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ON ESTIMATION THEORY FOR MULTIPLICATIVE CASCADES

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SUMMARY. The notion of multiplicative cascade was introduced into the statistical theory of turbulence by A.N. Kolmogorov as a phenomenological framework intended to accommodate the intermittency and large fluctuations observed in turbulent fluid flows. The basic idea is that energy is redistributed from larger to smaller scales via a splitting mechanism involving random multiplicative factors known as *cascade generators*. Primarily owing to the scaling structure of this class of models, applications have been extended to a wide variety of other naturally occurring phenomena such as rainfall, internet packet traffic, market prices, etc. which exhibit intermittent and highly variable behavior in space and time. The probability distribution of the cascade generators represents a hidden parameter which is reflected in the fine scale limiting behavior of certain scaling exponents calculated from a single sample realization. In this paper we describe the underlying statistical theory for estimation of the distribution of the generators, discuss examples, and provide a number of open problems in the general theory. Some new results involving estimation of an important intermittency parameter, the Hausdorff dimension of the support set, are also included. We then proceed to identify an outstanding open statistical problem from turbulence data.

1. Introduction

Following L.F. Richardson (1922), the notion of multiplicative cascade was developed in the statistical theory of turbulence by A.N. Kolmogorov (1941, 1962) as a phenomenological framework intended to accommodate the intermittency and large fluctuations observed in flows. The basic idea

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as applied to turbulent fluids is that under large scale stirring motions, for example, energy will be redistributed randomly to smaller scales by the splitting off of eddies. This process naturally defines a random measure representing the amounts of energy occupied by various subregions in the limit of repeated splittings. The rich geometric structure intrinsic to such random measures was the focus of early papers by Yaglom (1966), Mandelbrot (1974), and Kahane and Peyrière (1976). In his phenomenology Kolmogorov (1962) also assumed a lognormal distribution for the redistribution factors, called *cascade generators*. However the probability distribution of the cascade generators represents a hidden element which is reflected in the fine scale limiting behavior of sample moments. Data analysis and physical theory subsequent to Kolmogorov's lognormal hypothesis have led to interesting statistical questions which will be addressed in this paper.

The scaling structure of random cascade models has encouraged their use as models for a wide variety of other natural phenomena such as rainfall, internet packet traffic, market prices, etc. which exhibit intermittency and high variability in space and time; e.g. see Gupta and Waymire (1993), Resnick, Samorodnitsky, Gilbert and Willinger (2002), and Mandelbrot (1998), respectively. In this paper however, we focus on describing an outstanding statistical problem specific to turbulence theory.

Certain important scaling and dimension exponents of the cascade measure can be estimated from a single sample realization by exploitation of a.s. fine scale convergence. In the next section we give a precise definition of the cascade measure and somewhat terse outline of the basic elements of current convergence theory and related statistical developments.

The outline of the statistical theory will be followed by some illustrative examples and some new theoretical results on the statistics of the fine scale structure of multiplicative cascades. While various open problems for the statistical theory are identified along the way, a main focus is the application to turbulence theory elaborated upon in a final section of this paper.

2. Theoretical Foundations: An Overview and Some Open Problems

In this section we provide precise statements of the basic results underlying current statistical theory. Proofs of results given in this section may be found in Ossiander and Waymire (2000).

Let $b \geq 2$ be a natural number and let $\mathbf{T}$ denote the product space

$$\mathbf{T} = \{0, 1, 2, \ldots, b-1\}^{\mathbf{N}} \tag{1}$$

equipped with the metric $\rho(s,t) = b^{-|s\wedge t|}, s,t \in \mathbf{T}$, where $\mathbf{N}$ denotes the set of natural numbers and $|s\wedge t| = \inf\{n \geq 0 : s_{n+1} \neq t_{n+1}\}, s = (s_1, s_2, \ldots), t = (t_1, t_2, \ldots) \in \mathbf{T}$. Denote the corresponding Borel sigmafield on $\mathbf{T}$ by $\mathcal{B}(\mathbf{T})$. For $t = (t_1, t_2, \ldots) \in \mathbf{T}$ let $t|n = (t_1, t_2, \ldots, t_n)$. If points $t \in \mathbf{T}$ are viewed as paths through a b-ary tree then $v = t|n$ denotes the nth generation vertex along t and we write $|v| = n$.

For $s \in \mathbf{T}, n \in \mathbf{N}$, denote the closed ball of radius $r = b^{-n}$ centered at s by

$$\Delta_n(s) \equiv \Delta_n(s|n) = B_{b^{-n}}(s) = \{t \in \mathbf{T} : t_i = s_i, i \leq n\}. \tag{2}$$

The normalized Haar measure λ on $\mathbf{T}$, viewed as a countable product of cyclic groups of order b, is specified by

$$\lambda(\Delta_n(s)) = b^{-n}, s \in \mathbf{T}, n \geq 1. \tag{3}$$

The *cascade generators* are given by a denumerable family of i.i.d. non-negative mean one random variables $\{W_v : v \in \{0, 1, \ldots, b-1\}^n, n \geq 1\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{F}_n, n \geq 1$, denote the filtration defined by

$$\mathcal{F}_n = \sigma\{W_v : |v| \leq n\}, n \geq 1. \tag{4}$$

The cascade generators define a sequence of random measures $\{\lambda_n : n \geq 1\}$ on $(\mathbf{T}, \mathcal{B}(\mathbf{T}))$ for $n \geq 1$ via

$$\frac{d\lambda_n}{d\lambda}(t) = Q_n(t) = \prod_{i=0}^{n} W_{t|i} = W_\emptyset \prod_{i=1}^{n} W_{t|i}, t \in \mathbf{T}, \tag{5}$$

where $W_\emptyset$, referred to as the *cascade initiator,* is an a.s. positive random variable independent of $\mathcal{F}_n, n \geq 1$.

