Kinetic equations for the pure jump models of k-nary interacting particle systems.

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Abstract. Hydrodynamic limit of k-nary interacting particles with interaction of pure jump type is described by a system of kinetic equations that generalize classical Smoluchovski's coagulation equations and many other models. Existence and uniqueness theorems for these equations are proved and the propagation of chaos property is established.

Key words. Interacting particles, k-nary interaction, measure-valued limits, kinetic equation, mass exchange processes, coagulation-fragmentation, propagation of chaos.

Running Head: Kinetic equations for jump-type interactions.

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Content. 1. Introduction. 2. Preliminaries. 3. Heuristic derivation of kinetic equations and examples. 4. Existence of solutions and convergence of stochastic approximations. 5. Approximation by Markov chains. 6. Uniqueness and continuous dependence on initial data.

1. Introduction.

1. Aims of the paper. The paper is devoted to the description of the deterministic hydrodinamic limits (kinetic equations) of the general pure jump Markov modelds of k-nary interacting particle systems. The major particular cases are given by (i) the mass exchange processes which include the Smoluchovski coagulation model and its extensions with not necessary binary coagulations or fragmentation and also more general processes, where, say, the rate of coagulation or fragmentation of two particles can be increased or decreased by the presence of a third particle, or where a particle can split another particle in pieces and coagulate with one of them, (2) k-nary collision processes including Boltzmann's collisions, (iii) various combinations of the mass exchange and collision processes. The main objectives of the paper are (i) to show that as a number of particles go to infinity and under a natural scaling of interaction rates, these processes converge weakly to measure-valued deterministic processes, (ii) to derive general kinetic equations that describe the evolution of these limiting processes (equations (3.7),(3.8) below that include the Smoluchovski equations and the Boltzmann equation as particular cases), (iii) to prove the well-posedness of the Cauchy problem for these equations.

In [BK] we used a different method to obtain similar kinetic equations which was formal (i.e. without any rigorous convergence or existence results). In fact, we developed two such methods, one was suggested in [Be] and was based on the study of the evolution of the generating functionals and another was based on the idea of propagation of chaos (see e.g. [BM] or [Sz]). We shall return to the latter approach at the end of this paper proving the propagation of chaos property of our model under some assumptions thus giving a rigorous justification to formal calculations of [BK] in these cases.

2. Content of the paper. In the next section, we shall fix some notations used throughout the paper, in particular those connected with the state space of a system of interacting particles, and recall the basic notions and facts from the theory of pure jump Markov processes. In Section 3 we describe the general pure jump Markov processes that model the k-nary interacting exchangeable particle systems and give a heuristic derivation of the corresponding kinetic equations, discussing also various forms of these equations and their main examples. A rigorous analysis of our model starts at Section 4, where we justify the heuristic arguments of the previous section proving (under an appropriate scaling and subject to some additional assumptions) that a subsequence of the family of Markov processes describing our k-nary interacting exchangeable particle systems converges to the deterministic measure-valued process defined by the integral version of the weak kinetic equations (3.11), proving in particular, the existence of solutions to these kinetic equations. It is worth noting that an application of this result to binary interactions yields already a seemingly new rigorous result on the convergence of natural stochastic approximations to the solutions of the kinetic equations of the standard coagulation- fragmentation model filling the gap between the corresponding results from [No1], [No2] (pure coagulation model) and [Je] (coagulation-fragmentation with discrete mass distribution). We conclude this section by an important regularity result showing that our weak solution turn out to be also strong solution under mild additional assumptions. Section 5 is devoted to a slightly different approximation of kinetic equations, namely to an approximation by Markov chains obtained by the discretization of the state space X. This approximation seems to be of interest for applications, say, for computer simulations of our models, and it can be considered as a stochastic counterpart of deterministic discritizations of kinetic models (for the latter see [LM2] and references therein). In Section 6 we obtain the main result of this paper, Theorem 6.1, proving the well-posedness of the Cauchy problem for our kinetic equations (in particular, the continuous dependence of the solutions on the initial conditions) under some reasonable assumptions. In particular, we improve the results on uniqueness of the standard fragmentation-coagulation model obtained in [DS], where the uniqueness is proved only under a restrictive assumption of the existence of finite exponential moments (and for measures with a density with respect to Lebesgue measure). On the other hand, we extend more general result on uniqueness from [No1], [No2] obtained there for a binary coagulation model to a large variety of models including the standard coagulation-fragmentation model, collision breakage model, the Boltzmann equation and their combinations and generalizations with k-nary interaction. Our method of proving uniqueness can be considered as a non-trivial extension of the method of proving uniqueness developed for discrete models in [BC], [LW], [Ko3]. This extension is based on a measure theoretic result presented as Lemma A in the Appendix. As a corollary of our main result, we prove for conclusion a version of the propagation of chaos property for our model of k-nary interacting particle systems (cf. [Sz] for the corresponding result for the Boltzmann equation). As another corollary, let us stress that our uniqueness result essentially enlarges the class of initial distributions for which the discrete approximation scheme discussed in [LM] converges to the solution of the fragmentation-coagulation equation.

Concluding remarks. 1. Mass exchange processes with only discrete mass distribution were considered in [Ko3] following the analysis of finite-dimensional limits in [Ko1], [Ko2]. 2. Dealing only with pure jump models we do not include spatially non-trivial models, where the particles are characterized by their position in space or by other parameters that are changing according to some given law, for example as a Brownian motion. We are going to start the analysis of such a generalization in the next publication, see [Ko4]. In case of classical coagulation-fragmentation process, the first rigorous mathematical results on such spatially non-trivial models have been obtained recently, see [BW], [CP], [W] and references therein for discrete mass distribution and [Am], [LM1] for continuous masses.

2. Preliminaries.

1. Some notations. We list here a few notations that will be used throughout the paper without further reminder: $\mathbf{1}$ denotes the function that equals identically 1 or the identity operator in a Banach space; $\mathbf{1}_M$ for a set M denotes the indicator function of M that equals 1 for $x \in M$ and vanishes otherwise; $o(1)_{x\to a}$ denotes a function depending on x that tends to zero as $x \to a$;

for a measurable space Y, $B(Y)$ denotes the Banach space of real bounded measurable functions on Y equipped with the usual sup-norm; if Y is a topological space, $C_b(Y)$ denotes the Banach subspace of $B(Y)$ consisting of continuous functions; $\mathcal{M}(Y)$ is the space of finite (signed) measures on Y, considered as a Banach space with the norm $\|.\|$ on $\mathcal{M}(Y)$ being the total variation norm; for $\mu \in \mathcal{M}(Y)$ we shall denote by $|\mu|$ the total variation measure of μ so that $\|\mu\| = \int_Y |\mu|(dy)$;

if X is a locally compact space, then $C_0(X)$ (respectively $C_c(X)$) is the Banach space of continuous functions vanishing at infinity and equipped with sup-norm (respectively its subspace consisting of functions with a compact support);

the upper subscript "+" for all these spaces (e.g. $C_0^+(X)$, $\mathcal{M}^+(X)$) will denote the corresponding cones of non-negative elements;

by a symmetric function of n (vector) variables we shall understand a function, which is symmetric with respect to all permutations of these variables, and by a symmetric operator on the space of functions of n variables we shall understand an operator that preserves the set of symmetric functions;

for a finite subset $I = \{i_1, ..., i_k\}$ of a countable set J, we denote by $|I|$ the number of elements in I, by $\bar{I}$ its complement $J \setminus I$, by $\mathbf{x}_I$ the collection of the variables $x_{i_1}, ..., x_{i_k}$ and by $d\mathbf{x}_I$ the measure $dx_{i_1}...dx_{i_k}$.

A transitional kernel from X to Y means, as usual, a measurable function $\mu(x, .)$ from $x \in X$ to the cone of positive finite measures on Y.

We shall denote as usual by $D_X[0,\infty)$ the Skorokhod space of càdlàg paths $[0,\infty) \mapsto X$ equipped with the standard filtration $\mathcal{F}_t$. We shall denote by bald $\mathbf{P}$ and $\mathbf{E}$ the probability and the expectation of events and functions.

2. State space and observables of systems of interacting particles. Throughout the paper we shall denote by X a locally compact metric space equipped with its Borel sigma algebra. Denoting by X^0 a one-point space and by X^j the powers $X \times ... \times X$ (j-times) considered with their product topologies, we shall denote by $\mathcal{X}$ their disjoint union $\mathcal{X} = \cup_{j=0}^{\infty} X^j$, which is again a metric locally compact space. In applications, X specifies the state space of one particle and $\mathcal{X} = \cup_{j=0}^{\infty} X^j$ stands for the state space of a random number of similar particles. We shall denote by $B_{sym}(\mathcal{X})$ (resp. $C_{sym}(\mathcal{X})$) the Banach spaces of symmetric bounded measurable (resp. continuous) functions on $\mathcal{X}$ and by $B_{sym}(X^k)$ (resp. $C_{sym}(X^k)$) the corresponding spaces of functions on the finite power X^k. The space of symmetric measures will be denoted by $\mathcal{M}_{sym}(\mathcal{X})$. The elements of $\mathcal{M}^+_{sym}(\mathcal{X})$ and $C_{sym}(\mathcal{X})$ are called respectively the (mixed) states and observables for a Markov process on $\mathcal{X}$. We shall denote the elements of $\mathcal{X}$ by bold letters, e.g. $\mathbf{x}$, $\mathbf{y}$. Sometimes it is convenient to consider the factor spaces SX^k and $S\mathcal{X}$ obtained by the factorization of X^k and $\mathcal{X}$ with respect to all permutations, which allows for the identifications $C_{sym}(\mathcal{X}) = C(S\mathcal{X})$ and likewise. Symmetrical laws on X^k (which are uniquely defined by their projections to SX^k) are called exchangeable systems of k particles. A key observation for the theory of measure-valued limits is the inclusion $S\mathcal{X}$ to $\mathcal{M}^+(X)$ given by

$$\mathbf{x} = (x_1, ..., x_l) \mapsto \delta_{x_1} + ... + \delta_{x_l}, \tag{2.1}$$

which defines a bijection between $S\mathcal{X}$ and the space $\mathcal{M}^+_\delta(X)$ of finite linear combinations of δ-measures (notice that our inclusion (2.1) differs by a normalization from the form of this inclusion discussed in detail, e.g. in [Da]).

Clearly each $f \in B_{sym}(\mathcal{X})$ is defined by its components f^k on X^k so that for $\mathbf{x} = (x_1, ..., x_k) \in X^k \subset \mathcal{X}$, say, one can write $f(\mathbf{x}) = f(x_1, ..., x_k) = f^k(x_1, ..., x_k)$ (the upper index k at f is optional and is used to stress the number of variables in an expression). Similar notations are for measures. In particular, the pairing between $C_{sym}(\mathcal{X})$ and $\mathcal{M}(\mathcal{X})$ can be written as

$$(f, \rho) = \int f(\mathbf{x}) \rho(d\mathbf{x}) = f^0 \rho_0 + \sum_{n=1}^{\infty} \int f(x_1, ..., x_n) \rho(dx_1 ... dx_n),$$

$$f \in C_{sym}(\mathcal{X}),\ \rho \in \mathcal{M}(\mathcal{X}), \tag{2.2}$$

so that $\|\rho\| = (\mathbf{1}, \rho)$ for $\rho \in \mathcal{M}^+(\mathcal{X})$.

