \mathcal{V}_L^{\alpha,p}(T) \rightarrow \mathcal{U}_M^\alpha(T)$ is linear bounded.

5. THE REPRESENTATION MAP

The section is devoted to the study of the properties of the representation map \mathcal{L} , defined as

$$\mathcal{L}(u)(t, x) = \mathbf{E}[U_t^{x,t} \psi(X_t^{x,t})] + \int_0^t \mathbf{E}[U_t^{x,t} g(t-s, X_s^{x,t})] ds,$$

where $\psi = \psi(x)$, $g = g(t, x)$ and $X_s^{x,t}$ are the Lagrangian paths, defined in (5.3), and $U_s^{x,t}$ are the deformation matrices, defined in (5.4).

In the first part, some regularity properties of the Lagrangian paths and of the deformation matrices are obtained. In the second part we show that \mathcal{L} maps the space $\mathcal{U}^\alpha(T)$ in $\mathcal{V}^{\alpha,p}(T)$ (for the definition of the spaces, see (2.1) and (2.2)). Finally, in the third part, we prove that \mathcal{L} is Lipschitz-continuous from $\mathcal{U}^\alpha(T)$ to $\mathcal{V}^{\alpha,p}(T)$.

5.1. Regularity of the Lagrangian paths. In this section we study some regularity properties of the Lagrangian paths

$$\begin{cases} dX_s^x = u(s, X_s^x) ds + \sqrt{2\nu} dW_s, & s \in [0, T], \\ X_0^x = x, \end{cases}$$

and of the deformation matrices

$$\begin{cases} dU_s^x = U_s^x \mathcal{D}(s, X_s^x) ds, & s \in [0, T], \\ U_0^x = I, \end{cases}$$

where $u \in C([0, T]; C_b^1(\mathbf{R}^3, \mathbf{R}^3))$ and $\mathcal{D} \in C([0, T]; C_b^\alpha(\mathbf{R}^3, \mathbf{R}^{3 \times 3}))$ are given. Notice that both equations have unique strong solutions. Hence, fixed a 3D Brownian motion $((W_s)_{s \geq 0}, (\mathcal{F}_s)_{s \geq 0})$ on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, for each $x \in \mathbf{R}^3$ there is a process $(X_s^x, U_s^x)_{s \geq 0}$ that solves the corresponding equations, and the solution is unique up to indistinguishability. The equations can be solved path-wise, choosing the $\omega \in \Omega$ for which $s \rightarrow W_s(\omega)$ is a continuous function. Hence, the statements of this section are true for all such ω s, independently of x and s . First define

$$\|v\|_{\infty, s} = \sup_{0 \leq r \leq s} \|v(r)\|_\infty.$$

Lemma 5.1. *Assume $u \in C([0, T]; C_b^1(\mathbf{R}^3, \mathbf{R}^3))$. Then*

$$(5.1) \quad |X_s^x - X_s^y| \leq |x - y| e^{s \|\nabla u\|_{\infty, s}}, \quad s \geq 0, x, y \in \mathbf{R}^3.$$

Moreover, if $\operatorname{div} u = 0$, then for all $s \geq 0$ and $\omega \in \Omega$, the map

$$x \in \mathbf{R}^3 \mapsto X_s^x(\omega) \in \mathbf{R}^3$$

is a diffeomorphism, the determinant of its Jacobian is everywhere equal to 1 and

$$(5.2) \quad \int_{\mathbf{R}^3} \varphi(X_s^x(\omega)) dx = \int_{\mathbf{R}^3} \varphi(x) dx \quad \varphi \in L^1(\mathbf{R}^3).$$

Proof. First we prove (5.1). By easy computations,

$$|X_s^x - X_s^y| \leq |x - y| + \|\nabla u\|_{\infty, s} \int_0^s |X_r^x - X_r^y| dr$$

and applying Gronwall's lemma, we can conclude.