One may easily check that for any bounded Borel measurable function $f : \mathbf{T} \to \mathbf{R}$, the sequence of random variables $\{\int_{\mathbf{T}} f d\lambda_n\}_{n=1}^{\infty}$ is an L_1-bounded martingale with respect to $\mathcal{F}_n$, and thus has an a.s. limit as $n \to \infty$. This leads to a random measure λ_∞ on $(\mathbf{T}, \mathcal{B}(\mathbf{T}))$ such that

$$P(\lambda_n \Rightarrow \lambda_\infty \, as \, n \to \infty) = 1, \tag{6}$$

where $\Rightarrow$ denotes vague convergence; e.g. see Kahane and Peyrière (1976). Indeed, for any countable family Φ of bounded Borel measurable functions, cf. Kahane (1989),

$$P\left(\lim_{n\to\infty} \int_{\mathbf{T}} f(t)\lambda_n(dt) = \int_{\mathbf{T}} f(t)\lambda_\infty(dt), f \in \Phi\right) = 1. \tag{7}$$

The random measure λ_∞ is known as the *multiplicative cascade measure.* The following basic structure theorem for λ_∞ is also well-known; see Kahane and Peyrière (1976). First let

$$\chi_b(h) = \log_b \mathbf{E}[W^h \mathbf{1}[W > 0]] - (h-1), \tag{8}$$

where W is a generic cascade generator distributed as W_v for $v \neq \emptyset$. The structure function $\chi_b(h)$ is defined for all real numbers h but may be infinite, with the conventions that $0^0 = 0, 0 \cdot \infty = 0$. Notice that χ_b is a modified version of the cumulant generating function of $\ln W$. The use of the indicator function $\mathbf{1}[W > 0]$ allows incorporation of the case $h < 0$ into the general theory.

THEOREM 2.1 *(Kahane and Peyrière (1976))*
(i.) (Nondegeneracy) $\mathbf{E}\lambda_\infty(\mathbf{T}) > 0$ *iff* $\chi_b'(1-) < 0$.
(ii.) (Convergence of moments) $\mathbf{E}\lambda_\infty^h(\mathbf{T}) < \infty$ *for* $0 \leq h \leq 1$, *and, if* $h_c := \sup\{h \geq 1 : \chi_b(h) \leq 0\} > 1$, *then* $\mathbf{E}\lambda_\infty^h(\mathbf{T}) < \infty$ *for* $1 < h < h_c$.
(iii.) (Support dimension) If $\lambda_\infty(\mathbf{T}) > 0$, *then* λ_∞ *is a.s. supported on a sub-set of* $\mathbf{T}$ *with Hausdorff dimension* $-\chi_b'(1-)$.

Notice that the last part of Theorem 2.1 delineates an important intermittency parameter associated with the cascade measure λ_∞, $-\chi_b'(1-)$, the a.s. Hausdorff dimension of the sub-set of $\mathbf{T}$ that supports λ_∞; see Waymire and Williams (1995) for a proof in this generality. In Section 4 we define two estimators of this parameter, one in use by physicists, and one new, and show that they both converge to $-\chi_b'(1-)$ with probability 1.

PROBLEM 2.1: (Dependent Generators) Theorem 2.1 has been extended to a large class of statistically dependent cascade generators in Waymire and Williams (1994, 1996, 1997). They consider classes of generators such that (i) for each fixed $t \in \mathbf{T}$, $\{W_{t|n} : n = 0, 1, 2, ...\}$, are identically distributed non-negative random variables with $\mathbf{E}[W_{t|n+1}|\mathcal{F}_n] = 1$ and (ii) for $s, t \in \mathbf{T}$ fixed and $m, n > |s \wedge t|$, $W_{s|m}$ and $W_{t|n}$ are conditionally independent given $\mathcal{F}_{|s\wedge t|}$. However the statistical theory presented below remains to be extended to this generality. Another class of dependent generators is that of i.i.d. non-negative *vector generators*, $\mathbf{W}_v = (W_{<v,0>}, W_{<v,1>}, \ldots, W_{<v,b-1>})$, $v \in \cup_{n=0}^\infty \{0, 1, \ldots, b-1\}^n$, where $\mathbf{E}[\frac{1}{b}\sum_{k=0}^{b-1} W_{<v,k>}] = 1$. In general, Theorem 2.1 holds for i.i.d vector generators with $\chi_b(h)$ replaced by $\tilde{\chi}_b(h) = \log_b \mathbf{E}[\tilde{W}^h \mathbf{1}[\tilde{W} > 0]] - (h-1)$, where $\tilde{W} = W_k$ with probability $\frac{1}{b}$, for $k \in \{0, 1, \ldots, b-1\}$. An important family of special cases here are the *conservative cascades* defined by the almost sure equality $\frac{1}{b}\sum_{k=0}^{b-1} W_{<v,k>} = 1$. In this latter case the non-triviality problem (Theorem 2.1 (i)) is a non-issue, as $\lambda_n(\mathbf{T}) = \lambda_\infty(\mathbf{T})$ for all n. Some analysis issues for conservative cascades are treated in Gilbert, Willinger and Feldman (1999) and further statistical theory for symmetric conservative cascades with binary branching is developed using wavelet methods in Resnick et al. (2002), but the extension of the statistical theory presented below to more general settings remains to be completed.

In most applications the probability distribution of the cascade generators is not known apriori. However, one may have data in the form of a single sample realization of the cascade measure of pixels at some prescribed fine scale of resolution. The following result shows that, asymptotically, such data consistently determines the distribution of the cascade generators.

Theorem 2.2 *(Ossiander and Waymire (2000)) Assume that* $\chi_b'(1-) < 0$. *If* $\mathbf{E}[W^h \mathbf{1}[W > 0]]$ *exists and is finite for* h *belonging to some neighborhood of* 0, *then* $\{\lambda_\infty(\Delta_n(v)) : v \in \{0, 1, \ldots, b-1\}^n, n \geq 0\}$ *uniquely determines the distribution of the cascade generator* W.

Problem 2.2 (Consistency) If $\ln W \mathbf{1}[W > 0]$ does not have a finite moment generating function in a neighborhood of the origin, then $\chi_b(h)$ may not uniquely determine the distribution of W. In this case the problem of consistently determining the distribution of the generators remains open.