A useful class of measures (and mixed states) on $\mathcal{X}$ is given by the decomposable measures of the form $Y^{\otimes}$ and $Y^{\tilde{\otimes}}$, which are defined for an arbitrary finite measure $Y(dx)$ on X by their components

$$(Y^{\otimes})_n (dx_1 ... dx_n) = Y^{\otimes n}(dx_1 ... dx_n) = Y(dx_1) ... Y(dx_n) \tag{2.3}$$

and

$$(Y^{\tilde{\otimes}})_n (dx_1 ... dx_n) = \frac{1}{n!} Y^{\otimes n}(dx_1 ... dx_n) = \frac{1}{n!} Y(dx_1) ... Y(dx_n). \tag{2.4}$$

Notice that unlike $Y^{\tilde{\otimes}}$ the measure $Y^{\otimes}$ need not be a finite measure on $\mathcal{X}$ even when Y is finite. Similarly the decomposable observables (or exponential vectors) are defined for an arbitrary $Q \in C(X)$ as

$$(Q^{\otimes})^n (x_1, ..., x_n) = Q^{\otimes n}(x_1, ..., x_n) = Q(x_1) ... Q(x_n). \tag{2.5}$$

We shall use also the additively decomposable observables defined for an arbitrary $Q \in C(X)$ as

$$(Q^+)(x_1, ..., x_n) = Q(x_1) + ... + Q(x_n) \tag{2.6}$$

(Q^+ vanishes on X^0). In particular, if $Q = \mathbf{1}$, then $Q^+ = \mathbf{1}^+$ is the number of particles: $\mathbf{1}^+(x_1, ..., x_n) = n$.

We shall deduce now a simple combinatorial identity for symmetric functions that will be used in our derivation of kinetic equations (which is a variation of similar identities from [Da]) . We shall denote by δ_x the Dirac measure at x. By a Young scheme Γ we mean a collection $\Gamma = \{\gamma_1, ..., \gamma_l\}$ of natural numbers such that $1 \le \gamma_1 \le ... \le \gamma_l$. Let $h > 0$ be a positive parameter.

Proposition 2.1 *For any natural k, there exist constants α_Γ parametrized by all Young schemes $\Gamma = \{\gamma_1, ..., \gamma_l\}$ with $\gamma_1 + ... + \gamma_l = k$ such that for any natural n, $f \in B_{sym}(X^k)$ and a collection of points $x_1, ..., x_n$ in X*

$$h^k \sum_{I \subset \{1,...,n\}, |I|=k} f(x_I) = \frac{1}{k!} \int f(y_1, ..., y_k) \prod_{j=1}^k (h\delta_{x_1} + ... + h\delta_{x_n})(y_j)$$

$$+ \sum_\Gamma \alpha_\Gamma h^{k-l} \int f(y_1, ..., y_1, y_2, ..., y_2, ..., y_l, ..., y_l) \prod_{j=1}^l (h\delta_{x_1} + ... + h\delta_{x_n})(y_j), \tag{2.7}$$

where $\sum_\Gamma$ is the sum over all Young schemes $\Gamma = \{\gamma_1, ..., \gamma_l\}$ such that $\gamma_1 + ... + \gamma_l = k$ and $\gamma_l > 1$ (or, equivalently, $l < k$), and $f(y_1, ..., y_1, y_2, ..., y_2, ..., y_l, ..., y_l)$ means that the first γ_1 arguments of f are equal to y_1, the next γ_2 arguments are equal to y_2, etc. Moreover, if f is non-negative, then the l.h.s. of (2.7) does not exceed the first term of the r.h.s. of (2.7).

Proof. The proof is obtained by trivial induction in k from the following evident formula

$$h^k \sum_{i_1,...i_k=1}^n f(x_{i_1}, ..., x_{i_k}) = \int f(y_1, ...y_k) \prod_{j=1}^k (h\delta_{x_1} + ... + h\delta_{x_n})(y_j). \tag{2.8}$$

To avoid long expressions, we shall just deduce (2.7) for $k = 2$ and $k = 3$. We have

$$h^2 \sum_{I \subset \{1,...,n\}, |I|=2} f(x_I) = \frac{h^2}{2} \sum_{i_1 \ne i_2} f(x_{i_1}, x_{i_2}) = \frac{h^2}{2} [\sum_{i_1,i_2=1}^n f(x_{i_1}, x_{i_2}) - \sum_{i=1}^n f(x_i, x_i)]$$

$$= \frac{1}{2} \int f(y_1, y_2) \prod_{j=1}^2 (h\delta_{x_1} + ... + h\delta_{x_n})(y_j) - \frac{h}{2} \int f(y, y)(h\delta_{x_1} + ... + h\delta_{x_n})(y).$$

Similarly,

$$h^3 \sum_{I \subset \{1,...,n\}, |I|=3} f(x_I) = \frac{h^3}{3!} \sum_{i_1,i_2,i_3: i_1 \ne i_2, i_1 \ne i_3, i_2 \ne i_3} f(x_{i_1}, x_{i_2}, x_{i_3})$$

$$= \frac{h^3}{3!}(\sum_{i_1,i_2,i_3} f(x_{i_1},x_{i_2},x_{i_3}) - \frac{h^3}{2} \sum_{i_1\neq i_2} f(x_{i_1},x_{i_1},x_{i_2}) - \frac{h^3}{3!} \sum_i f(x_i,x_i,x_i))$$

$$= \frac{1}{3!} \int f(y_1,y_2,y_3) \prod_{j=1}^3 (h\delta_{x_1}+...+h\delta_{x_n})(y_j) - \frac{h}{2} \int f(y_1,y_1,y_2) \prod_{j=1}^2 (h\delta_{x_1}+...+h\delta_{x_n})(y_j)$$

$$+\frac{h^2}{3} \int f(y,y,y)(h\delta_{x_1}+...+h\delta_{x_n})(y).$$

The last assertion of the Proposition is evident, because one sees from the above that, whenever f is non-negative, the first term of the asymptotics (2.7) is obtained by increasing the sum from the l.h.s. of (2.7).

3. Basic facts about jump processes. By a pure jump Markov process on X we mean a Markov process with a generator of the form

$$(Gf)(x) = \int (f(y) - f(x))q(x,dy) = \int f(y)q(x;dy) - f(x)q(x), \tag{2.9}$$

where $(q(x), q(x,.))$ is a totally stable conservative q-pair, which means that (see e.g. [Ch]) $q(x,.)$ is a transition kernel from X to X such that $q(x,\{x\}) = 0$, and $q(x) = q(x,X)$ called the intensity of the q-pair, is everywhere finite. An important question is whether one can construct a Markov process with a given generator of type (2.9) and whether such a process is unique. We refer to [Ch] and [EK] for the general theory. We shall need the following corollary of this theory, which proof we sketch for completeness.

Proposition 2.2. *(i) Suppose $x \mapsto q(x,.)$ is a uniformly bounded function $X \mapsto \mathcal{M}^+(X)$ that is continuous with respect to the weak topology of $\mathcal{M}^+(X)$. Then for any $x \in X$ there exists a unique Markov process Z_t^x on X with generator (2.9), whose domain is the whole Banach space $C_b(X)$, and moreover Z_t^x can be characterized as the unique solution to the corresponding martingale problem, i.e. as the unique distribution on the Skorohod space $D_X[0,\infty)$ such that*

$$M_t = f(Z_t^x) - f(x) - \int_0^t Gf(Z_s^x)\, ds \tag{2.10}$$

is an $\mathcal{F}_t$-martingale for all $f \in C_c(X)$ such that $M_0 = 0$ almost surely. The corresponding semigroup $T_t f(x) = Ef(Z_t^x)$ is a strongly continuous semigroup of contractions on $C_b(X)$ and (2.10) is a martingale for any non-negative continuous f on X such that $Gf(x) \leq b + cf(x)$ for all x and some constants $b, c \geq 0$.

(ii) Suppose again that $x \mapsto q(x,.)$ is weakly continuous, but may be not bounded, and that there exists a continuous non-negative function ψ on X such that the intensity $q(x)$ is uniformly bounded on all sets $U_a = \{x : \psi(x) \leq a\}$, and moreover $G\psi(x) \leq b + c\psi(x)$ for all x and some constants $b, c \geq 0$. Then there exists a unique Markov process Z_t^x on X starting at x having a generator that is an extension of (2.9) considered on the domain $C_c(X)$, and moreover Z_t^x can be characterized as the unique distribution on the Skorohod space $D_X[0,\infty)$ such that (2.10) is a martingale for all $f \in C_c(X)$. The corresponding

semigroup $T_t f(x) = Ef(Z_t^x)$ is a semigroup of contractions on $C_b(X)$ (but not necessarily strongly continuous). At last, for any positive T and r

$$\mathbf{P}\left(\sup_{t\in[0,T]} \psi(Z_t^x) > r\right) \le \frac{C(T)}{r}, \tag{2.11}$$

where $C(T)$ depends only on T,b and $\psi(x)$.

Sketch of the proof. (i) In this case, G is a bounded operator on $C_b(X)$ and the semigroup is uniquely obtained as $T_t = e^{Gt}$ (defined as a converging series). This implies the well-posedness of the martingale problem for G with the domain C_b (see e.g. Theorem 4.1 of Chapter 4 from [EK]). To get the well-posedness of the martingale problem for G with the domain C_c one can, for example, approximate G by operators G_n obtained by multiplication of $q(x,.)$ by continuous functions $X \mapsto [0,1]$ with compact support (such G_n define a strongly continuous semigroup in $C_0(X)$ with a core C_c and consequently the martingale problem is well posed for G_n defined on $C_c(X)$), and then apply the localization theory (see e.g. Theorem 6.2 of Chapter 4 from [EK]). At last, let a continuous f be non-negative and $Gf \le b + cf$. Let $f_n(x) = \min(n, f(x))$. Then $Gf_n \le b + cf_n$ for all n (in particular, $Gf_n(x) \le 0$ whenever $f_n(x) = n$). As $0 \le f_n \le n$, (2.10) is a martingale for f_n playing the role of f. By Gronwall's lemma this implies

$$\mathbf{E} f_n(Z_t^x) \le (f(x) + tb)e^{ct}. \tag{2.13}$$

By the monotone convergence theorem this impies the same estimate for f on the place of f_n. Hence approximating f by f_n in (2.10) we can conclude that (2.10) holds by the dominated convergence theorem.

(ii) Instead of G, consider its approximation G_n defined as

$$(G_n f)(x) = \int (f(y) - f(x))\tilde{\mathbf{1}}_{\psi(.)\le n} q(x, dy),$$

where $\tilde{\mathbf{1}}_{\psi(.)\le n}$ is a continuous function $X \mapsto [0,1]$ that coincides with $\mathbf{1}_{\psi(.)\le n}$ on the set of all x where $\psi(x) \notin [n, n+1]$. Each G_n satisfies conditions (i). Moreover, as G_n are obtained from G by the multiplication on $\mathbf{1}_{\psi(x)\le n}$, they enjoy the same property $G_n\psi(x) \le b + c\psi(x)$. Hence

$$\psi(Z_t^{x,n}) - \psi(x) - \int_0^t G_n \psi(Z_s^{x,n})\, ds \tag{2.12}$$

is a martingale and

$$\mathbf{E}\psi(Z_t^{x,n}) \le (\psi(x) + tb)e^{ct}. \tag{2.13}$$

Hence, for any T and with probability arbitrary close to one, the processes $\psi(Z_t^{x,n})$ are uniformly bounded for $t \in [0,T]$, and hence G_n form a localizing sequence for G such that Theorem 6.3 from Chapter 4 of [EK] can be applied to get the well-posedness of the martingale problem for G and hence the existence and uniqueness of the corresponding Markov process. At last, passing to the limit as $n \to \infty$ in (2.12),(2.13) yields the same properties for $\psi(Z_t^x)$, and hence using the Doob maximal inequality for martingales yields (2.11).

In the future, we shall be interested in pure jump processes on $\mathcal{X}$, whose semigroup and the generator preserves the space C_{sym} of continuous symmetric functions so that the q-pair is given by symmetric transition kernels $q(\mathbf{x}; d\mathbf{y})$ that could be thus considered as kernels on the factor space $S\mathcal{X}$.