Using Theorem 4.6.5 of Kunita [18] (actually the assumption of Hölder continuity on u is useless for our aim, since we deal with an additive noise, see also Theorem 4.1.1 of Busnello

[5]) one can easily deduce that $x \mapsto X_t^x$ is a diffeomorphism and the determinant of its Jacobian is constant. Moreover, since $\operatorname{div} u = 0$, the determinant of its Jacobian is equal to 1 for all times, so that, by a change of variables and a density argument, also (5.2) can be deduced. \square

Lemma 5.2. *Assume $u \in C([0, T]; C_b^1(\mathbf{R}^3, \mathbf{R}^3))$ and $\mathcal{D} \in C([0, T]; C_b^\alpha(\mathbf{R}^3, \mathbf{R}^{3 \times 3}))$. Then*

$$|U_s^x| \leq e^{s\|\mathcal{D}\|_{\infty, s}} \quad x \in \mathbf{R}^3, \quad s \in [0, T]$$

and for $x, y \in \mathbf{R}^3$ and $s \in [0, T]$,

$$|U_s^x - U_s^y| \leq s e^{2s\|\mathcal{D}\|_{\infty, s} + s\|\nabla u\|_{\infty, s}} [\mathcal{D}]_{\alpha, s} |x - y|^\alpha.$$

Proof. The first property derives from (3.8). As regards the second, from (3.8) and (5.1) we get

$$\begin{aligned} |U_s^x - U_s^y| &\leq \int_0^s \|\mathcal{D}(r)\|_{\infty} |U_r^x - U_r^y| dr + \int_0^s e^{s\|\mathcal{D}\|_{\infty, s}} |\mathcal{D}(r, X_r^x) - \mathcal{D}(r, X_r^y)| dr \\ &\leq \|\mathcal{D}\|_{\infty, s} \int_0^s |U_r^x - U_r^y| dr + s e^{s\|\mathcal{D}\|_{\infty, s} + s\|\nabla u\|_{\infty, s}} [\mathcal{D}]_{\alpha, s} |x - y|^\alpha \end{aligned}$$

and, by Gronwall's lemma, the claim follows. \square

Let $\mathcal{B}_b(\mathbf{R}^3, \mathbf{R}^3)$ be the space of all bounded Borel-measurable functions and define the operator

$$Q_s \varphi(x) = \mathbf{E}[U_s^x \varphi(X_s^x)], \quad x \in \mathbf{R}^3.$$

Lemma 5.3. *Let $s \geq 0$, then*

1. $Q_s \in \mathcal{L}(\mathcal{B}_b(\mathbf{R}^3, \mathbf{R}^3))$ and $\|Q_s\|_{\mathcal{L}(\mathcal{B}_b)} \leq e^{s\|\mathcal{D}\|_{\infty, s}}$
2. $Q_s \in \mathcal{L}(C_b^\alpha(\mathbf{R}^3, \mathbf{R}^3))$ and $\|Q_s\|_{\mathcal{L}(C_b^\alpha)} \leq e^{2s\|\mathcal{D}\|_{\infty, s} + s\|\nabla u\|_{\infty, s}} (1 + s[\mathcal{D}]_{\alpha, s})$

Moreover, if $\operatorname{div} u = 0$, then

3. $Q_s \in \mathcal{L}(L^p(\mathbf{R}^3, \mathbf{R}^3))$ and $\|Q_s\|_{\mathcal{L}(L^p)} \leq e^{s\|\mathcal{D}\|_{\infty, s}}$

Proof. First property is an obvious consequence of the previous lemma. About the second, using the two lemmas above,

$$\begin{aligned} |\mathbf{E}[U_s^x \varphi(X_s^x) - U_s^y \varphi(X_s^y)]| &\leq \mathbf{E}|U_s^x - U_s^y| \cdot |\varphi(X_s^x)| + \mathbf{E}|U_s^y| \cdot |\varphi(X_s^x) - \varphi(X_s^y)| \\ &\leq (s[\mathcal{D}]_{\alpha, s} + 1) e^{2s\|\mathcal{D}\|_{\infty, s} + s\|\nabla u\|_{\infty, s}} \|\varphi\|_{C_b^\alpha} |x - y|^\alpha. \end{aligned}$$

Finally, assume $\operatorname{div} u = 0$. Using (5.2), Hölder inequality and the previous lemma, we get

$$\int_{\mathbf{R}^3} |Q_s \varphi(x)|^p \leq e^{ps\|\mathcal{D}\|_{\infty, s}} \mathbf{E} \int_{\mathbf{R}^3} |\varphi(X_s^x)|^p \leq e^{ps\|\mathcal{D}\|_{\infty, s}} \|\varphi\|_p^p.$$