Throughout the remainder of the paper we will restrict our consideration to cascade generators for which

$$\chi_b'(1-) < 0, \tag{9}$$

so that $\mathbf{E}\lambda_\infty(\mathbf{T}) > 0$; cf Theorem **??**. A basic family of statistics which we consider are the *nth-scale sample moments* defined by

$$M_n(h) = \sum_{|v|=n} \lambda_\infty^h(\Delta_n(v)), h \in \mathbf{R}. \tag{10}$$

These are a natural family of functionals which are computable from observing the multiplicative cascade on pixels, or balls, $\Delta_n(v)$. Two different estimators of $\chi_b(h)$, $\widehat{\tau}_n(h)$ and $\widetilde{\tau}_n(h)$, can be defined in terms of the $M_n(h)$'s as follows:

$$\widehat{\tau}_n(h) = n^{-1} \log_b M_n(h) \tag{11}$$

and

$$\widetilde{\tau}_n(h) = \log_b(M_{n+1}(h)/M_n(h)). \tag{12}$$

It is convenient to introduce a class of random measures $\lambda_\infty(h;dt), h \in \mathbf{R}$, which we refer to as *h-cascades,* and define via the *h-cascade generators*

$$W_v(h) = \frac{W_v^h}{\mathbf{E}W_v^h}, h \in \mathbf{R}. \tag{13}$$

The h-cascades are related to the original cascades via

$$\frac{\lambda_n^h(\Delta_n(v))}{b^{n\chi_b(h)}} = \lambda_n(h;\Delta_n(v)), \tag{14}$$

where the sequence of *nth level h-cascades,* $n = 1, 2, \ldots$ is defined by

$$\frac{d\lambda_n(h;\cdot)}{d\lambda}(t) \equiv Q_n(h;t) = \prod_{i=0}^{n} W_{t|i}(h), t \in \mathbf{T}. \tag{15}$$

The following proposition points to the usefulness of h-cascades in this context.

Proposition 2.1 *For* $h \in \mathbf{R}, n \geq 1,$ *one has*

$$\frac{M_n(h)}{b^{n\chi_b(h)}} = \sum_{|v|=n} Z_\infty^h(v)\lambda_n(h;\Delta_n(v)) = \int_{\mathbf{T}} Z_\infty^h(t|n)\lambda_n(h;dt)$$

where a.s.

$$Z_\infty(v) = \lim_{N\to\infty} \sum_{|u|=N-n} \prod_{i=1}^{N-n} W_{v*(u_1\ldots u_i)} b^{-(N-n)},$$

and * *denotes the concatenation*

$$(v_1, \ldots, v_n) * (u_1, \ldots, u_N) = (v_1, \ldots, v_n, u_1, \ldots, u_N).$$

Notice that this proposition gives an implicit decomposition of the limiting cascade measures. This decomposition is crucial in reaching an understanding of the properties of λ_∞ and functionals thereof.

The next proposition results from an application of Theorem 2.1 to describe the limiting behavior of the h-cascades. The structure function of the $W_v(h)$'s is given by

$$\chi_{b,h}(r) = \chi_b(hr) - r\chi_b(h).$$

This yields

$$\chi'_{b,h}(1) = h\chi'_b(h) - \chi_b(h).$$

It is easy to check that, as a function of h, $\chi'_{b,h}(1)$ is convex. In light of Theorem 2.1, this gives the following.

Proposition 2.2 *Assume that* $\chi_b'(1-) < 0$ *and let*

$$H_c^+ = \sup\{h \geq 1 : h\chi_b'(h) - \chi_b(h) < 0\}$$

and

$$H_c^- = \inf\{h \leq 0 : h\chi_b'(h) - \chi_b(h) < 0\}$$

Then $H_c^- \leq 0 < 1 \leq H_c^+$, *with* $h\chi_b'(h) - \chi_b(h) < 0$ *for all* $H_c^- < h < H_c^+$. *Furthermore, for* $h \in [0,1] \cup (H_c^-, H_c^+)$, $\lambda_n(h;\mathbf{T}) \to \lambda_\infty(h;\mathbf{T})$ $P-$*a.s., where* $\mathbf{E}\lambda_\infty(h;\mathbf{T}) = 1$ *and for* $h \in (H_c^+, h_c)$, $\lambda_n(h;\mathbf{T}) \to 0$ $P-$*a.s. as* $n \to \infty$.

A discussion of the structure and relationship of the support sets in $\mathbf{T}$ of the limiting h-cascades can be found in Ossiander (2000).

The following theorem is the basis of our derivation of the a.s. convergence of both $\widehat{\tau}_n(h)$ and $\widetilde{\tau}_n(h)$. A generalization of the first part of this theorem is given in Section 4, where it is instrumental in verifying convergence of estimators of the Hausdorff dimension of the supporting set of λ_∞.

Theorem 2.3 *(Ossiander and Waymire (2000))*
(i.) For $h \in [0,1] \cup (H_c^-, H_c^+)$,

$$\frac{M_n(h)}{b^{n\chi_b(h)}} \to \lambda_\infty(h,\mathbf{T})\mathbf{E}\lambda_\infty^h(\mathbf{T})$$

$P-$*a.s. as* $n \to \infty$.
(ii.) For $h \in (H_c^+, h_c)$,

$$\frac{M_n(h)}{b^{n\chi_b(h)}} \to 0$$

$P-$*a.s. as* $n \to \infty$.

Convergence of the estimators $\widehat{\tau}_n(h)$ and $\widetilde{\tau}_n(h)$ to the structure function $\chi_b(h)$ for h inside the critical interval (H_c^-, H_c^+) is delineated in Corollaries 2.1 and 2.2 and Corollary 2.3 respectively.

Corollary 2.1 *For any* $h \in [0,1] \cup (H_c^-, H_c^+)$, *the following hold* $P-$*a.s. as* $n \to \infty$ *on the set* $[\lambda_\infty(\mathbf{T}) > 0]$*:*
(i.) $(\log_b M_n(h) - n\chi_b(h)) \to \log_b \lambda_\infty(h,\mathbf{T}) + \log_b \mathbf{E}\lambda_\infty^h(\mathbf{T})$
and
(ii.) $\widehat{\tau}_n(h) \to \chi_b(h)$.

Corollary 2.2 *On the set* $[\lambda_\infty(\mathbf{T}) > 0]$,

$$\{\widehat{\tau}_n(h) : h \in [0,1] \cup (H_c^-, H_c^+)\} \to \{\chi_b(h) : h \in [0,1] \cup (H_c^-, H_c^+)\}$$

$P-$*a.s. as* $n \to \infty$.