For conclusion, let us notice that though we have chosen to work with locally compact metric spaces X for simplicity and transparency (for the same purpose we also assume the continuity of all kernels), most of the results can be generalized (with minor modifications) to Polish spaces X or even to general measurable spaces X, as considered in [No2] for coagulation models.

3. Heuristic derivation of kinetic equations and examples.

1. Pure jump Markov models for k-nary interacting particles.

A k-nary interaction is specified by a transition kernel

$$P^k(x_1, ..., x_k; d\mathbf{y}) = \{P^k_m(x_1, ..., x_k; dy_1...dy_m)\}$$

from SX^k to $S\mathcal{X}$ such that $P^k(\mathbf{x}; \{\mathbf{x}\}) = 0$ for all $\mathbf{x} \in \mathcal{X}$, and with the intensity

$$P^k(x_1, ..., x_k) = \int P^k(x_1, ..., x_k; d\mathbf{y}) = \sum_{m=0}^{\infty} \int P^k_m(x_1, ..., x_k; dy_1...dy_m).$$

The intensity defines the rate of decay of any collection of k particles $x_1, ..., x_k$ and the measure $P^k(x_1, ..., x_k; d\mathbf{y})$ defines the distribution of possible outcomes. Supposing that any k particles from a given set of $n \geq k$ particles can interact, we arrive to the following generator of k-nary interacting particles defined by the kernel P^k:

$$(G_k f)(x_1, ..., x_n)$$

$$= \sum_{m=0}^{\infty} \sum_{I \subset \{1,...,.n\}, |I|=k} \int (f^{l+(n-k)}(\mathbf{x}_{\bar{I}}, y_1, ..., y_m) - f(x_1, ..., x_n)) P^k_m(x_I; dy_1...dy_m)$$

$$= \sum_{I \subset \{1,...,.n\}, |I|=k} \int (f_{\mathbf{x}_{\bar{I}}}(\mathbf{y}) - f(x_1, ..., x_n)) P^k(x_I, d\mathbf{y}), \tag{3.1}$$

where we have introduced the notation $f_{\mathbf{x}}(\mathbf{y}) = \mathbf{f}(\mathbf{x}, \mathbf{y})$. Taking into account all possible interactions of order $\leq k$ given by the kernels $P = \{P^l_m\}$, $l = 0, ..., k$, $m = 0, 1, ...$, with the intensity

$$P(\mathbf{x}) = \int P(\mathbf{x}; d\mathbf{y}) = \int \sum_{m=0}^{\infty} P^l_m(x_1, ..., x_l; dy_1, ..., dy_m) \tag{3.2}$$

whenever $\mathbf{x} = (x_1, ..., x_l)$ with $l \leq k$, and $P(\mathbf{x}) = 0$ whenever $\mathbf{x} \in X^l$ with $l > k$, yields the generator of a general pure jump Markov processes with the interaction of order $\leq k$.

$$(G_{\leq k} f)(x_1, ..., x_n) = \sum_{I \subset \{1,...,n\}} \int (f_{\mathbf{x}_{\bar{I}}}(\mathbf{y}) - f(x_1, ..., x_n)) P(\mathbf{x}_I, d\mathbf{y}). \tag{3.3}$$

Changing the state space by (2.1) yields the corresponding Markov process on $\mathcal{M}_\delta^+(X)$. Choosing a positive parameter h, we shall perform now the following scaling: firstly, we scale the empirical measures $\delta_{x_1} + ... + \delta_{x_n}$ by a factor h, secondly we scale all l-nary interactions by a factor h^l, and thirdly we scale the whole generator by $1/h$ (which effectively means the scaling of time as in the theory of superprocesses, see e.g. [Dy]). After this scaling the operator (3.3) takes the form

$$\Lambda^h f(h\nu) = \frac{1}{h} \sum_{l=0}^k h^l \sum_{I \subset \{1,...,n\}, |I|=l} \sum_{m=0}^\infty$$

$$\times \int [f(h\nu - \sum_{i \in I} h\delta_{x_i} + h\delta_{y_1} + ... + h\delta_{y_m}) - f(h\nu)] P(\mathbf{x}_I; dy_1 ... dy_m) \tag{3.4}$$

(we have omitted the index k from the notation of Λ for brevity) and acts on the space $B(\mathcal{M}_{h\delta}^+(X))$ of functions defined on the set $\mathcal{M}_{h\delta}^+(X)$ of measures of the form $h\nu = h\delta_{x_1} + ... + h\delta_{x_n}$. This generator defines our basic Markov model of (h-scaled) k-nary interacting exchangeable particles with a pure jump interaction.

2. Derivation of kinetic equations. By Proposition 2.1, operator (3.4) can be written in the form

$$\frac{1}{h} \sum_{l=0}^k \frac{1}{l!} \sum_{m=0}^\infty \int [f(h\nu - h\delta_{z_1} - ... - h\delta_{z_l} + h\delta_{y_1} + ... + h\delta_{y_m}) - f(h\nu)]$$

$$\times P(z_1, ..., z_l; dy_1 ... dy_m) \prod_{j=1}^l (h\delta_{x_1} + ... + h\delta_{x_n})(z_j)$$

$$+ \frac{1}{h} \sum_\Gamma \alpha_\Gamma h^{l-p} \int [f(h\nu - h\gamma_1 \delta_{z_1} - ... - h\gamma_p \delta_{z_p} + h\delta_{y_1} + ... + h\delta_{y_m}) - f(h\nu)]$$

$$\times P(z_1, ..., z_1, ..., z_p, ..., z_p; dy_1 ... dy_m) \prod_{j=1}^p (h\delta_{x_1} + ... + h\delta_{x_n})(z_j), \tag{3.5}$$

where $\sum_\Gamma$ is the sum over all Young schemes $\Gamma = \{1 \leq \gamma_1 \leq ... \leq \gamma_p\}$ with $\gamma_p > 1$ and $\gamma_1 + ... + \gamma_p = l \leq k$, and $P(z_1, ..., z_1, ... z_p, ... z_p; dy_1 ... dy_m)$ means that the first γ_1 arguments of P equal z_1, the next γ_2 arguments equal z_2 etc. As $h \to 0$ and $h\delta_{x_1} + ... + h\delta_{x_n}$ tends to some finite measure μ (i.e. the number of particles tends to infinity, but the "whole mass" remains finite due to the scaling of each atom), this tends to

$$(\Lambda f)(\mu) = \sum_{l=0}^{k} \frac{1}{l!} \sum_{m=0}^{\infty} \int [-\frac{\delta f}{\delta \mu(z_1)} - ... - \frac{\delta f}{\delta \mu(z_l)} + \frac{\delta f}{\delta \mu(y_1)} + ... + \frac{\delta f}{\delta \mu(y_m)}]$$

$$\times P(z_1, ..., z_l; dy_1 ... dy_m) \prod_{j=1}^{l} \mu(dz_j). \qquad (3.6)$$

The equation $\dot{f} = \Lambda f$ describing the evolution of functions on $\mathcal{M}^+(X)$ is a first order partial differential equation in functional derivatives. The kinetic equations we are looking for are the characteristics of this equation describing the evolution of measures themselves. Formally these characteristics can be found by writing down the corresponding evolution of a measure-valued function $f(\mu) = \mu$. As formally

$$\frac{\delta \mu}{\delta \mu(z)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon}(\mu + \epsilon \delta_z - \mu) = \delta_z,$$

the evolution of measures is given by the equation

$$\frac{d}{dt} \mu_t(dz) = \sum_{l=0}^{k} \frac{1}{l!} \sum_{m=0}^{\infty} \int [-\delta_{z_1}(z) - ... - \delta_{z_l}(z) + \delta_{y_1}(z) + \delta_{y_m}(z)]$$

$$\times P(z_1, ..., z_l; dy_1 ... dy_m) \prod_{j=1}^{l} \mu_t(dz_j)$$

(where it would be better for consistency to write $\delta_y(dz)$ instead of $\delta_y(z)$, but we kept here the standard convention denoting the Dirac measure as a generalized function). Using notations (2.6) and (2.2), one can rewrite this equation in the following concise form

$$\frac{d}{dt} \mu_t(.) = \int (\mathbf{1}^+_{(.)}(\mathbf{y}) - \mathbf{1}^+_{(.)}(\mathbf{z})) P(\mathbf{z}; d\mathbf{y}) \mu_t^{\tilde{\otimes}}(d\mathbf{z}) \qquad (3.7)$$

(where (.) denotes an arbitrary Borel set), which is precisely our general kinetic equation in the strong form.

3. Various forms of kinetic equations. Equation (3.7) can be equivalently written as

$$\frac{d}{dt} \mu_t(dz) = -\sum_{l=1}^{k} \frac{1}{(l-1)!} \mu_t(dz) \int_{z_1, ..., z_l} \mu_t(dz_1) ... \mu_t(dz_{l-1}) P(z, z_1, ..., z_{l-1})$$

$$+ \sum_{l=0}^{k} \frac{1}{l!} \sum_{m=1}^{\infty} m \int_{z_1, ..., z_l, y_1, ..., y_{m-1}} \mu_t(dz_1) ... \mu_t(dz_l) P(z_1, ..., z_l; dz dy_1 ... dy_{m-1}). \qquad (3.8)$$

It is often convenient to work with a weak form of these equations. In order to write it down explicitly, observe that for a linear function

$$f_g(\mu) = \int g(x)\mu(dx) \tag{3.9}$$

on $\mathcal{M}(X)$ one has $\frac{\delta f_g}{\delta \mu(z)} = g(z)$ and hence

$$(\Lambda f_g)(\mu) = \sum_{l=0}^{k} \frac{1}{l!} \int [g^+(\mathbf{y}) - g(z_1) - ... - g(z_l)] P(z_1, ..., z_l; d\mathbf{y}) \prod_{j=1}^{l} \mu(dz_j), \tag{3.10}$$

where we used notation (2.6). Hence, multiplying both sides of (3.7) by a function $g(z) \in C_b(X)$ and integrating over z, yields

$$\frac{d}{dt} \int g(z)\mu_t(dz) = \sum_{l=0}^{k} \frac{1}{l!} \int_{z_1,...,z_l,\mathbf{y}} [g^+(\mathbf{y}) - g(z_1) - ... - g(z_m)]$$

$$\times P(z_1, ..., z_l; d\mathbf{y}) \prod_{j=1}^{l} \mu_t(dz_j).$$

Using notation (2.4), one can rewrite this equation in a more concise form:

$$\frac{d}{dt} \int g(z)\mu_t(dz) = \int (g^+(\mathbf{y}) - g^+(\mathbf{z})) P(\mathbf{z}; d\mathbf{y}) \mu_t^{\tilde{\otimes}}(d\mathbf{z}). \tag{3.11}$$

We say that $\mu_t(dx)$ is a *weak solution* of the kinetic equation (3.7) if (3.11) holds for any $g(z) \in C_c(X)$, where the integral is well defined in the Lebesgue sense.