\square

5.2. Definition of the representation map. Here we prove that \mathcal{N} maps $\mathcal{U}^\alpha(T)$ in $\mathcal{V}^{\alpha, p}(T)$. Before proving such claim, we need some preliminary definitions and results. For each $u \in \mathcal{U}^\alpha(T)$, consider for all $x \in \mathbf{R}^3$ and $t \in [0, T]$ the Lagrangian paths

$$(5.3) \quad \begin{cases} dX_s^{x, t} = -u(t - s, X_s^{x, t}) ds + \sqrt{2\nu} dW_s, & s \in [0, t], \\ X_0^{x, t} = x, \end{cases}$$

and the deformation matrices

$$(5.4) \quad \begin{cases} dU_s^{x, t} = U_s^{x, t} \mathcal{D}_u(t - s, X_s^{x, t}) ds, & s \in [0, T], \\ U_0^{x, t} = I, \end{cases}$$

where $\mathcal{D}_u = \nabla u$ or $\mathcal{D}_u = \frac{1}{2}(\nabla u + \nabla u^T)$.

Lemma 5.4. *Let $u \in \mathcal{U}^\alpha(T)$ and $\psi \in C_b(\mathbf{R}^3, \mathbf{R}^3) \cap L^p(\mathbf{R}^3, \mathbf{R}^3)$. The function*

$$(s, t) \in \{0 \leq s \leq t \leq T\} \mapsto \mathbf{E}[U_s^{x,t} \psi(X_s^{x,t})] \in L^p(\mathbf{R}^3, \mathbf{R}^3) \cap C_b(\mathbf{R}^3, \mathbf{R}^3)$$

is continuous with respect to both variables.

Proof. First we show the continuity in C_b . If $0 \leq s \leq t \leq T$ and $0 \leq r \leq v \leq T$, with $t \leq v$, then for each $x \in \mathbf{R}^3$,

$$(5.5) \quad \begin{aligned} & |\mathbf{E}[U_s^{x,t} \psi(X_s^{x,t})] - \mathbf{E}[U_r^{x,v} \psi(X_r^{x,v})]| \leq \\ & \leq \mathbf{E}|U_s^{x,t} - U_s^{x,v}| \cdot |\psi(X_s^{x,t})| + \mathbf{E}|U_s^{x,v}| \cdot |\psi(X_s^{x,t}) - \psi(X_s^{x,v})| \\ & \quad + \mathbf{E}|U_s^{x,v} - U_r^{x,v}| \cdot |\psi(X_s^{x,v})| + \mathbf{E}|U_r^{x,v}| \cdot |\psi(X_s^{x,v}) - \psi(X_r^{x,v})|. \end{aligned}$$

In order to estimate the different terms of the above inequality, we see that from the equations

$$|U_s^{x,v} - U_r^{x,v}| = \left| \int_s^r U_\sigma^{x,v} \mathcal{D}_u(v - \sigma, X_\sigma^{x,v}) d\sigma \right| \leq e^{v\|\mathcal{D}_u\|_{\infty,v}} \|\mathcal{D}_u\|_{\infty,v} |r - s|,$$

and

$$(5.6) \quad |X_r^{x,v} - X_s^{x,v}| \leq \|u\|_{\infty} |s - r| + \sqrt{2\nu} |W_r - W_s|.$$

Moreover

$$\begin{aligned} |X_s^{x,t} - X_s^{x,v}| & \leq \int_0^s |u(t - \sigma, X_\sigma^{x,t}) - u(v - \sigma, X_\sigma^{x,v})| d\sigma \\ & \leq \int_0^s \|u(t - \sigma) - u(v - \sigma)\|_{\infty} + \int_0^s \|\nabla u(v - \sigma)\|_{\infty} |X_\sigma^{x,t} - X_\sigma^{x,v}| \end{aligned}$$

and, by Gronwall's lemma,

$$|X_s^{x,t} - X_s^{x,v}| \leq e^{v\|\nabla u\|_{\infty,v}} \int_0^s \|u(t - \sigma) - u(v - \sigma)\|_{\infty} d\sigma.$$