COROLLARY 2.3 *On the set* $[\lambda_\infty(\mathbf{T}) > 0]$ *one has* $P-a.s.$ *that*

$$\{\widetilde{\tau}_n(h) : h \in [0,1] \cup (H_c^-, H_c^+)\} \to \{\chi_b(h) : h \in [0,1] \cup (H_c^-, H_c^+)\}$$

as $n \to \infty$.

Although both $\widehat{\tau}_n(h)$ and $\widetilde{\tau}_n(h)$ converge to $\chi_b(h)$ for h within the critical regime, in practice $\widetilde{\tau}_n(h)$ is a more useful estimate of $\chi_b(h)$ for moderate values of n. This can be seen from the following heuristic calculations motivated by (i.) of Corollary 2.1:

$$\begin{aligned} \widehat{\tau}_n(h) - \chi_b(h) \quad &= n^{-1}\log_b M_n(h) - \chi_b(h) \\ &\approx n^{-1}(\log_b \lambda_\infty(h,\mathbf{T}) + \log_b \mathbf{E}\lambda_\infty^h(\mathbf{T})). \end{aligned}$$

The term $n^{-1}(\log_b \lambda_\infty(h,\mathbf{T}) + \log_b \mathbf{E}\lambda_\infty^h(\mathbf{T}))$ can be thought of as an asymptotically negligible bias term. On the other hand, $\widetilde{\tau}_n(h)$ is given by a weighted differencing of $\widehat{\tau}_n(h)$ in a way that decreases the bias term:

$$\widetilde{\tau}_n(h) - \chi_b(h) = (n+1)\widehat{\tau}_{n+1}(h) - n\widehat{\tau}_n(h) \approx 0. \tag{16}$$

The following theorem reveals that the limiting behavior of $\widehat{\tau}_n(h) = n^{-1}\log_b M_n(h)$, viewed as a function of h, is different outside the set $[0,1] \cup (H_c^-, H_c^+)$; i.e. when the low frequency h-cascade $\lambda_n(h,\mathbf{T})$ dies out a.s. with respect to P. It also makes clear that for estimation purposes the interval (H_c^-, H_c^+) is a true critical interval. Indeed, it shows that for h outside this interval $\widehat{\tau}_n(h)$ estimates a linear extension of $\chi_b(h)$ rather than $\chi_b(h)$ itself. Weaker versions of this result appear in Lovejoy and Schertzer (1991), Holley and Waymire (1992), Collet and Koukiou (1992), Franchi (1995), and Molchan (1996).

THEOREM 2.4 *(Ossiander and Waymire (2000)) Let*

$$\overline{\chi}_b(h) = \begin{cases} h\chi_b'(H_c^-) & \text{if } h \le H_c^- < 0 \\ \chi_b(h) & \text{if } h \in (H_c^-, H_c^+) \cup [0,1] \\ h\chi_b'(H_c^+) & \text{if } h \ge H_c^+. \end{cases}$$

If $H_c^+ < h_c$ *and* $H_c^- < 0$, *with* $\mathbf{E}W^h\mathbf{1}[W > 0] < \infty$ *for some* $h < H_c^-$, *then on* $A = [\lambda_\infty(\mathbf{T}) > 0]$,

$$\{\widehat{\tau}_n(h) : h \in \mathbf{R}\} \to \{\overline{\chi}_b(h) : h \in \mathbf{R}\}$$

$P-a.s.$ *as* $n \to \infty$. *If* $H_c^- = 0$ *and* $H_c^+ < h_c$, *then on* A,

$$\{\widehat{\tau}_n(h) : h \ge 0\} \to \{\overline{\chi}_b(h) : h \ge 0\}$$

$P-a.s.$ *as* $n \to \infty$.

PROBLEM 2.3. Simulations given in Ossiander and Waymire (2000) strongly suggest that the estimator $\widetilde{\tau}_n(h) = \log_b(M_{n+1}(h)/M_n(h))$ converges to a value that equals neither $\chi_b(h)$ nor $\overline{\chi}_b(h)$ for $h > H_c^+$ and $h < H_c^-$. On the other hand, using the representation of $\widetilde{\tau}_n(h)$ given in (**??**), summing over n, and averaging over h, we see that if $\widetilde{\tau}_n(h)$ converges a.s. it must converge to $\overline{\chi}_b(h)$. This leaves the identification of the limiting behavior of $\widetilde{\tau}_n(h)$ open for h outside the critical region $[H_c^-, H_c^+]$.

PROBLEM 2.4. (Transform Inversion/Density Estimation) The estimators given here provide estimators of moment generating functions of the distribution of $\log W$. No satisfactory inversion theory has been developed, even for the case in which there is an analytic continuation to an integrable Fourier transform.

Martingale central limit theory may be exploited to obtain asymptotic error distributions for the estimators $\widehat{\tau}_n(h)$ and $\widetilde{\tau}_n(h)$ for h within the scaled critical interval $(H_c^-/2, H_c^+/2)$. The central limit theorem for the estimator $\widetilde{\tau}_n(h)$ is given in Corollary 2.5. The key result for these error distributions may be stated as follows.

For each $n \geq 1$, let $\{X_n(v) : |v| = n\}$ be a collection of independent random variables which are also independent of $\mathcal{F}_n$. Define

$$S_n(h) = \sum_{|v|=n} X_n(v)\lambda_n(h; \Delta_n(v)). \tag{17}$$

Also let

$$R_n(h) = \frac{S_n(h)}{(\sum_{|v|=n} \lambda_n^2(h; \Delta_n(v)))^{\frac{1}{2}}}. \tag{18}$$

THEOREM 2.5 *(Ossiander and Waymire (2000))*
If $\mathbf{E}X_n^2(v) = 1$ *and* $\mathbf{E}X_n(v) = 0$ *for each* v, *and if*

$$\sup_n \sup_{|v|=n} \mathbf{E}|X_n(v)|^{2(1+\delta)} < \infty$$

for some $\delta > 0$, *then for* $h \in (H_c^-/2, H_c^+/2)$,

$$\lim_{n\to\infty} \mathbf{E}[e^{izR_n(h)}|\mathcal{F}_n] = \mathbf{1}[\lambda_\infty(\mathbf{T}) = 0] + e^{-\frac{1}{2}z^2}\mathbf{1}[\lambda_\infty(\mathbf{T}) > 0], \tag{19}$$

with the convention that $R_n(h) = 0$ *if* $\lambda_n(\mathbf{T}) = 0$.