Another form of (3.7), (3.8) can be obtained if one is looking for measures having densities with respect to some given reference measure dx on X (say, Lebesgue measure on $\mathbf{R}^n$). The equation on densities can be obtained whenever the mapping $\pi(\mu_t)$ on the r.h.s. of (3.8) preserves the subspace of measures having densities with respect to dx, for example, if there exist measures $Q(y; dz_1...dz_l)$ such that

$$\sum_{m=1}^{\infty} m\, dz_1...dz_l \int_{y_1,...,y_m} P(z_1, ..., z_l; dy dy_1...dy_m) = Q(y; dz_1...dz_l) dy \tag{3.12}$$

for all l (a criterion for the existence of such Q is given in [BK]) . Then clearly equation (3.8) holds for measures of the form $\mu_t(dz) = \phi_t(z)dz$ whenever the densities ϕ_t satisfy the following kinetic equation on densities

$$\dot{\phi}_t(z) = -\sum_{l=1}^{k} \frac{1}{(l-1)!} \phi_t(z) \int_{z_1,...,z_{l-1}} \phi_t(z_1)...\phi_t(z_{l-1}) q(z, z_1, ..., z_{l-1}) dz_1...dz_{l-1}$$

$$+\sum_{l=0}^{k}\frac{1}{l!}\int_{z_1,...,z_l}\phi_t(z_1)...\phi_t(z_l)Q(z;dz_1...dz_l). \tag{3.13}$$

In discrete case, when $X = \mathbf{Z}^+ = \{1,2,...\}$, say, (3.8) takes the form

$$\dot{\nu}_j = -\sum_{l=1}^{k}\frac{1}{(l-1)!}\sum_{m=0}^{\infty}\sum_{i_1,...,i_{l-1}}\sum_{j_1,...,j_m} P^{j_1,...,j_m}_{j,i_1,...,i_{l-1}}\nu_j\nu_{i_1}...\nu_{i_l}$$

$$+\sum_{l=0}^{k}\frac{1}{l!}\sum_{m=1}^{\infty} m\sum_{i_1,...,i_l}\sum_{j_1,...,j_{m-1}} P^{j,j_1,...,j_{m-1}}_{i_1,...,i_l}\nu_{i_1}...\nu_{i_l}, \tag{3.14}$$

where we denoted by $\nu = \{\nu_j(t)\}$ the unknown measure on $\mathbf{Z}^+$ and omitted the dependence on t for brevity. Equivalently, by shifting the indexes, (3.14) can be written in the form

$$\dot{\nu}_j = \sum_{l=0}^{k}\frac{1}{l!}\sum_{m=0}^{\infty}\sum_{i_1,...,i_l}\sum_{j_1,...,j_m}\nu_{i_1}...\nu_{i_l}[(m+1)P^{j,j_1,...,j_m}_{i_1,...,i_l} - P^{j_1,...,j_m}_{j,i_1,...,i_l}\nu_j], \tag{3.15}$$

which is equation (4.9) from [BK]. A slightly different form of this equation was given in [Ko3].

4. Some examples of general kinetic equations. The most important particular case of discrete equations (3.15) correspond to the mass preserving (or mass exchange) processes, where the indexes in (3.15) are interpreted as masses and it is supposed that the rates $P^{j_1,...,j_m}_{i_1,...,i_l}$ may not vanish only if $i_1+...+i_l = j_1+...+j_m$. The mathematical theory of the equations of these kind was developed in [Ko3]. These equations correspond to a variety of processes including classical Smoluchovski's coagulation fragmentation equations with discrete mass distribution and its generalization with not necessary binary coagulations, coagulation equations with collision breakage, some known models of evolution of raindrops size spectra etc, see a fuller discussion and the relevant bibliography in [Ko3].

As a basic nontrivial example of general, not discrete, model, let us distinguish the processes that combine pure coagulations of no more than k particles, spontaneous fragmentation in no more than k pieces, and collisions (or collision breakages) of no more than k particles (with an arbitrary k). These processes are specified by the transition kernels $P^l_1(z_1,...,z_l) = K_l(z_1,...,z_l;dy)$, $l = 2,...,k$, called the coagulation kernels, the transition kernels $P^1_m(z;dy_1...dy_m) = F_m(z;dy_1...dy_m)$, $m = 2,...,k$, called the fragmentation kernels and the kernels

$$P^l_l(z_1,...,z_l;dy_1...dy_l) = C_l(z_1,...,z_l;dy_1...dy_2), \quad l = 2,...,k,$$

called the collision kernels, all other P^m_l are supposed to vanish. Then equation (3.11) takes the form

$$\frac{d}{dt}\int g(z)\mu_t(dz) = \sum_{l=2}^{k}\frac{1}{l!}\int_{z_1,...,z_l,y}[g(y)-g(z_1)-...-g(z_m)]K_l(z_1,...,z_l;dy)\prod_{j=1}^{l}\mu_t(dz_j)$$

$$+\sum_{m=2}^{k}\int_{z,y_1,...,y_m}[g(y_1)+...+g(y_m)-g(z)]F_m(z;dy_1...dy_m)\mu_t(dz)$$

$$+\sum_{l=2}^{k}\int[g(y_1)+...+g(y_l)-g(z_1)-...-g(z_l)]C_l(z_1,...,z_l;dy_1...dy_l)\prod_{j=1}^{l}\mu_t(dz_j). \quad (3.16)$$

In usual models, some function L on X is given that is preserved by the kernels K, F, C, which means that, say, the measures $K_l(z_1,...,z_l;.)$ are supported on the set $\{y : L(y) = L(z_1)+...+L(z_l)\}$.

In particular, in case of pure mass exchange processes, a particle is characterized by its mass only, $X = \mathbf{R}^+$ and $L(x) = x$ is interpreted as the mass of a particle. On the other hand, to get a (spatially homogeneous) Boltzmann equation, one sets all K_l and F_l to be zero, $X = \mathbf{R}^3$ (the space of velocities of particles) and takes the function L to be the kinetic energy v^2 of a particle with velocity v. The resulting equation can be called k-nary Boltzmann equation. In case when $C_l \neq 0$ only for $l \neq 2$ and (3.12) holds, the corresponding equation on densities is

$$\dot{\phi}_t(z) = -\phi_t(z)\int\phi_t(w)P(z,w)\,dw + \int\phi(z_1)\phi(z_2)Q(z;dz_1dz_2). \quad (3.17)$$

A particular case of this equation is given by the classical spatially homogeneous Boltzmann equation

$$\dot{\phi}_t(z) = \int_{n\in S^2, w:(n,z-w)\geq 0}[\phi(z-n(z-w,n))\phi(z+n(z-w,n))-\phi(z)\phi(w)]Q(z-w;dn)dw,$$

where $v, w \in \mathbf{R}^3$ and $Q(z;dn)$ is a certain measure on the unit sphere $S^2 \subset \mathbf{R}^3$. Application of our general results (given below) to this classical Boltzmann equation reproduces the classical well posedness results for this equation (see [Ce]) . As was found recently (see [MW]), a more carefull analysis of this particular equation yields a bit more precise results.

Classical Smoluchovski's coagulation equation corresponds to the case when only $P_1^2 = K_2$ in (3.16) does not vanish. The mathematical theory of such equation (3.16) (existence and uniqueness of solutions, convergence of stochastic approximations) was developed recently in [No1], [No2]. The existence and uniqueness of solutions for the case with vanishing C, $X = \mathbf{R}^+$ and K_l and F_l being non-vanishing only for $l = 2$, i.e. for the classical coagulation-fragmentation model

$$\frac{d}{dt}\int g(z)\mu_t(dz) = \int_{z_1,z_2}[g(z_1+z_2)-g(z_1)-g(z_2)]K(z_1,z_2)\mu_t(dz_1)\mu_t(dz_2)$$

$$+\int_0^z dy\int_0^\infty \mu_t(dz)[g(y)+g(z-y)-g(z)]F(z,y) \quad (3.18)$$

was obtained in [DS] for solution measures μ_t having densities with respect to Lebesgue measure and also finite exponential moments. Though in physical literature the model with collision breakage is well known (see e.g. [Du],[CR],[KK],[Sa]), mathematical results on even simplest pure binary collision breakage (only C_2 does not vanish in (3.16)) are available only for simplified models of discrete mass distribution (see [LW], [Ko3]). The full situation with non-vanishing F, K, C was not seemingly investigated mathematically even for pure binary case. As we noted in our general setting (when X is not necessarily $\mathbf{R}_+$ and L is not necessarily the mass) this model comprises also the Boltzmann collisions.

4. Existence of solutions and convergence of stochastic approximations.

This section is devoted to a justification of heuristic arguments of the previous section. Under some additional assumptions we are going to prove the existence of global solutions to (3.11) and on the convergence of approximating exchangeable particle systems to these solutions. The following definitions will play the crucial role in our analysis.

Let L be a non-negative function on X. As this function can be often interpreted as a mass or a size of a particle, we shall call the number $L(x)$ the size of a particle x. We say that the transition kernel $P(\mathbf{x}; d\mathbf{y})$ in (3.3) and the corresponding generators (3.3) or (3.6) are *L-subcritical* (respectively *L-critical*), if

$$\int (L^+(\mathbf{y}) - L^+(\mathbf{x}))P(\mathbf{x}; d\mathbf{y}) \leq 0 \tag{4.1}$$

for all $\mathbf{x}$ (respectively if the equality holds). We say that $P(\mathbf{x}; d\mathbf{y})$ in (3.3) is *L-preserving* (respectively *L-non-increasing*) if the measure $P(\mathbf{x}; d\mathbf{y})$ is supported on the set $\{\mathbf{y} : L^+(\mathbf{y}) = L^+(\mathbf{x})\}$ (respectively $\{\mathbf{y} : L^+(\mathbf{y}) \leq L^+(\mathbf{x})\}$). We say that $P(\mathbf{x}; d\mathbf{y})$ is *l-nary L-subcritical* (L-critical, L-preserving, L-non-increasing), if the corresponding property holds only for $\mathbf{x} \in X^l$. Clearly, if $P(\mathbf{x}; d\mathbf{y})$ is L-preserving (respectively L-non-increasing), then it is also L-critical (respectively L-subcritical). We shall say that the intensity (3.2) is *multiplicatively L-bounded* or *$L^{\otimes}$-bounded* (respectively *additively L-bounded* or *L^+-bounded*) whenever $P(\mathbf{x}) \leq cL^{\otimes}(\mathbf{x})$ (respectively $P(\mathbf{x}) \leq cL^+(\mathbf{x})$) for all $\mathbf{x}$ and some constant $c > 0$, where we used the notations (2.5), (2.6). In the future, we shall always take $c = 1$ here for brevity. We shall say that the intensity (3.2) is *strongly multiplicatively L-bounded* whenever $P(\mathbf{x}) = o(1)_{\mathbf{x}\to\infty} L^{\otimes}(\mathbf{x})$, i.e. if

$$P(x_1, ..., x_l) = o(1)_{(x_1,...,x_l)\to\infty} \prod_{j=1}^{l} L(x_j) \tag{4.2}$$

for $l = 1, ..., k$. We shall need also some conditions that forbid the creation of dust, i.e. of a large number of small particles. Various conditions of this kind are possible. We choose the following two no dust conditions (for a given L):

(ND1) for any $C > 0$ there exists $M_C > 0$ such that not more than M_C particles $y_1, ..., y_l$ of the size $L(y_j) < C$ can be created during one act of the interaction of any family of particles $\mathbf{x}$, i.e. for any m and $I \subset \{1, ..., m\}$ with $|I| > M_C$, the support of the measure $P(\mathbf{x}; .)$ has no intersection with the set $\{\mathbf{y} = (y_1, ..., y_m) : L(y_{i_j}) < C \, \forall \, j \in I\}$,

(ND2) there exists $r > 0$ or $r = \infty$ such that the kernel $P(\mathbf{x}; d\mathbf{y})$ is l-nary $\mathbf{1}_{L(.)<r}$-subcritical for all $l \geq 2$ (i.e. the expectation of the number of particles of size less than r

does not increase by l-nary interactions with $l \geq 2$); moreover, in case $r = \infty$, the number of particles created by an act of l-nary interaction with $l = 1$ is uniformly bounded, i.e. $P^1_m(x; dy_1...dy_m) = 0$ for all x and $m > m_0$ with some m_0.

(ND3) if $k > 2$, then either P is $\mathbf{1}$-non-increasing (in particular, no fragmentation is allowed), or $L(x) \geq \epsilon$ for all x with some $\epsilon > 0$ (i.e. no particle of size less than ϵ can appear in the system).