Finally,

$$\begin{aligned} |U_s^{x,t} - U_s^{x,v}| & \leq \int_0^s |U_\sigma^{x,t}| \cdot |\mathcal{D}_u(t - \sigma, X_\sigma^{x,t}) - \mathcal{D}_u(t - \sigma, X_\sigma^{x,v})| d\sigma \\ & \quad + \int_0^s |U_\sigma^{x,t}| \cdot |\mathcal{D}_u(t - \sigma, X_\sigma^{x,v}) - \mathcal{D}_u(v - \sigma, X_\sigma^{x,v})| d\sigma \\ & \quad + \int_0^s |\mathcal{D}_u(v - \sigma, X_\sigma^{x,v})| \cdot |U_\sigma^{x,t} - U_\sigma^{x,v}| d\sigma \\ & \leq e^{v\|\nabla u\|_{\infty,v}} \|\mathcal{D}_u\|_{\infty,v} \sup_{[0,v]} |X_\sigma^{x,t} - X_\sigma^{x,v}| \\ & \quad + v e^{v\|\nabla u\|_{\infty,v}} \sup_{[0,v]} \|\mathcal{D}_u(t - \sigma) - \mathcal{D}_u(v - \sigma)\|_{\infty} \\ & \quad + \|\nabla u\|_{\infty,v} \int_0^s |U_\sigma^{x,t} - U_\sigma^{x,v}| d\sigma \\ & \leq A(t, v) + C \int_0^s |U_\sigma^{x,t} - U_\sigma^{x,v}| d\sigma, \end{aligned}$$

where $\mathbf{E}A(t, v) \rightarrow 0$ as $|t - v| \rightarrow 0$, and by Gronwall's lemma,

$$|U_s^{x,t} - U_s^{x,v}| \leq A(t, v)e^{Cv}.$$

Using the above estimates in (5.5), it is easy to show continuity with values in C_b . In order to show continuity in L^p , we remark that the above estimates ensure convergence for all $x \in \mathbf{R}^3$, so that we need only to show uniform integrability. To this end, notice that, by a change of variables,

$$\begin{aligned} \int_{|x| \geq K} |\mathbf{E}[U_s^{x,t} \psi(X_s^{x,t})]|^p &\leq C \mathbf{E} \int_{X_s^{x,t}(B_K^c)} |\psi(y)|^p dy \\ &\leq C \int_{|y| \geq \frac{K}{2}} |\psi|^p \mathbf{P}[|X_s^{x,t} - x| \leq \frac{K}{2}] \\ &\quad + C \|\psi\|_p^p \mathbf{P}[|X_s^{x,t} - x| \geq \frac{K}{2}], \\ &\leq C \int_{|y| \geq \frac{K}{2}} |\psi|^p + C \|\psi\|_p^p \mathbf{P}[|X_s^{x,t} - x| \geq \frac{K}{2}], \end{aligned}$$

where $C = Te^{T\|\nabla u\|_{\infty, T}}$, and, because of (5.6), for $K \rightarrow \infty$, the above quantity converges to 0 independently of s, t . \square

Now it is possible to prove the above mentioned result on the map \mathcal{L} .

Proposition 5.5. *Given $1 \leq p < \frac{3}{2}$ and $0 < \alpha < 1$, let $\psi \in C_b^\alpha(\mathbf{R}^3, \mathbf{R}^3) \cap L^p(\mathbf{R}^3, \mathbf{R}^3)$ and $g \in \mathcal{V}^{\alpha, p}(T)$, then \mathcal{L} maps $\mathcal{U}^\alpha(T)$ in $\mathcal{V}^{\alpha, p}(T)$ and*

$$(5.7) \quad \|\mathcal{L}(u)(t)\|_{C_b^\alpha \cap L^p} \leq e^{3t\|\nabla u\|_{\infty, t}} (1 + t\|\nabla u\|_{C_b^\alpha}) (\|\psi\|_{C_b^\alpha \cap L^p} + \int_0^t \|g(s)\|_{C_b^\alpha \cap L^p} ds)$$