COROLLARY 2.4 *For* $h \in (H_c^-/2, H_c^+/2)$,

$$R_n(h) \to^d \eta N_h$$

where $\eta = \mathbf{1}[\lambda_\infty(\mathbf{T}) = 0]$ *and* N_h *is an independent standard normal random variable.*

Note: it can also be shown that the N_h's are independent for different values of h.

The estimator $\widetilde{\tau}_n(h)$ of $\chi_b(h)$ is obtained by differencing the logarithms of the $h-$th sample moments at scales of resolution $n+1$ and n; namely $\widetilde{\tau}_n(h) = \log_b(M_{n+1}(h)/M_n(h))$. In view of Corollary 2.3 we have asymptotic consistency of this estimator for $h \in (H_c^-, H_c^+)$. The following gives an observable normalization of this estimator, which allows computation of asymptotically exact confidence intervals for $\chi_b(h)$ for observation of a single realization of the random cascade.

Define

$$V_n^2(h) = \sum_{|v|=n} \left(\frac{\lambda_\infty^h(\Delta_n(v))}{M_n(h)} - \sum_{i=0}^{b-1} \frac{\lambda_\infty^h(\Delta_{n+1}(v*i))}{M_{n+1}(h)} \right)^2 . \tag{20}$$

Problem 2.5. (Least Squares Estimates) Under certain bounds on the cascade generators, the statistic $V_n^2(h)$ has been shown to be an ordinary least squares variance estimator for sufficiently small h in Troutman and Vecchia (1999). One might expect this to hold in the more general setting laid out in this paper as well.

The following corollary then gives a central limit theorem for a completely observable statistic whose asymptotic distribution does not depend on the distributions of the unobservable generator variables W or the unknown distribution of the cascade itself, $\lambda_\infty(\mathbf{T})$. The independence in h of the N_h's noted above indicates that the errors in this estimator of $\chi_b(h)$, namely $\widetilde{\tau}_n(h)$, are asymptotically independent.

Corollary 2.5 *For* $h \in (H_c^-/2, H_c^+/2)$,

$$\frac{\widetilde{\tau}_n(h) - \chi_b(h)}{V_n(h)} \to^d (\log b)^{-1} \eta N_h. \tag{21}$$

Problem 2.6. (Fluctuation Law/Scaling Outside Critical Interval) Neither the nature of the fluctuations nor the appropriate scaling is known outside the scaled critical interval $(H_c^-/2, H_c^+/2)$.

3. **Some Basic Examples**

In this section we provide a selection of examples to aid in illustrating the theory described in the previous section.

EXAMPLE 3.1. (Zero/Nonzero Bernoulli Generators) This example is explicitly related to the Galton-Watson branching process. Let W take values p^{-1} and 0 with probability p and $1-p$ respectively; i.e. W is a Bernoulli r.v. scaled to have mean one. The structure function χ_b is then linear with slope $\log_b(1/bp)$ and the random measure λ_n has total mass $\lambda_n(\mathbf{T})$ obtained as a sum of b^n terms which each have value 0 or value $(bp)^{-n}$. The number of non-zero terms in the sum is the number X_n in the nth generation of a Galton-Watson branching process with a Binomial(b,p) offspring distribution; that is

$$\lambda_n(\mathbf{T}) = \frac{X_n}{(bp)^n}. \tag{22}$$

$\lambda_n(\mathbf{T})$ is then the non-negative mean one martingale associated with X_n, and $\lambda_\infty(\mathbf{T})$ is its a.s. limit. Part (i) of Theorem 2.1 implies that $P(\lambda_\infty(\mathbf{T}) > 0) > 0$ iff the slope of χ_b is less than 0; i.e. $p > b^{-1}$. This coincides exactly with the well-known condition which guarantees that the mean-normalized Galton-Watson branching process $(bp)^{-n}X_n$ lives with positive probability.

EXAMPLE 3.2. (Beta-distributed Generators) Consider generators W which are uniformly distributed on $[0,2]$ and take the branching number $b=2$. In this case it is simple to check using Proposition 2.1 that the total mass $Z_\infty := \lambda_\infty(\mathbf{T})$ has the Gamma distribution specified by

$$P(Z_\infty \in dx) = 4xe^{-2x}dx, \quad x \geq 0. \tag{23}$$

This is a special case of the following. Fix $b \geq 2$, and take W/b to have a Beta [1] distribution with parameters r and $(b-1)r$ for some $r > 0$. Then the distribution of Z_∞ is that of a Gamma r.v. with shape parameter br and scale parameter $(br)^{-1}$. In particular an exact simulation of the limit cascade may be achieved for models in this family.

EXAMPLE 3.3. (logNormal Generators) The logNormal generators are of the form $W = e^{\sigma Z - \frac{\sigma^2}{2}}$, where Z has a standard normal distribution. In this case one obtains the quadratic structure function

$$\chi_b(h) = \frac{\sigma^2}{2\ln b}h^2 - \left(\frac{\sigma^2}{2\ln b} + 1\right)h + 1. \tag{24}$$

with critical values given by the roots

$$H_c^+ = \frac{\sqrt{2\ln b}}{\sigma} \quad \text{and} \quad H_c^- = -\frac{\sqrt{2\ln b}}{\sigma} \tag{25}$$

[1] In the physics literature the model described in Example 3.1 is commonly referred to as the beta-model. This terminology is at odds with the terminology used in the statistics literature, and thus the reader is duly cautioned.

of $h\chi_b'(h) - \chi_b(h)$.

Example 3.4. (logPoisson Generators) The logPoisson generators are of the form $W = e^{aY - c(e^a - 1)}$, where Y has a Poisson distribution with parameter $c > 0$ and

$$\chi_b(h) = \frac{c}{\ln b}(e^{ah} - he^a) + \left(\frac{c}{\ln b} - 1\right)(h-1). \tag{26}$$

We assume that the choice of a and c provides $\chi_b'(1) < 0$. H_c^+, H_c^- are respectively positive and negative solutions of the equation

$$\frac{c}{\ln b}(ah-1)e^{ah} + \frac{c}{\ln b} - 1 = 0. \tag{27}$$

For $c \le \ln b$ and $a > 0$, it is easy to check that $H_c^- = -\infty$. If $c = \ln b$ and $0 < a < 1$, then $H_c^+ = \frac{1}{a}$. Similarly for $c \ne \ln b$, H_c^+ may or may not be finite depending on the parameter a.