Remarks. The notion of L-subcriticality is borrowed from the theory of branching processes and superprocesses, where it is used usually with $L = \mathbf{1}$ (see also [Ko2]). The assumptions of $L^{\otimes}$- and L^{+}-boundedness constitute natural generalizations of two basic assumptions on the coagulation kernel used in the theory of Smoluchovski's equation (see e.g. [BC] and [No1]). Condition (ND1) is trivially satisfied for coagulation- fragmentation processes whenever the number of particles created by one act of fragmentation is assumed to be bounded. Condition (ND2) deals only with $l \geq 2$, so it trivially holds for the standard Smoluchovski coagulation-fragmentation model. Condition (ND2) with $r = \infty$ means that the expectation of the number of particles does not increase for interactions with $l \geq 2$. On the other hand, since r can be arbitrary small, (ND2) can be considered generally as not very restrictive technical assumption. Notice at last that (ND1) and (ND2) (with $r = \infty$) hold trivially for our basic example (3.16). At last, let us stress that (ND3) is void for usual binary interactions and is the strongest additional condition that helps to overcome some additional technical difficulties arising for interactions of order more than 2.

We begin now our analysis of the Markov processes defined by (3.3) and (3.4).

Proposition 4.1. *Suppose the transition kernel $P(\mathbf{x}, .)$ is a continuous function from $S\mathcal{X}$ to $\mathcal{M}^+(S\mathcal{X})$, where $\mathcal{M}^+(S\mathcal{X})$ is considered with its weak topology, the family P^l_m in (3.3) is L-subcritical for some continuous non-negative function L on X such that every set $\{x : L(x) \leq a\}$ is a compact for any a (for example, $L(x) \to \infty$ whenever $x \to \infty$), the intensity (3.2) is $((1 + L)^{\otimes}$-bounded and conditions (ND1), (ND2) hold. Suppose also for simplicity that $P^0(dy) = 0$, i.e. that no spontaneous input in the system is possible. Then*

(i) the corresponding Markov processes $Z^{h\nu}(t)$ ($h\nu$ denotes as usual the starting point) in $\mathcal{M}^+_{h\delta}(X) \subset \mathcal{M}^+(X)$ are uniquely defined by their generators (3.3), (3.4), and the process $f_L(Z^{h\nu}(t))$ is a non-negative supermartingale, where f_L is defined by (3.9);

(ii) given arbitrary $b > 0$, $T > 0$, for a family $Z^{h\nu}(t)$ with $h \in (0, 1]$ and initial $h\nu$ with $h \int (\mathbf{1} + L)(x)\nu(dx) \leq b$

$$\mathbf{P}\left(\sup_{t\in[0,T]} f_{L+1}(Z^{h\nu}(t)) > r\right) \leq \frac{C(T, b, M_2)}{r} \tag{4.3}$$

for all $r > 0$, where the constant $C(T, b, M_2)$ depends on T,b and M_2 from (ND1) (but not on h); moreover, such a family $Z^{h\nu}(t)$ enjoys the compact containment condition, i.e. for arbitrary $\eta > 0$, $T > 0$ there exists a compact subset $\Gamma_{\eta,T} \subset \mathcal{M}^+(X)$ for which

$$\inf_{h\nu} \mathbf{P}(Z^{h\nu}(t) \in \Gamma_{\eta,T} \quad for \quad 0 \leq t \leq T) \geq 1 - \eta; \tag{4.4}$$

(iii) if the family of measures $\nu = \nu(h)$ is such that $f_{1+L}(h\nu)$ is uniformly bounded, then the family of processes $Z^{h\nu}(t)$ from (ii) is tight as a family of processes with sample paths in $D_{\mathcal{M}^+(X)}[0, \infty)$.

Proof. (i) By L-subcriticallity and by (3.4), (4.1)

$$\Lambda^h f_L(h\nu) \le 0, \quad \nu \in \mathcal{M}^+_\delta(X).$$

By (ND2), (ND1) and Proposition 2.1

$$\Lambda^h f_{\mathbf{1}_{L(.)\le r}}(h\nu) \le \int (\mathbf{1}^+_{L(.)\le r}(\mathbf{y}) - \mathbf{1}_{L(.)\le r}(z)) P(z; d\mathbf{y}) h\nu(dz) \le M f_{1+L}(h\nu),$$

where $M = M_r$ from (ND1) in case $r < \infty$, and $M = m_0$ from (ND2) in case $r = \infty$. Moreover, the intensity q of the q-pair corresponding to the generator (3.4) equals

$$q(h\nu) = \frac{1}{h}\sum_{l=1}^k h^l \sum_{I\subset\{x_1,\dots,x_n\},|I|=l} \int P(\mathbf{x}_I; d\mathbf{y}) \le \frac{1}{h}\sum_{l=1}^k \frac{1}{l!}(f_{1+L}(h\nu))^l. \tag{4.5}$$

for $\nu = (\delta_{x_1} + \dots + \delta_{x_n})$. Hence the conditions of Proposition 2.2 (ii) hold with $f_{\mathbf{1}_{L\le r}+L}$ playing the role of the barrier ψ.

(ii) Estimate (4.3) follows from (i) and formula (2.11) of Proposition 2.1. Condition (4.4) follows from (4.3) and the standard Prokhorov criterion of compactness for a family of measures.

(iii) By the well known Jacubovski criterion (see e.g. [EK] or [Da]), when the compact containment condition (4.4) is proved, in order to get tightness it is enough to show the tightness of the family of the real valued processes $f(Z^{h\nu}(t))$ (as a family of processes with sample paths in $D_{\mathbf{R}}[0,\infty)$) for any f from a dense subset (in the topology of uniform convergence on compact sets) of $C_b(\mathcal{M}^+(X))$. By Weierstrass theorem, it is thus enough to verify tightness of $f(Z^{h\nu}(t))$ for f from the algebra generated by f_g of type (3.9) with $g \in C_c(X)$. Let us show it for $f = f_g$ (as seen from the proof, it is straightforward to generalize it to the sums and products of these functions). By Propositions 2.2 and 4.1, $f_g(Z^{h\nu}(t))$ is a supermartingale and moreover (by (2.9)), the process

$$M_g(t) = f_g(Z^{h\nu}(t)) - f_g(h\nu) - \int_0^t \Lambda^h f_g(Z^{h\nu}(s))\, ds \tag{4.6}$$

is a martingale for any $g \in C_c(X)$. By standard criteria of tightness of real valued supermartingales, to get a tightness for $f_g(Z^{h\nu}(t))$, one needs to estimate the oscillations of the predictable finite variation component of $f_g(Z^{h\nu}(t))$, which is given by the integral in (4.6) and for which such an estimate is thus obvious, and for the quadratic variation $[M_g(t)]$ of the martingale (4.6). In fact, we shall show that for an arbitrary ϵ,

$$\mathbf{P}\left([M_g(t)] - [M_g(s)] \le \sigma h(t-s)\right) \ge 1 - \epsilon \tag{4.7}$$

with some constant σ uniformly for all $0 \le s \le t \le T$ and an arbitrary T, and this implies the required tightness.

Since the process $Z^{h\nu}(t)$ is a pure jump process,

$$[M_g(t)] = \sum_{s\le t} (\Delta f_g(Z^{h\nu}(s)))^2, \tag{4.8}$$

where $\Delta Z(s) = Z(s) - Z(s_-)$ denotes the jump of a process $Z(s)$. Since g has a compact support and due to (ND1), the number of particles created inside the support of g is bounded by some constant C, it follows from (3.4) that

$$|\Delta f_g(Z^{h\mu}(s))|^2 \le \|g\|^2 (C+k)^2 h^2$$

for any s. Clearly now in order to get (4.7) it is enough to show that the number of jumps on any interval $[s,t]$ is of order $(t-s)/h$ uniformly for all $s \le t \le T$ with any T with probability arbitrary close to one. But as our Markov process is a pure jump process, the number of jumps on the interval $[s,t]$ is controlled by the product of $(t-s)$ and the maximal intensity on $[0,T]$ (for example, by the standard Levy formula for (4.8), see e.g. [Br]). So, it is enough to prove that the maximal intensity is of order $1/h$. But for $\nu = \delta_{x_1} + ... + \delta_{x_n}$, one gets from (4.5) that $q(h\nu)$ is of order $1/h$ with the probability arbitrary close to one due to the compact containment condition. This proves (4.6) and hence completes the proof of the Proposition 4.1.

The next result addresses the properties of L^+-bounded intensities.

Proposition 4.2. *Under the conditions of Proposition 4.1, suppose additionally that (3.3) is L-non-increasing and* $(1+L)^+$*-bounded and (ND3) holds. Then for any* $\beta \ge 1$

$$\mathbf{E}(f_{L^\beta}(Z^{h\nu}(s))) \le a(t) f_{L^\beta}(Z^{h\nu}(0)) + b(t) \tag{4.9}$$

uniformly for all $s \in [0,t]$ *with an arbitrary* t*, and with some constants* $a(t), b(t)$ *depending on* t *(and* β*), but not on* h*.*

Proof. As the process is L-non-increasing, the process $Z^{h\nu}(s)$ lives on measures with support on a compact space $\{x \in X : L(x) \le c/h\}$ with some constant c. Hence $f_{L^\beta}(Z^{h\nu}(s))$ is uniformly bounded and one can apply Dynkin's formula, i.e. the fact that (4.6) is a martingale for $g = L^\beta$. Hence

$$\mathbf{E}(f_{L^\beta}(Z^{h\nu}(t))) \le f_{L^\beta}(h\nu) + \int_0^t \mathbf{E}\Lambda^h(f_{L^\beta}(Z^{h\nu}(s)))\, ds. \tag{4.10}$$

Since (3.3) is L-non-increasing,

$$\int (L^\beta)^+(\mathbf{y}) P(\mathbf{x}, d\mathbf{y}) \le \int (L^+(\mathbf{y}))^\beta P(\mathbf{x}, d\mathbf{y}) \le (L^+(\mathbf{x}))^\beta P(\mathbf{x}, d\mathbf{y}),$$

and consequently for $\nu = \delta_{x_1} + ... + \delta_{x_n}$

$$(\Lambda^h f_{L^\beta})(h\nu) \le \frac{1}{h} \sum_{l=1}^k h^l \sum_{|I|=l} [(L^+(\mathbf{x}_I))^\beta - (L^\beta)^+(\mathbf{x}_I)] P(\mathbf{x}_I)$$

$$\le \sum_{l=1}^k \frac{1}{l!} \int [(L(y_1) + ... + L(y_l))^\beta - L^\beta(y_1) - .... - L^\beta(y_l)] P(y_1, ..., y_l) \prod_{j=1}^l (h\nu(dy_j)),$$

where we used Proposition 2.1. Using the symmetry with respect to permutations of $x_1, ..., x_n$, we conclude that $(\Lambda^h f_{L^\beta})(h\nu)$ does not exceed

$$\sum_{l=2}^{k} \frac{1}{(l-1)!} \int [(L(y_1) + ... + L(y_l))^\beta - L^\beta(y_1) - .... - L^\beta(y_l)](1 + L(y_1)) \prod_{j=1}^{l} (h\nu(dy_j)).$$

Using the elementary inequalities

$$(a+b)^\beta - a^\beta - b^\beta \leq w^1_\beta (ab^{\beta-1} + ba^{\beta-1}),$$

$$a((a+b)^\beta - a^\beta - b^\beta) \leq w^2_\beta (ab^\beta + ba^\beta)$$

(that hold for all $\beta \geq 1$ and positive a, b with some constants w^1_β, w^2_β depending only on β, see e.g. [Ca] for a proof of the second one) with $a = L(y_1)$, $b = L(y_2) + ... + L(y_l)$, and again the symmetry yields for the last expression the estimate

$$\sum_{l=2}^{k} \kappa^l_\beta (L^\beta(y_1) + L^{\beta-1}(y_1)) L(y_2) \prod_{j=1}^{l} (h\nu(dy_j)) \tag{4.11}$$

with some constants κ^l_β. In case $k > 2$, which does not exceed $\tilde{a} f_{L^\beta}(h\nu) + \tilde{b}$ with some $\tilde{a}, \tilde{b}$, as (ND3) implies that $f_{L+1}(h\nu)$ is uniformly bounded almost surely. From this estimate, (4.10), and Gronwall's lemma, one obtains (4.9). In case $k = 2$ one sees that $\|Z^{h\nu}(t)\|$ is not involved in (4.11) and we get the same conclusion as for $k > 2$ using only the boundedness of $f_L(Z^{h\nu}(t))$ (and induction in β if necessarily).