Proof. First, $\mathcal{L}(u) \in C_b^\alpha \cap L^p$ follows by Lemma 5.3, moreover also estimate (5.7) can be easily deduced. Finally, from the previous lemma it follows that

$$t \mapsto \mathcal{L}(u)(t) \in C_b^\alpha \cap L^p$$

is continuous. \square

5.3. Lipschitz continuity of the representation map. Let $g \in \mathcal{V}^{\alpha, p}(T)$ and $\psi \in C_b^\alpha(\mathbf{R}^3, \mathbf{R}^3) \cap L^p(\mathbf{R}^3, \mathbf{R}^3)$, and consider the map

$$\mathcal{L} : \mathcal{U}^\alpha(T) \longrightarrow \mathcal{V}^{\alpha, p}(T)$$

defined in the previous section. The aim of the present section is to show that such map is locally Lipschitz-continuous. In order to do this, we will use Girsanov formula. First we rewrite \mathcal{L} in a more appropriate form, namely

$$\mathcal{L}(u)(t, x) = \mathbf{E}[F_{t,u}(X^{x,t,u})],$$

where for each trajectory $w \in C([0, T]; \mathbf{R}^3)$,

$$F_{t,u}(w) = V_t^{t,u}(w)\psi(w_t) + \int_0^t V_s^{t,u}(w)g(t-s, w_s) ds$$

and $V^{t,u}(w)$ is the solution of the following differential equation

$$\begin{cases} \dot{V}_s^{t,u} = V_s^{t,u} \mathcal{D}_u(t-s, w_s), & s \leq t, \\ V_0^{t,u}(w) = I. \end{cases}$$

Notice that $U_s^{x,t,u}(\omega) = V_s^{t,u}(X^{x,t,u}(\omega))$, for each $\omega \in \Omega$, and we have made an explicit reference to the dependence from u in the Lagrangian paths $X^{x,t,u}$ and in the deformation matrices $U^{x,t,u}$.

By Girsanov formula, we have

$$\mathbf{E}[F_{t,u}(X^{x,t,u})] = \mathbf{E}[Z_t^{x,t,u} F_{t,u}(x + \sqrt{2\nu}W_t)],$$

where

$$Z_s^{x,t,u} = \exp\left[\frac{1}{\sqrt{2\nu}} \int_0^s \langle u(t-r, x + \sqrt{2\nu}W_r), dW_r \rangle - \frac{1}{4\nu} \int_0^s |u(t-r, x + \sqrt{2\nu}W_r)|^2 dr\right],$$

with $s \leq t$, so that for each u ,

$$\begin{aligned} \mathcal{N}(u)(t, x) &= \mathbf{E}[Z_t^{x,t,u} V_t^{t,u}(x + \sqrt{2\nu}W_t) \psi(x + \sqrt{2\nu}W_t)] + \\ &+ \int_0^t \mathbf{E}[Z_t^{x,t,u} V_s^{t,u}(x + \sqrt{2\nu}W_s) g(t-s, x + \sqrt{2\nu}W_s)] ds. \end{aligned}$$

Using this representation, we will prove the following proposition.

Proposition 5.6. *Given $1 \leq p < \frac{3}{2}$ and $0 < \alpha < 1$, let $\psi \in C_b^\alpha(\mathbf{R}^3, \mathbf{R}^3) \cap L^p(\mathbf{R}^3, \mathbf{R}^3)$ and $g \in \mathcal{V}^{\alpha,p}(T)$ and set*

$$\varepsilon_0 = \|\psi\|_{C_b^\alpha \cap L^p} + \int_0^T \|g(s)\|_{C_b^\alpha \cap L^p} ds.$$

For each $u, v \in \mathcal{U}_M^\alpha(T)$,

$$\sup_{t \leq T} \|\mathcal{N}(u) - \mathcal{N}(v)\|_{C_b^\alpha \cap L^p} \leq C(\nu, p) C_M(T) \varepsilon_0 \sup_{t \leq T} \|u - v\|_{C_b^{1,\alpha}},$$

where $C(\nu, p)$ is a constant depending only on p and ν and $\lim_{T \rightarrow 0} C_M(T) = 0$.