The significance of these examples in applications to statistical turbulence theory will be elaborated upon in Section 5.

4. An Approach to Dimension Estimates: Some New Results

The intermittency of the cascade measure is reflected in the Hausdorff dimension of the supporting set of the measure. This important geometric parameter is given by $-\chi_b'(1^-) = 1 - \mathbf{E}W\log_b W$, the derivative of the structure function at $h = 1$, whenever the cascade survives. Two natural estimators for the Hausdorff dimension are

$$\widehat{D}_n = -(n\lambda_\infty(\mathbf{T}))^{-1} \sum_{|v|=n} \lambda_\infty(\Delta_n(v)) \log_b \lambda_\infty(\Delta_n(v)), \tag{28}$$

defined for $n \ge 1$, and

$$\widetilde{D}_n = (\lambda_\infty(\mathbf{T}))^{-1} \left(\sum_{|v|=n} \lambda_\infty(\Delta_n(v)) \log_b \lambda_\infty(\Delta_n(v)) - \sum_{|v|=n+1} \lambda_\infty(\Delta_{n+1}(v)) \log_b \lambda_\infty(\Delta_{n+1}(v)) \right), \tag{29}$$

defined for $n \ge 0$. Notice that $\widehat{D}_n = -\widehat{\tau}_n'(1)/\lambda_\infty(\mathbf{T})$ and $\widetilde{D}_n = -\widetilde{\tau}_n'(1)/\lambda_\infty(\mathbf{T})$. $\widehat{D}_n$ is used by physicists as an estimator of the Hausdorff dimension of the support set of a measure; see Chhabra and Jensen(1989). To our knowledge, $\widetilde{D}_n$ is a new estimator. The following theorem shows that both $\widehat{D}_n$ and $\widetilde{D}_n$ are strongly consistent estimators of $-\chi_b'(1)$.

THEOREM 4.6 *If $h_c > 1$, then both*
(i.) $\widetilde{D}_n \to -\chi_b'(1)$
and
(ii.) $\widehat{D}_n \to -\chi_b'(1)$
$P-$a.s. as $n \to \infty$.

A central limit theorem for the estimator $\widetilde{D}_n$ is also obtainable. The observable normalization is given by

$$\widetilde{V}_n^2 = \sum_{|v|=n} \Big(\lambda_\infty(\Delta_n(v)) \log_b \lambda_\infty(\Delta_n(v)) - \sum_{i=0}^{b-1} \lambda_\infty(\Delta_{n+1}(v*i)) \log_b \lambda_\infty(\Delta_{n+1}(v*i)) - \lambda_\infty(\Delta_n(v)) \widetilde{D}_n \Big)^2. \tag{30}$$

THEOREM 4.7 *(Central Limit Theorem for $\widetilde{D}_n$) If $H_c^+ > 2$, then*

$$\frac{(\widetilde{D}_n + \chi_b'(1))\lambda_\infty(\mathbf{T})}{\widetilde{V}_n} \to^d \eta N,$$

where $\eta = \mathbf{1}[\lambda_\infty(\mathbf{T}) = 0]$ and N is an independent standard normal random variable.

The proof of Theorem 4.1 above depends on the following generalization of Theorem 2.3.

THEOREM 4.8 *(Ossiander and Waymire (2000)) Suppose that $\{X(v) : |v| = n, n \geq 1\}$ is a collection of identically distributed non-negative random variables defined on $(\Omega, \mathcal{F}, P)$ with $\mathbf{E}X^{1+\epsilon}(v_0) < \infty$ for some $\epsilon > 0$ and, for each $n \geq 1$, $\{X(v) : |v| = n\}$ is a collection of independent random variables which is also independent of $\mathcal{F}_n$. Then for $h \in [0,1] \cup (H_c^-, H_c^+)$,*

$$\sum_{|v|=n} X(v)\lambda_n(h; \Delta_n(v)) \to \lambda_\infty(h; \mathbf{T})\mathbf{E}X(v_0)$$

$P-$ a.s. as $n \to \infty$.

This theorem can be thought of a strong law of large numbers for a collection of i.i.d. non-negative r.v.'s $\{X(v)\}$ with the random weights $\lambda_n(\Delta_n(v))$ playing the role of the usual deterministic weights b^{-n}. It can be extended to general i.i.d. $X(v)$ by breaking each of these r.v.'s into its positive and negative parts and treating them separately. The following is immediate:

Corollary 4.6 *Suppose that* $\{X(v) : |v| = n, n \geq 1\}$ *is a collection of identically distributed random variables defined on* $(\Omega, \mathcal{F}, P)$ *with* $\mathbf{E}|X(v_0)|^{1+\epsilon} < \infty$ *for some* $\epsilon > 0$ *and, for each* $n \geq 1$, $\{X(v) : |v| = n\}$ *is a collection of independent random variables which is also independent of* $\mathcal{F}_n$. *Then for* $h \in [0,1] \cup (H_c^-, H_c^+)$,

$$\sum_{|v|=n} X(v)\lambda_n(h; \Delta_n(v)) \to \lambda_\infty(h; \mathbf{T})\mathbf{E}X(v_0)$$

$P-$ *a.s. as* $n \to \infty$.

This sets the stage for the proof of Theorem 4.1.