We can now prove the main result of this section.

Theorem 4.1. *Under assumptions of Proposition 4.1 suppose additionally that the family of initial measures $h\nu$ converges (weakly) to a measure $\mu \in \mathcal{M}^+(X)$ and that either*

(i) the intensity (3.2) is strongly $(1+L)^\otimes$-bounded, or

(ii) the initial conditions $h\nu$ are such that $\int (L^\beta)(x) h\nu(dx) \leq C$ for all $h\nu$ and some constants $\beta > 1, C > 0$, the intensity (3.2) is $(1+L^\alpha)^+$-bounded with some $\alpha \in [0,1]$ and (3.3) is L-non-increasing and (ND3) holds.

Then there is a subsequence of the family of processes $Z^{h\nu}(t)$ that weakly converges to a global non-negative solution μ_t of the integral version of the weak equation (3.11), i.e.

$$\int g(z)\mu_t(dz) - \int g(z)\mu_0(dz) - \int_0^t ds \int (g^+(\mathbf{y}) - g^+(\mathbf{z})) P(\mathbf{z}; d\mathbf{y}) \mu_s^{\tilde{\otimes}}(d\mathbf{z}) = 0 \tag{4.12}$$

for all t and all $g \in C_c(X)$. Under assumptions (ii)

$$\sup_{t\in[0,T]} \int (\mathbf{1} + L^\beta)(x)\mu_t(dx) \leq C(T) \tag{4.13}$$

with some constant $C(T)$ for an arbitrary T.

Moreover, if under assumptions (ii) the intensity is not only L-non-increasing but also L-preserving, then the obtained solution μ_t is also L-preserving, i.e. for all t

$$\int L(x)\mu_t(dx) = f_L(\mu_t) = f_L(\mu_0) = \int L(x)\mu_0(dx).$$

Proof. By Proposition 4.1 we can choose a sequence of positive numbers tending to zero such that the family $Z^{h\nu}(s)$ is weakly converging as $h \to 0$ and belong to this sequence. Let us denote the limit by μ_t and prove that it satisfies (4.12). By Skorohod's theorem, we can and will assume that $Z^{h\nu}(s)$ converges to μ_t almost surely. The idea now is to pass to the limit in equation (4.6). Due to (4.7), the martingale on the left hand side of (4.6) tends to zero almost surely. Moreover, due to our estimates of (4.8), any limiting process μ_t has continuous sample paths almost surely. The positivity of μ_t and the estimate (4.13) follow from the corresponding properties of $Z^{h\nu}(t)$.

Clearly, the first two terms on the r.h.s. of (4.6) tend to the first two terms on the l.h.s. of (4.12). So, we need to show that the integral on the r.h.s. of (4.6) tends to the last integral on the l.h.s. of (4.12). Let us show first that

$$|\Lambda^h f_g(Z^{h\nu}(s)) - \Lambda f_g(Z^{h\nu}(s))| \to 0, \quad h \to 0, \tag{4.14}$$

uniformly for $s \in t$. As by (3.5) for any $\eta = \delta_{v_1} + ... + \delta_{v_m}$

$$\Lambda^h f_g(h\eta) = \sum_{l=1}^k h^l \sum_{I \subset \{1,...,m\}, |I|=l} \int \left(g^+(\mathbf{y}) - \sum_{i \in I} g(v_i) \right) P(\mathbf{v}_I; d\mathbf{y})$$

$$= \sum_{l=0}^k \frac{1}{l!} \int [g^+(\mathbf{y}) - g(z_1) - ... - g(z_l)] P(z_1, ..., z_l; d\mathbf{y}) \prod_{j=1}^l (h\eta(dz_j))$$

$$+ \sum_\Gamma \alpha_\Gamma h^{l-p} \int (g^+(\mathbf{y}) - \gamma_1 g(z_1) - ... - \gamma_p g(z_p)) P(z_1, ..., z_1, ..., z_p, ..., z_p; d\mathbf{y}) \prod_{j=1}^p (h\eta(dz_j)),$$

in order to prove (4.14), one needs to show that all

$$h^{l-p} \int_{z_1,...,z_p,\mathbf{y}} P(z_1, ..., z_1, ..., z_p, ..., z_p; d\mathbf{y}) \prod_{j=1}^p (h\eta(dz_j)) \tag{4.15}$$

tend to zero as $h \to 0$ uniformly for all $\eta \in \mathcal{M}_\delta^+(X)$ with uniformly bounded $f_L(h\eta)$.

In case when (3.3) is $(1 + L)^+$ bounded, this is evident, as each integral here is uniformly bounded for all ν with uniformly bounded $f_{L+1}(h\nu)$.

Consider the case when (3.3) is strongly $(1 + L)^\otimes$-bounded. As the boundedness of $f_L(h\eta)$ implies, in particular, that the support of all η is contained in a set where L is bounded by c/h with some constant c, in order to prove that all terms (4.15) tend to zero it is enough to show that for any i

$$h^i \int_{z:L(z) \le ch^{-1}} (1 + L(z))^{i+1} o(1)_{z\to\infty} (h\eta)(dz) \to 0, \quad h \to 0. \tag{4.16}$$

To this end, let us write the integral in (4.16) as the sum of two integrals over the set $|z| \leq K$ and $|z| \geq K$ with some $K > 0$. Then the first term clearly tends to zero, as $L(z)$ is bounded. The second term does not exceed

$$\int_{z:L(z)\geq K} (1 + L(z))o(1)_{K\to\infty}(h\eta)(dz)$$

and tends to zero as $K \to \infty$.

When (4.14) is proved, it remains to show that

$$|\Lambda f_g(Z^{h\nu}(t)) - \Lambda f_g(\mu_t)| \to 0,$$

or more explicitly that the integral

$$\int_0^t ds \int (g^+(\mathbf{y}) - g^+(\mathbf{z}))P(\mathbf{z}; d\mathbf{y})[\tilde{\mu}_s^{\otimes}(d\mathbf{z}) - \tilde{Z}^{h\nu}(s)^{\otimes}(d\mathbf{z})] \tag{4.17}$$

tends to zero as $h \to 0$. But from a weak convergence (and the fact that μ_t has continuous sample paths) it follows (see e.g. [EK]) that $Z^{Nh}(s)$ converges to μ_s for all $s \in [0, t]$. Hence we need to show that

$$\int P(\mathbf{z}; d\mathbf{y})[\tilde{\mu}_s^{\otimes}(d\mathbf{z}) - \tilde{Z}^{h\nu}(s)^{\otimes}(d\mathbf{z})] \tag{4.18}$$

tends to zero as $h \to 0$ under condition that $Z^{h\nu}(t)$ weakly converges to μ_t.

To prove (4.18) consider first the case when our intensity is strongly $(1+L)^{\otimes}$-bounded. Decompose the integral in (4.18) into the sum of two integrals by decomposing the domain of integration into the domains $\{\mathbf{z} = (z_1, ..., z_m) : \max L(z_j) \geq K\}$ and its complement. By the condition of $(1+L)^{\otimes}$-boundedness, both integrals from (4.18) over the first domain tend to zero as $K \to \infty$. And on the second domain, the integrand is uniformly bounded and hence the weak convergence of $Z^{h\nu}(s)$ to μ_t ensures the smallness of the l.h.s. of (4.18).

Suppose now conditions (ii) of the Theorem hold. As above, we decompose the domain of integration into two parts. The second part is dealt with precisely as above. Hence we need only to show that for any l, by choosing K arbitrary large, we can make

$$\int_{L(z_1)\geq K} (1 + L(z_1))[\prod_{j=1}^{l} \mu_s(dz_j) + \prod_{j=1}^{l} Z^{h\nu}(s)(dz_j)]$$

arbitrary small. But this integral does not exceed

$$K^{-(\beta-1)} \int (1 + L(z_1))(L(z_1))^{\beta-1}[\prod_{j=1}^{l} \mu_s(dz_j) + \prod_{j=1}^{l} Z^{h\nu}(s)(dz_j)],$$

which is finite and tends to zero as $K \to \infty$ due to Proposition 4.2.

It remains to show that μ_t is L-preserving whenever the transition kernel $P(\mathbf{x}; d\mathbf{y})$ is L-preserving. As the approximations $Z^{h\nu}$ are then L-preserving, we only need to show that

$$\lim_{h\to 0} \int L(x)(\mu_t - Z^{h\nu}(t))(dx) = 0. \tag{4.19}$$

But this is done as above. Decomposing the domain of integration into two parts: $\{x : L(x) \leq K\}$ and its complement, we observe that the required limit for the first integral is zero due to the weak convergence of $Z^{h\nu}(t)$ to μ_t, and on the second part both integrals from (4.19) can be made arbitrary small by choosing K large enough, due to the Proposition 4.2.

For conclusion, let us discuss the regularity of the solutions of (4.12) constructed in Theorem 4.1 showing in particular that they are in fact solutions to both weak and strong equations (3.11) and (3.7). More generally, the following holds.

Theorem 4.2. *Let μ_t be a solution to (4.12) satisfying (4.13) with some $\beta > 1$. Moreover, let the intensity (3.2) be $(\mathbf{1} + L^\alpha)^+$-bounded for some $\alpha \in [0,1]$ and (3.3) be L-non-increasing. Then*

(i) (4.12) holds not only for $g \in C_c(X)$ but for any measurable g on X such that $|g| \leq C(1 + L^{\beta-\alpha})$ with some constant C,

(ii) the function $\int g(x)\mu_t(dx)$ is an absolutely continuous function of t for all g from (i) and it is continuous for any measurable g such that $|g(x)| \leq C(\mathbf{1} + L^\beta)(x)$,

(iii) for any g from (i), the function $\int g(x)\mu_t(dx)$ is a continuously differentiable function of t and (3.11) holds,

(iv) the function $t \mapsto \mu_t$ is a continuously differentiable function in the sense of the total variation norm topology of $\mathcal{M}(X)$ and (3.7) holds in the strong sense, i.e. the derivative is again understood in the sense of the total variation norm of $\mathcal{M}(X)$.

Proof. (i) One shows that (4.12) holds for all bounded measurable functions with a compact support by a straightforward limit argument. Hence, it is enough to prove the claim for measurable g such that $|g| \leq L^{\beta-\alpha}$. This is obtained again by approximating g by functions with a compact support using the dominated convergence theorem and the estimate

$$\int g^+(\mathbf{y})P(\mathbf{z}; d\mathbf{y})\mu_s^{\tilde{\otimes}}(d\mathbf{z}) \leq \int (L^{\beta-\alpha})^+(\mathbf{y})P(\mathbf{z}; d\mathbf{y})\mu_s^{\tilde{\otimes}}(d\mathbf{z})$$

$$\leq \int (L^{\beta-\alpha})^+(\mathbf{z})(\mathbf{1} + L^\alpha)^+(\mathbf{z})\mu_s^{\tilde{\otimes}}(d\mathbf{z})$$

when passing to the limit on both sides of (4.12).