The proof of the above proposition will be carried on using the subsequent lemmas. In order to make the explanations easier, we introduce the following notations. We define $\Delta_{xy}f = f(x) - f(y)$ for any function f . Notice that

$$(5.8) \quad \Delta_{xy}(fg) = (\Delta_{xy}f)g(x) + f(y)(\Delta_{xy}g).$$

If the functions depends on two variables, we define Δ_{uvxy} as $\Delta_{uv}\Delta_{xy}$ and, by applying twice the above formula,

$$\begin{aligned} \Delta_{uvxy}(fg) &= \Delta_{uv}[(\Delta_{xy}f)g(\cdot, x) + f(\cdot, y)(\Delta_{xy}g)] \\ (5.9) \quad &= (\Delta_{uvxy}f)g(u, x) + [\Delta_{xy}f(v)][\Delta_{uv}g(x)] \\ &+ [\Delta_{uv}f(y)][\Delta_{xy}g(u)] + f(v, y)(\Delta_{uvxy}g). \end{aligned}$$

Lemma 5.7. *Let $u, v \in \mathcal{U}_M^\alpha(T)$, then for each $w, w' \in C([0, T]; \mathbf{R}^3)$ and for all $s \leq t$,*

$$\begin{aligned} |V_s^{t,u}(w)| &\leq e^{tM}, \\ |\Delta_{uv} V_s^{t,\cdot}(w)| &\leq t e^{2tM} \|u - v\|_{C_b^1}, \\ |\Delta_{ww'} V_s^{t,u}(\cdot)| &\leq 2M t e^{2tM} \|w - w'\|_\infty^\alpha, \\ |\Delta_{uvww'} V_s^{t,\cdot}(\cdot)| &\leq (1 + 3tM) t e^{3tM} \|w - w'\|_\infty^\alpha \|u - v\|_{C_b^{1,\alpha}}. \end{aligned}$$

Proof. The proofs of these properties are similar, we just give the proof of the last one. Indeed, using formula (5.9),

$$\begin{aligned} \frac{d}{ds}(\Delta_{uvww'} V_s^{t,\cdot}(\cdot)) &= \Delta_{uvww'} \left(\frac{d}{ds} V_s^{t,\cdot}(\cdot) \right) = \Delta_{uvww'} (V_s^{t,\cdot}(\cdot) \mathcal{D}_\cdot(t-s, \cdot)) \\ &= [\Delta_{uvww'} V_s^{t,\cdot}(\cdot)] \mathcal{D}_u(t-s, w_s) + V_s^{t,v}(w') [\Delta_{ww'} \mathcal{D}_{u-v}(t-s, \cdot)] \\ &\quad + [\Delta_{ww'} V_s^{t,v}(\cdot)] \mathcal{D}_{u-v}(t-s, w_s) + [\Delta_{uv} V_s^{t,\cdot}(w')] [\Delta_{ww'} \mathcal{D}_u(t-s, \cdot)] \end{aligned}$$

so that, by using the other inequalities of this lemma,

$$\begin{aligned} |\Delta_{uvww'} V_s^{t,\cdot}(\cdot)| &\leq M \int_0^s |\Delta_{uvww'} V_r^{t,\cdot}(\cdot)| dr + \|w - w'\|_\infty^\alpha \|u - v\|_{C_b^{1,\alpha}} \int_0^s |V_r^{t,v}(w')| dr \\ &\quad + \|u - v\|_{C_b^{1,\alpha}} \int_0^s |\Delta_{ww'} V_r^{t,v}(\cdot)| dr + M \|w - w'\|_\infty^\alpha \int_0^s |\Delta_{uv} V_r^{t,\cdot}(w')| dr \\ &\leq M \int_0^s |\Delta_{uvww'} V_r^{t,\cdot}(\cdot)| dr + (1 + 3tM) s e^{2tM} \|w - w'\|_\infty^\alpha \|u - v\|_{C_b^{1,\alpha}} \end{aligned}$$

and, by the Gronwall's lemma, the inequality follows. \square

Using the previous lemma and formulas (5.8) and (5.9), we can easily deduce similar properties for the functional F .