Proof. We first derive the a.s. convergence of $\widetilde{D}_n$. Decompose $\lambda_\infty(\Delta_n(v)) = \lambda_n(\Delta_n(v))Z_\infty(v)$ as suggested by Proposition 2.1 and rewrite

$$\begin{aligned} \widetilde{D}_n \, \lambda_\infty(\mathbf{T}) &= \sum_{|v|=n} \sum_{i=0}^{b-1} \lambda_\infty(\Delta_{n+1}(v*i)) \log_b(\lambda_\infty(\Delta_n(v))/\lambda_\infty(\Delta_n(v*i))) \\ &:= \sum_{|v|=n} \lambda_n(\Delta_n(v))X(v) \end{aligned} \tag{31}$$

where the r.v.'s $X(v)$ are defined by

$$X(v) := b^{-1} \sum_{i=0}^{b-1} W_{v*i} Z_\infty(v*i) \log_b(bZ_\infty(v)/W_{v*i}Z_\infty(v*i)). \tag{32}$$

It is easy to check that the $X(v)$'s are i.i.d. and independent of $\mathcal{F}_n$, with $\mathbf{E}X(v) = 1 - \mathbf{E}W \log_b W = -\chi_b'(1)$. Since $h_c > 1$, both $\mathbf{E}W^{1+\epsilon} < \infty$ and $\mathbf{E}Z_\infty^{1+\epsilon} < \infty$ for some $\epsilon > 0$. This gives $\mathbf{E}|X(v)|^{1+\epsilon} < \infty$ for some $\epsilon > 0$ as well. Applying Corollary 4.1, we have the result (i).

The convergence of $\widehat{D}_n$ is as follows. It is easy to see that $\widetilde{D}_k = (k+1)\widehat{D}_{k+1} - k\widehat{D}_k$ for $k \geq 1$ and $\widetilde{D}_0 = \widehat{D}_1 - \log_b \lambda_\infty(\mathbf{T})$. Collapsing this telescoping sum and solving for $\widehat{D}_n$ gives

$$\widehat{D}_n = n^{-1} \left(\sum_{k=0}^{n-1} \widetilde{D}_k + \log_b \lambda_\infty(\mathbf{T}) \right).$$

Using the a.s. convergence of $\widetilde{D}_n$ and Kronecker's Lemma the result follows. □

The proof of Theorem 4.2, the central limit theorem for $\widetilde{D}_n$, follows from Theorem 2.5.

PROOF. Since $H_c^+ > 2$, both $EZ_\infty^2(v)$ and $EX^2(v)$ are finite where the $X(v)$'s are as defined above in the proof of Theorem 4.1. Set

$$Y_n(v) = X(v) + Z_\infty(v)\chi_b'(1).$$

The $Y_n(v)$'s are identically distributed with $EY_n(v) = 0$ and $\sigma^2 := VarY_n(v) < \infty$. In addition, for each fixed n, the $Y_n(v)$'s are mutually independent as well as being independent of $\mathcal{F}_n$. From Corollary 2.4 we have

$$\frac{\sum_{|v|=n} \lambda_n(\Delta_n(v))Y_n(v)}{\sigma(\sum_{|v|=n} \lambda_n^2(\Delta_n(v)))^{1/2}} \to^d \eta N_1.$$

Some simplification gives

$$\begin{aligned} \sum_{|v|=n} \lambda_n(\Delta_n(v))Y_n(v) &= \sum_{|v|=n} \lambda_n(\Delta_n(v))X(v) + \lambda_\infty(\mathbf{T})\chi_b'(1) \\ &= (\widetilde{D}_n + \chi_b'(1))\lambda_\infty(\mathbf{T}). \end{aligned}$$

We can rewrite $\widetilde{V}_n^2$ as follows:

$$\begin{aligned} \widetilde{V}_n^2 &= \sum_{|v|=n} (\lambda_n(\Delta_n(v))X(v) - \lambda_n(\Delta_n(v))Z_\infty(v)\widetilde{D}_n)^2 \\ &= \sum_{|v|=n} (\lambda_n^2(\Delta_n(v))X^2(v) - 2\widetilde{D}_n \sum_{|v|=n} \lambda_n^2(\Delta_n(v))X(v)Z_\infty(v) \\ &\qquad + \widetilde{D}_n^2 \sum_{|v|=n} \lambda_n^2(\Delta_n(v))Z_\infty^2(v). \end{aligned}$$

Using Corollary 4.1, one has both

$$\begin{aligned} \frac{\sum_{|v|=n} \lambda_n^2(\Delta_n(v))\sigma^2}{b^{n\chi_b(2)}} &= \sum_{|v|=n} \lambda_n(2;\Delta_n(v))\sigma^2 \\ &\to \lambda_\infty(2;\mathbf{T})\sigma^2 \end{aligned} \tag{33}$$

and

$$\begin{aligned} \frac{\widetilde{V}_n^2}{b^{n\chi_b(2)}} &= \sum_{|v|=n} \lambda_n(2;\Delta_n(v))X^2(v) - 2\widetilde{D}_n \sum_{|v|=n} (\lambda_n(2;\Delta_n(v))X(v)Z_\infty(v) \\ &\qquad + \widetilde{D}_n^2 \sum_{|v|=n} (\lambda_n(2;\Delta_n(v))Z_\infty^2(v) \qquad (34) \\ &\to \lambda_\infty(2;\mathbf{T})(EX^2(v) + 2\chi_b'(1)EX(v)Z_\infty(v) + (\chi_b'(1))^2 EZ_\infty^2(v) \end{aligned}$$

$P-$a.s. as $n \to \infty$. Notice that

$$\begin{aligned} EX^2(v)+ \quad & 2\chi_b'(1)EX(v)Z_\infty(v) + (\chi_b'(1))^2 EZ_\infty^2(v) \\ & = E(X(v) + \chi_b'(1)Z_\infty(v))^2 \\ & = \sigma^2. \end{aligned} \tag{35}$$

This gives

$$\frac{\widetilde{V}_n^2}{\sum_{|v|=n} \lambda_n^2(\Delta_n(v))\sigma^2} \to 1$$

$P-$a.s.. The result follows. □

Problem 4.7. (Consistent Estimation of Other Parameters and CLTs) The consistent statistical estimation of other parameters of the fine scale structure, e.g. other points of the singularity spectrum and multifractal dimensions, has not been developed.

5. Applications to Turbulence: A Statistics Problem

In this section we describe a major outstanding problem for statistics in the physical sciences, namely statistical inference for cascade generators from a single sample realization.