(ii) It follows directly from (4.12) that the function $\int g(x)\mu_t(dx)$ is absolutely continuous for any g from (i). The required continuity for more general g is obtained again by approximating these g by functions with a compact support using the observation that the integrals of g and its approximations over the set $\{x : L(x) \geq K\}$ can be made uniformly arbitrary small for all μ_t, $t \in [0,T]$, by choosing K large enough.

(iii) For g from (i), equation (3.11) holds almost everywhere, which follows from (4.12).

But as it follows from (ii), the integral

$$\int \phi(z_1, ..., z_l) \prod_{j=1}^{l} \mu_t(dz_j)$$

is a continuous function of t for any symmetric measurable function ϕ on X^l such that

$$|\phi(z_1, ..., z_l)| \leq C \prod_{j=1}^{l} (\mathbf{1} + L^\beta)(z_j)$$

(for all $|z_j| \leq K$ with an arbitrary K we can approximate these ϕ by products, and outside this domain all integrals are uniformly small). Consequently the r.h.s. of (3.11) is continuous, which implies the claim in (iii).

(iv) Choosing g to be characteristic functions of Borel sets, we get (3.7) set-wise. As

$$\|\mu_t - \mu_s\| = \sup_{g:|g|\leq 1} \int g(x)(\mu_t - \mu_s)(dx) = O(t-s)$$

uniformly for $0 \leq s \leq t \leq T$ with an arbitrary T, we conclude that $\|\mu_t\|$ is an absolutely continuous function of t in the sense of the norm topology on $\mathcal{M}(X)$. By the above reasoning we conclude that the norm continuity of the map $t \mapsto \mu_t$ implies the norm continuity of the map $t \mapsto L^\alpha(x)\mu_t(dx)$, which implies the norm continuity of the r.h.s. of (3.7). To complete the proof of the Theorem, it remains to notice that if $\dot{\mu}_t = \nu_t$ set-wise and if $t \mapsto \nu_t$ is continuous in the sense of the norm in $\mathcal{M}(X)$, then $\dot{\mu}_t = \nu_t$ holds in the sense of the norm topology. In fact,

$$\mu_t(B) = \mu_s(B) + (t-s)\nu_s(B) + \int_s^t (\nu_\tau(B) - \nu_s(B))\, ds,$$

and the norm of the integral on the r.h.s. is of order $o(t-s)$ as $t \to s$, whenever $\|\nu_t - \nu_s\| = 0(1)$ as $t \to s$.

To conclude this section let us notice that the regularity of the solution μ_t (the number of continuous derivatives) increases with the growth of β in (4.13). As by Proposition 4.2 estimate (4.13) for μ_t follows from the corresponding estimate for μ_0, the regularity of μ_t depends on the rate of decay of μ_0 at infinity. A criterion for infinite smoothness is given in the next statement.

Proposition 4.3. *If μ_0 in (4.12) has a finite exponential moment, i.e.*

$$\int \exp\{\omega L(x)\}\mu_0(dx) < \infty$$

for some $\omega > 0$, the solution μ_t obtained by Theorem 4.1 (ii) (which is the unique solution by Theorem 6.1 below) is infinitely differentiable in t (with respect to the norm topology in $\mathcal{M}^+(X)$).

Proof. As in Proposition 4.2 one shows that the condition of Proposition 4.3 implies the estimate

$$\sup_{t\in[0,T]}\int \exp\{\omega L(x)\}\mu_t(dx) < \infty.$$

And this estimate implies that the derivatives of the r.h.s. of (3.7) of all orders are finite and continuous functions.

5. Approximation by Markov chains.

This short section is devoted to a possible approximation of the dynamics given by kinetic equations by means of Markov chains obtained by a discretization of X. For a positive ω, a *marked ω-partition* of X, is a pair (X^ω, U^ω) where $X^\omega = \{M_j\}$, $j = 1, 2, ...$, is a countable subset of X and $U^\omega = \{U_j^\omega\}$ is a sequence of pairwise disjoint Borel subsets of X with the diameters not exceeding ω forming its partition, i.e. $X = \cup_{j=1}^\infty U_j$, such that $M_j \in U_j$ for all j. We shall write sometimes $U(M_j)$ for U_j that contains M_j. Clearly the countable set $X^\omega = \{M_1, M_2, ...\} \subset X$ is a ω-net of X, i.e. for any $x \in X$ there exists M_j such that $d(x, M_j) \leq \omega$.

Remark. If $X = \mathbf{R}^d$, the lattice $\omega\mathbf{Z}^d$ is the most natural candidate for X^ω.

We shall say that a marked ω-partition of X is *L-admissible* if (i) for all j

$$L(M_j) = \min\{L(y) : y \in U_j\}$$

and (ii) $L(y) < r$ for all $y \in U_j$ whenever $L(M_j) < r$, where r is from condition (ND2). It is easy to see that L-admissible marked ω-partitions exist for any ω.

To an integral operator of type (3.3) on $B(S\mathcal{X})$ there corresponds a finite-difference operator $G^\omega_{\leq k}$ on $B(S\mathcal{X}^\omega)$ (also of type (3.3)) defined by

$$G^\omega_{\leq k} f(x_1, ..., x_n) = \sum_{I\subset\{1,...,n\}} \sum_{m=0}^{\infty} \sum_{q_1,...,q_m} P^{q_1,...,q_m}_{\mathbf{x}_I}(\omega)(f(\mathbf{x}_{\bar I}, q_1, ..., q_m) - f(x_1, ..., x_m)), \tag{5.1}$$

$x_j, q_j \in X^\omega$, where

$$P^{q_1,...,q_m}_{x_1,...,x_l}(\omega) = P^l_m(x_1, ..., x_l; U(q_j) \times ... \times U(q_m)) \tag{5.2}$$

The corresponding scaled operator of type (3.4) has the form

$$\Lambda^{h,\omega} f(h\nu) = \frac{1}{h}\sum_{l=0}^{k} h^l \sum_{I\subset\{1,...,n\},|I|=l} \sum_{m=0}^{\infty}$$

$$\times \sum_{q_1,...,q_m} [f(h\nu - \sum_{i\in I} h\delta_{x_i} + h\delta_{y_1} + ... + h\delta_{y_m}) - f(h\nu)] P^{q_1,...,q_m}_{\mathbf{x}_I}(\omega). \tag{5.3}$$

Let us denote by $Z^{h\nu}_\omega$ the Markov chain on $\mathcal{M}^+_{h\delta}(X^\omega) \subset \mathcal{M}^+(X)$ specified by generator (5.3) and the initial condition $h\nu$. The L-admissibility of a marked ω-partition implies that if the assumptions of Proposition 4.1 hold for (3.4) and the corresponding transition kernel is L-non-increasing, then the discrete kernel (5.2) is also L-non-increasing and the assumptions of Proposition 4.1 hold also for (5.3). This implies, in particular, that $Z_\omega^{h\nu}$ are well defined, and moreover, as all estimates are uniform for $\omega \leq 1$, the family of processes $Z_\omega^{h\nu}$, $h \in (0,1]$, $\omega \in (0,1]$ is tight.

The following result is an analogue of Theorem 4.1 for Markov chain approximations.

Proposition 5.1. *Under the assumptions of Theorem 4.1 (ii) there is a subsequence of the family of processes $Z_\omega^{h\nu}$ that weakly converges (as $h \to 0$, $\omega \to 0$) to a global weak solution μ_t of the corresponding kinetic equation, i.e. μ_t satisfies (3.11).*

Proof. This is the same as the proof of Theorem 4.1 (supplemented by Proposition 4.2). The only difference is the necessity to prove

$$|\Lambda^{h,\omega} f_g(Z_\omega^{h\nu}(s)) - \Lambda f_g(Z_\omega^{h\nu}(s))| \to 0, \quad h \to 0, \quad \omega \to 0, \tag{5.4}$$

instead of (4.14). But as the terms of type (4.15) are small for small h, to get (5.4) one only needs to show that the difference between

$$= \sum_{l=1}^{k} \frac{1}{l!} \int [g^+(\mathbf{y}) - g(z_1) - ... - g(z_l)] P(z_1, ..., z_l; d\mathbf{y}) \prod_{j=1}^{l} (h\eta(dz_j))$$

and

$$= \sum_{l=1}^{k} \frac{1}{l!} \sum_{m=0}^{\infty} \sum_{q_1,...,q_m} [g(q_1) + ...g(q_m) - g(z_1) - ... - g(z_l)] P_{z_1,...,z_l}^{q_1,...,q_m} \prod_{j=1}^{l} (h\eta(dz_j))$$

tends to zero as $h \to 0$. But this holds by (5.2), as g has a compact support.

6. Uniqueness and continuous dependence on initial data.

The following theorem is the main result of this paper.

Theorem 6.1. *(i) Under the conditions of Theorem 4.1 (ii) with $\beta \geq \alpha + 1$, there exists a unique non-negative solution μ_t to (3.11) satisfying (4.13) and a given initial condition μ_0 such that $\int L^\beta(x)\mu_0(dx) < \infty$. This μ_t is a strong solution of the corresponding kinetic equation, i.e. it satisfies (3.7), where the derivative is understood in the sense of the variation norm in $\mathcal{M}(X)$.*

(ii) Moreover, the whole family of processes $Z^{h\nu}(t)$ from Theorem 4.2 (ii) (not just its subsequence) with the weak limit μ_0 of $Z^{h\nu}(0)$ converges weakly to μ_t. Also the family of Markov chains $Z_\omega^{h\nu}$ (defined in Section 5) converges weakly, as $h \to 0$, $\omega \to 0$, to the same limit.

(iii) Under the conditions of Theorem 4.2 (ii) with $\beta \geq \alpha + 1$ suppose μ_t and ν_t are solutions of (4.12) satisfying (4.13) and with initial conditions μ_0 and ν_0. Then

$$\int (\mathbf{1} + L)|\mu_t - \nu_t|\,(dx) \leq a e^{at} \int (\mathbf{1} + L)|\mu_0 - \nu_0|\,(dx) \tag{6.1}$$

for some constant a uniformly for all $t \in [0,T]$.

Proof. By the previous results we only need to prove the claim (iii). Clearly it is enough to prove (6.1) with the function $\mathbf{1}$ replaced by the function $\mathbf{1}_{L(.)\le r}$ with an arbitrary finite r. Next, as by Theorem 4.2 (iv), μ_t and ν_t are solutions of (3.7) with continuous right hand sides, Lemma A from the Appendix (see also Remark after this lemma) can be applied to the measure $(\mathbf{1}_{L(.)\le r}+L)(x)(\mu_t-\nu_t)(dx)$. Consequently, denoting by f_t a version of the density of $\mu_t-\nu_t$ with respect to $|\mu_t-\nu_t|$ from Lemma A yields

$$\int (\mathbf{1}_{L(.)\le r}+L)(x)|\mu_t-\nu_t|(dx) = \|(\mathbf{1}_{L(.)\le r}+L)(\mu_t-\nu_t)\|$$

$$= \int (\mathbf{1}_{L(.)\le r}+L)(x)|\mu_0-\nu_0|(dx) + \int_0^t ds \int_X f_s(x)(\mathbf{1}_{L(.)\le r}+L)(x)(\dot{\mu}_s-\dot{\nu}_s)(dx). \quad (6.2)$$

By (3.7) the last integral in (6.2) equals

$$\int_0^t ds \sum_{l=1}^k \int\int \left([f_s(\mathbf{1}_{L(.)\le r}+L)]^+(\mathbf{y}) - [f_s(\mathbf{1}_{L(.)\le r}+L)]^+(\mathbf{z})\right) P(\mathbf{z};d\mathbf{y})$$

$$\times \left(\prod_{j=1}^l \mu_s(dz_j) - \prod_{j=1}^l \nu_s(dz_j)\right)$$

$$= \int_0^t ds \sum_{l=1}^k \int\int \left([f_s(\mathbf{1}_{L(.)\le r}+L)]^+(\mathbf{y}) - [f_s(\mathbf{1}_{L(.)\le r}+L)]^+(\mathbf{z})\right) P(\mathbf{z};d\mathbf{y})$$

$$\times \sum_{j=1}^l \prod_{i=1}^{j-1} \nu_s(dz_i)(\mu_s-\nu_s)(dz_j) \prod_{i=j+1}^{l} \mu_s(dz_i). \quad (6.3)$$