Lemma 5.8. *Let $u, v \in \mathcal{U}_M^\alpha(T)$, then for each $w, w' \in C([0, T]; \mathbf{R}^3)$, and for all $t \in [0, T]$,*

$$\begin{aligned} |F_{t,u}(w)| &\leq e^{tM} [|\psi(w_t)| + \int_0^t |g(t-s, w_s)| ds] \\ |\Delta_{uv} F_{t,\cdot}(w)| &\leq t e^{2tM} \|u - v\|_{C_b^1} [|\psi(w_t)| + \int_0^t |g(t-s, w_s)| ds] \\ |\Delta_{ww'} F_{t,u}(\cdot)| &\leq (1 + 2tM) e^{2tM} \varepsilon_0 \|w - w'\|_\infty^\alpha \\ |\Delta_{uvww'} F_{t,\cdot}(\cdot)| &\leq (2 + 3tM) t e^{3tM} \varepsilon_0 \|w - w'\|_\infty^\alpha \|u - v\|_{C_b^{1,\alpha}}, \end{aligned}$$

where $\varepsilon_0 = \|\psi\|_{C_b^\alpha} + \int_0^t \|g(s)\|_{C_b^\alpha} ds$.

Finally, we estimate the same quantities on the process Z .

Lemma 5.9. *Let $u, v \in \mathcal{U}_M^\alpha(T)$ and $q \geq 2$. Then for all $s \leq t$,*

$$\begin{aligned} \mathbf{E} |Z_s^{x,t,u}|^q &\leq C e^{Ct^{q/2} M^q}, \\ \mathbf{E} |\Delta_{uv} Z_s^{x,t,\cdot}|^q &\leq C t^{q/2} e^{CM^q t^{q/2}} \|u - v\|_{C_b}^q, \\ \mathbf{E} |\Delta_{xy} Z_s^{\cdot,t,u}|^q &\leq C t^{q/2} M^q e^{CM^q t^{q/2}} |x - y|^{\alpha q}, \\ \mathbf{E} |\Delta_{uvxy} Z_s^{\cdot,t,\cdot}|^q &\leq C t^{3q/2} M^{2q} e^{CM^q t^{q/2}} |x - y|^{\alpha q} \|u - v\|_{C_b^{1,\alpha}}^q, \end{aligned}$$

where $C = C(q, \nu)$ is a constant depending only on q and ν .

Proof. From the definition, we see that $Z_s^{x,t,u}$ solves

$$\begin{cases} dZ_s^{x,t,u} = \frac{1}{\sqrt{2\nu}} Z_s^{x,t,u} u(t-s, x + \sqrt{2\nu} W_s) dW_s, & s \leq t, \\ Z_0^{x,t,u} = 1. \end{cases}$$

Again, the proofs of the four inequalities are similar, we prove only the last one. By applying formula (5.9), we get

$$\begin{aligned} d_s(\Delta_{uvxy} Z_s^{t,\cdot}) &= \Delta_{uvxy}(d_s Z_s^{t,\cdot}) = \\ &= \frac{1}{\sqrt{2\nu}} \left[(\Delta_{uvxy} Z_s^{t,\cdot}) u(t-s, Y_s^x) dW_s + Z_s^{y,t,v} \Delta_{xy}[u(t-s, Y_s^x) - v(t-s, Y_s^x)] dW_s \right. \\ &\quad \left. + (\Delta_{xy} Z_s^{t,v}) [u(t-s, Y_s^x) - v(t-s, Y_s^x)] dW_s + (\Delta_{uv} Z_s^{y,t,\cdot}) [\Delta_{xy} u(t-s, Y_s^x)] dW_s \right], \end{aligned}$$

where, for the sake of brevity, we have set $Y_s^x = x + \sqrt{2\nu} W_s$. By the Burkholder, Davis and Gundy inequality,

$$\begin{aligned} \mathbf{E} |\Delta_{uvxy} Z_s^{t,\cdot}|^q &\leq C \left[M^q \mathbf{E} \left[\int_0^s |\Delta_{uvxy} Z_r^{t,\cdot}|^2 dr \right]^{\frac{q}{2}} + \|u-v\|_{C_b^1}^q |x-y|^{\alpha q} \mathbf{E} \left[\int_0^s |Z_r^{y,t,v}|^2 dr \right]^{\frac{q}{2}} \right. \\ &\quad \left. + \|u-v\|_{C_b^1}^q \mathbf{E} \left[\int_0^s |\Delta_{xy} Z_r^{t,v}|^2 dr \right]^{\frac{q}{2}} + M^q |x-y|^{\alpha q} \mathbf{E} \left[\int_0^s |\Delta_{uv} Z_r^{y,t,\cdot}|^2 dr \right]^{\frac{q}{2}} \right], \end{aligned}$$