The *energy dissipation rate* ϵ defined by

$$\epsilon(x) = \frac{\nu}{2} \sum_{i,j=1}^{3} \left(\frac{\partial u_i}{\partial x_j} \right)^2, \quad x \in R^3, \tag{36}$$

is computed in terms of the fluid velocity $u = (u_1, u_2, u_3)$ as the local rate of decay of kinetic energy $\frac{d}{dt}\frac{1}{2}\int_V |u(x)|^2 dx$ from incompressible Navier-Stokes equation in a region V with viscosity parameter $\nu > 0$.

One begins with the assumption that the multiplicative cascade with *i.i.d.* non-negative mean one generators is a valid statistical model for the turbulent redistribution of energy in the statistical model of the random dissipation field $\epsilon(dx)$ over an appropriate range of length scales, referred to as the *Kolmogorov inertial range.* The Kolmogorov inertial range is an interval of length scales from the largest length scale at which energy enters the system down to the smallest length scale at which energy is dissipated by fluid viscosity. Actual observations of ϵ are one-dimensional cross sections wherein (**??**) is replaced by the surrogate measurement $15\nu(\frac{\partial u_1}{\partial x_1})^2$. As a result Jouault, Greiner, and Lipa (2000) have effectively argued that *i.i.d.* mean

Figure 1: Graph of the linearly corrected logNormal structure function $\overline{\chi}_b(h)$ with the Anselmet turbulence data superimposed.

one generators provide the appropriate model for measurements of energy dissipation rates from the point of view of conservation laws. In particular, taking one-dimensional cuts through the three dimensional energy dissipation field makes the measurements non-conservative in an almost sure sense. The statistical model with *i.i.d.* mean one generators provides conservation on average.

As calculated in Example 3.3, the Kolmogorov lognormal hypothesis leads to a quadratic structure function $\chi_b(h)$. Early data analysis revealed a departure from quadratic multiscaling exponents, which is now understood to be remarkably adjusted by the linear correction $\overline{\chi}_b(h)$ as depicted in Figure 1. This effect had already been anticipated by preliminary calculations in the physics literature by Lovejoy and Schertzer (1991), and Molchan (1997). Here the parameterizing ratio $\sigma^2/2\ln b$ is taken to be .1 as suggested by Anselmet et al. This gives $\chi_b(h) = .1h^2 - 1.1h + 1$ and $H_c^+ = \sqrt{10}$.

Largely prompted by discrepencies between the observed data and the quadratic structure function as illustrated in Figure 1, various adhoc alternatives to the lognormal hypothesis have been considered in the physics literature; see Frisch (Figure 8.8, p.132;1995). However, the logPoisson distribution surfaced as an alternate hypothesis as a somewhat indirect consequence of an analysis by She and Lèvêsque (1994), Dubrulle (1994), and She and Waymire (1994,1995). Specifically, She and Lèvêsque (1994) obtain the following second order, linear, nonhomogeneous difference equation for the scaling exponents.

$$\tau(h+2) - (1+\beta)\tau(h+1) + \beta\tau(h) + \frac{2}{3}(1-\beta) = 0, \tag{37}$$

where $\beta = \frac{2}{3}$, and $\tau(0) = \tau(1) = 0$ as a consequence of the following log-convexity hypothesis on the structure of the size-biased moments $\epsilon_l^{(h)} := E\epsilon_l^{h+1}/E\epsilon_l^h$ of energy dissipation:

$$\epsilon_l^{(h+1)} = A_h(\epsilon_l^{(h)})^\beta(\epsilon_l^{(\infty)})^{1-\beta}, \quad 0 < \beta < 1. \tag{38}$$

According to She (personal communication) this hypothesis was formulated in response to observations made in numerical simulations of Navier-Stokes

Figure 2: Graph of the linearly corrected logPoisson structure function of She and Lèvêsque with the Anselmet turbulence data superimposed. Here $H_c^+ \approx 4.14$.

equations. A plot of the scaling exponents with and without the linear correction for logPoisson generators against the Anselmet data is given in Figure 2.

Obviously if the $\widehat{\tau}_n(h)$'s as observed by Anselmet et al. gave consistent estimates of the structure function $\chi_b(h)$ for all values of h, then the need for extensive statistical theory to test a hypothesis of logNormal vs. logPoisson would hardly be justified. However, the problem of distinguishing between these models based on an understanding of the convergence properties of $\widehat{\tau}_n(h)$ as illustrated in Figures 1 and 2 requires the appropriate error bars as being developed in the present paper. It will also require more detailed data of measurements of the dissipation rates at small length scales. In the literature the reported data is often that of velocity exponents $\zeta(h)$ defined by $E(u_1(x+\lambda) - u_1(x))^h \sim \lambda^{\zeta(h)}$ from which $\tau(h)$, defined by $E\epsilon^h(dx) \sim (dx)^{\tau(h)}$, is obtained by the assumed relationship based on dimensional arguments that $\zeta(h) = \frac{h}{3} + \tau(\frac{h}{3})$.

Problem 5.8 (LogNormal vs LogPoisson in Turbulence). Construct an appropriate test of hypothesis of logNormal vs. logPoission hypotheses in turbulence energy dissipation.

For other applications, eg. rainfall, internet traffic, the precise nature of the cascade generators is not a central issue. However, the problem of reconstructing the distribution from the empirical cumulant generating function in a neighborhood of the origin $\chi_b(h)$ remains an interesting open question for applications in these areas. In these areas there is also interest in identifying further models which possess similar intermittancy and scaling properties as the multiplicative cascades.

In spite of practical problems associated with data collection as well as a still incomplete theoretical basis for precise statistical inference, we hope that this section communicates an exciting outstanding challenge to statistics in the physical sciences. While the direction of this research focuses on the multiscaling exponent data and theory available in the physics literature, other approaches are possible which are not reviewed here; eg. see Jouault, Greiner, Lipa (2000) and Barndorff-Nielsen, Jensen, and Sorensen (1990) for interesting alternative approaches to statistical inference on turbulence data.

Finally, we wish to record that the statistical theory is complimented on the purely mathematical side by foundational efforts which seek to explain observed qualitative structure in turbulence measurements directly from analysis of incompressible Navier-Stokes equations. An extensive collection of references in this regard may be found in the recent monograph by Foias, Manley, Rosa, and Temam (2001). Also, for a recent alternative probabilistic analysis of Navier-Stokes see LeJan and Sznitman (1997) and Bhattacharya et.al. (2002).

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