so that, by using the Hölder inequality and the other inequalities of this lemma, we get

$$\mathbf{E} |\Delta_{uvxy} Z_s^{t,\cdot}|^q \leq CM^q s^{\frac{q}{2}-1} \int_0^s \mathbf{E} |\Delta_{uvxy} Z_r^{t,\cdot}|^q dr + CM^{2q} t^q s^{q/2} e^{CM^q t^{q/2}} |x-y|^{\alpha q} \|u-v\|_{C_b^1}^q.$$

Finally, using the Gronwall's lemma, we obtain the required inequality. \square

We are now able to prove the main result of this section.

Proof of Proposition 5.6. Let $u, v \in \mathcal{U}_M^\alpha(T)$. We start with the estimates in C_b and L^p . Using formula (5.8) and Hölder inequality, we get, for each $x \in \mathbf{R}^3$ and $t > 0$,

$$\begin{aligned} (5.10) \quad |[\Delta_{uv} \mathcal{N}(\cdot)](t, x)| &= |\mathbf{E}[\Delta_{uv}(Z_t^{x,t,\cdot} F_{t,\cdot}(Y^x))]| \\ &\leq C(q) \left[(\mathbf{E} |\Delta_{uv} Z_t^{x,t,\cdot}|^{q'})^{1/q'} (\mathbf{E} |F_{t,u}(Y^x)|^q)^{1/q} \right. \\ &\quad \left. + (\mathbf{E} |Z_t^{x,t,v}|^{q'})^{1/q'} (\mathbf{E} |\Delta_{uv} F_{t,\cdot}(Y^x)|^q)^{1/q} \right] \end{aligned}$$

where $q \geq 1$, q' is the Hölder conjugate exponent of q and we have set $Y_s^x = x + \sqrt{2\nu} W_s$. Using the estimates in Lemma 5.8 and in Lemma 5.9, and the inequality above with $q = 2$, we obtain the estimate in the C_b norm,

$$\sup_{t \leq T} \|\Delta_{uv} \mathcal{N}(\cdot)\|_{C_b} \leq C \varepsilon_0 (T + \sqrt{T}) e^{(CM^2 + 2M)T} \|u-v\|_{C_b^{1,\alpha}}.$$

Using again Lemma 5.8 and 5.9 and the inequality (5.10) above, with $q = p$, we can obtain the estimate in the L^p norm,

$$\sup_{t \leq T} \|\Delta_{uv} \mathcal{N}(\cdot)\|_{L^p}^p \leq C \varepsilon_0^p (T^p + T^{p/2}) e^{2TM + CM^{p'} t^{p'/2}} \|u-v\|_{C_b^{1,\alpha}}^p.$$

To conclude the proof, we need the estimate in the C_b^α norm. For all $x, y \in \mathbf{R}^3$ and $t > 0$, by applying formula (5.9) we get

$$\begin{aligned} |\Delta_{uvxy}\mathcal{L}(\cdot)(t, \cdot)| &\leq \mathbf{E}[\Delta_{uvxy}(Z^{\cdot,t}, F_{t,\cdot}(Y^\cdot))] \\ &\leq \mathbf{E}\left[(\Delta_{uvxy}Z_t^{\cdot,t})F_{t,u}(Y^x) + Z_t^{y,t,v}[\Delta_{uvxy}F_{t,\cdot}(Y^\cdot)] \right. \\ &\quad \left. + (\Delta_{uv}Z_t^{y,t,\cdot})[\Delta_{xy}F_{t,u}(Y^\cdot)] + (\Delta_{xy}Z_t^{\cdot,t,v})[\Delta_{uv}F_{t,\cdot}(Y^x)]\right] \end{aligned}$$

Using the inequalities in Lemma 5.8 and Lemma 5.9, it follows that

$$|\Delta_{uvxy}\mathcal{L}(\cdot)(t, \cdot)| \leq C\varepsilon_0(\sqrt{T} + T + MT^{3/2} + MT^2)e^{3TM+CTM^2}|x - y|^\alpha\|u - v\|_{C_b^{1,\alpha}}.$$

□

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