Let us pick up arbitrary $l \le k$ and $j \le l$ and estimate the corresponding term in the sum (6.3). We have

$$\int\int \left([f_s(\mathbf{1}_{L(.)\le r}+L)]^+(\mathbf{y}) - [f_s(\mathbf{1}_{L(.)\le r}+L)]^+(\mathbf{z})\right) P(\mathbf{z};d\mathbf{y})(\mu_s-\nu_s)(dz_j)$$

$$\times \prod_{i=1}^{j-1} \nu_s(dz_i) \prod_{i=j+1}^{l} \mu_s(dz_i)$$

$$= \int\int \left([f_s(\mathbf{1}_{L(.)\le r}+L)]^+(\mathbf{y}) - [f_s(\mathbf{1}_{L(.)\le r}+L)]^+(\mathbf{z})\right) P(\mathbf{z};d\mathbf{y})$$

$$\times f_s(z_j)|\mu_s-\nu_s|(dz_j) \prod_{i=1}^{j-1} \nu_s(dz_i) \prod_{i=j+1}^{l} \mu_s(dz_i). \quad (6.4)$$

As L is non-increasing by $P(\mathbf{z}; d\mathbf{y})$,

$$\left([f_s(\mathbf{1}_{L(.)\leq r} + L)]^+(\mathbf{y}) - [f_s(\mathbf{1}_{L(.)\leq r} + L)]^+(\mathbf{z})\right) f_s(z_j)$$

$$\leq (\mathbf{1}_{L(.)\leq r} + L)^+(\mathbf{y}) - f_s(z_j)[f_s(\mathbf{1}_{L(.)\leq r} + L)]^+(\mathbf{z})$$

$$\leq M_r + k + L^+(\mathbf{z}) - L(z_j) - \sum_{i\neq j} f_s(z_j) f_s(z_i) L(z_i)$$

$$\leq M_r + k + 2\sum_{i\neq j} L(z_i),$$

where the constant M_r is from condition (ND1). Hence (6.4) does not exceed

$$\int (M_r + k + 2\sum_{i\neq j} L(z_i))(k + L^\alpha(z_j) + \sum_{i\neq j} L^\alpha(z_i))|\mu_s - \nu_s|(dz_j) \prod_{i=1}^{j-1} \nu_s(dz_i) \prod_{i=j+1}^{l} \mu_s(dz_i).$$

Consequently, as $1 + \alpha \leq \beta$ and (4.13) holds, the integral (6.3) does not exceed

$$a(r,T) \int_0^t ds \int (\mathbf{1}_{L(.)\leq r} + L)(x)|\mu_t - \nu_t|(dx)$$

with some constant $a(r,T)$, which implies (6.1) by Gronwall's lemma. Proof of the Theorem is complete.

As a consequence, we shall obtain now a version of the propagation of chaos property for interacting particle systems considered in Section 3. In general, this property means (see e.g. [Sz]) that the moment measures of some random measures tend to the product measures when passing to a certain limit. The moment measures μ_t^m of the jump processes $Z^{h\nu}(t)$ from Theorem 4.1 are defined (see e.g. [Da]) as

$$\mu_{t,h}^m(dx_1...dx_m) = \mathbf{E}\left(Z^{h\nu}(t)(dx_1)...Z^{h\nu}(t)(dx_m)\right). \tag{6.5}$$

Theorem 6.2 *Under the conditions of Theorem 4.1 (ii) with $\beta \geq \alpha + 1$, suppose additionally that (ND2) with $r = \infty$ holds in a stronger sense, i.e. that the number of particles does not increase by l-nary interactions with $l \geq 2$. Let the family of initial measures $h\nu = h\nu(h)$ converges weakly to a certain measure μ_0 as $h \to 0$. Then for any $m = 1, 2, ...,$ the moment measures $\mu_{t,h}^m$ converge weakly to the product measure $\mu_t^{\otimes m}$.*

Proof. By Theorem 6.1, for any $g \in C_c(SX^m)$ the random variables

$$\eta_h = \int g(x_1, ..., x_m) Z^{h\nu}(t)(dx_1)...Z^{h\nu}(t)(dx_m)$$

converge almost surely to the $\int g(x_1, ..., x_m) \prod_{j=1}^m \mu_t(dx_j)$. It is an easy consequence of the strong (ND2) property above that the random variables η_h have uniformly bounded variances which implies that η_h converge also in L_1 to its point-wise deterministic limit.

7. Appendix.

We shall prove here the following measure theoretic result.

Lemma A. *Let Y be a measurable space and the mapping $t \mapsto \mu_t$ from $[0,T]$ to $\mathcal{M}(Y)$ is continuously differentiable in the sense of the norm in $\mathcal{M}(Y)$ with a (continuous) derivative $\dot\mu_t = \nu_t$. Let σ_t denote a density of μ_t with respect to its total variation $|\mu_t|$, i.e. the class of measurable functions taking three values $-1,0,1$ and such that $\mu_t = \sigma_t|\mu_t|$ and $|\mu_t| = \sigma_t\mu_t$ almost surely with respect to $|\mu_t|$. Then there exists a measurable function $f_t(x)$ on $[0,T]\times Y$ such that f_t is a representative of class σ_t for any $t \in [0,T]$ and*

$$\|\mu_t\| = \|\mu_0\| + \int_0^t ds \int_Y f_s(y)\nu_s(dy). \tag{A1}$$

Proof.

Step 1. As μ_t is continuously differentiable, $\|\mu_t - \mu_s\| = O(t-s)$ uniformly for $0 \le s \le t \le T$. Hence $\|\mu_t\|$ is an absolutely continuous real-valued function. Consequently, this function has almost everywhere on $[0,T]$ a derivative, say ω_s, and has an integral representation $\|\mu_t\| = \|\mu_0\| + \int_0^t \omega_s\,ds$ valid for all $t \in [0,T]$. It remains to calculate ω_s. To simplify this calculation, we observe that as the right and left derivatives of an absolutely continuous function coincide almost everywhere (Lebesgue theorem) , it is enough to calculate only the right derivative of $\|\mu_t\|$. Hence from now on we shall consider only the limits $t \to s$ with $t \ge s$.

Step 2. For an arbitrary measurable $A \subset Y$ and an arbitrary representative σ_t of the density, we have

$$O(t-s) = \int_A |\mu_t|(dy) - \int_A |\mu_s|(dy) = \int_A \sigma_t\mu_t(dy) - \int_A \sigma_s\mu_s(dy)$$

$$= \int_A (\sigma_t - \sigma_s)\mu_s(dy) + \int_A \sigma_t(\mu_t - \mu_s)(dy).$$

As the second term here is also of order $O(t-s)$, we conclude that the first term is of order $O(t-s)$ uniformly for all A and s,t. Hence $\sigma_t \to \sigma_s$ almost surely with respect to $|\mu_s|$ as $t \to s$. As σ_t takes only three values $(0,1,-1)$, it follows that $\dot\sigma_s(x)$ exists and vanishes for almost all x with respect to $|\mu_s|$.

Remark. Writing now formally

$$\frac{d}{dt}\|\mu_t\| = \frac{d}{dt}\int_Y \sigma_t\mu_t(dy) = \int_Y \dot\sigma_t\mu_t(dy) + \int_Y \sigma_t\dot\mu_t(dy)$$

and noticing that the first term here vanishes (by Step 2) yields (A1). However, this formal calculation can not be justify for an arbitrary choice of σ_t.

Step 3. Let us choose now an appropriate representative of σ_t. For this purpose let us perform the Lebesgue decomposition of ν_t into the sum $\nu_t = \nu_t^s + \nu_t^a$ of a singular and an absolutely continuous measure with respect to $|\mu_t|$. Let η_t be the density of ν_t^s with respect to its total variation measure $|\nu_t^s|$, i.e. the class of functions taking values $0,1,-1$

and such that $\nu_t^s = \eta_t |\nu_t^s|$ almost surely with respect to $|\nu_t^s|$. Now let us pick up an f_t from the intersection $\sigma_t \cap \eta_t$, i.e. f is a representative of both density classes simultaneously. Such a choice is possible, because μ_t and ν_t^s are mutually singular. From this definition of f_t and from Step 2 it follows that $f_t \to f_s$ as $t \to s$ (and $t \geq s$) almost surely with respect to both μ_s and ν_s. In fact, let B be either a positive part in the Hahn decomposition of ν_s^s or any its measurable subset. Then $f_s = 1$ on B. Moreover,

$$\mu_t(B) = (t-s)\nu_s^s(B) + o(t-s), \quad t-s \to 0,$$

and is positive and hence $f_t = 1$ on B for t close enough to s.

Step 4. By definition,

$$\frac{d}{ds}\|\mu_s\| = \lim_{t \to s} \frac{\|\mu_t\| - \|\mu_s\|}{t-s} = \lim_{t \to s} \int \frac{f_t - f_s}{t-s} \mu_s(dy) + \lim_{t \to s} \int f_t \frac{\mu_t - \mu_s}{t-s}(dy) \qquad (A2)$$

(if both limits exist, of course). It is easy to see that the second limit here always exists and equals $\int f_s \nu_s(dy)$. In fact,

$$\int f_t \frac{\mu_t - \mu_s}{t-s}(dy) = \int f_s \frac{\mu_t - \mu_s}{t-s}(dy) + \int (f_t - f_s)\left(\frac{\mu_t - \mu_s}{t-s} - \nu_s\right)(dy) + \int (f_t - f_s)\nu_s(dy),$$

and the limit of the first integral equals $\int f_s \nu_s(dy)$, the second integral is of order $o(t-s)$ by the definition of the derivative and hence vanishes in the limit, and the limit of the third integral is zero because (due to our choice of f_t in step 3) $f_t - f_s \to 0$ as $t \to s$ $(t \geq s)$ almost surely with respect to ν_s. Consequently, to complete the proof it remains to show that the first term on the r.h.s. of (A2) vanishes.

Remark. As we showed in Step 2, the function $(f_t - f_s)/(t-s)$ under the integral in this term tends to zero almost surely with respect to μ_s, but unfortunately this is not enough to conclude that the limit of the integral vanishes, and consequently an additional argument is required.

Step 5. In order to show that the first term in (A2) vanishes, it is enough to show that the corresponding limits of the integrals over the sets A^+ and A^- vanish, where $Y = A^+ \cup A^- \cup A^0$ is the Hahn decomposition of Y with respect to the measure μ_s. Let us consider only A^+ (A^- is considered similarly). Hence, as $f_s = 1$ on A^+ almost surely, we need to show that

$$\lim_{t \to s} \int_A \frac{f_t - 1}{t-s} \mu_s(dy) = 0 \qquad (A3)$$

where A is a measurable subset of Y such that $(\mu_s)|_A$ is a positive measure. Using now the Lebesgue decomposition of $(\nu_s)|_A$ into the sum of a singular and absolutely continuous parts with respect to μ_s, we can and will reduce the discussion to the case when ν_s is absolutely with respect to μ_s on A.

Introducing the set $A_t = \{y \in A : f_t(y) \leq 0\}$ one can clearly replace A by A_t in (A3). Consequently, to get (A3) it is enough to show that $\mu_s(A_t) = o(t-s)$ as $t \to s$. This will be done in the next final step.

Step 6. From the definition of A_t it follows that $\mu_t(B) \leq 0$ for any $B \subset A_t$ and hence

$$\mu_s(B) + (t-s)\nu_s(B) + o(t-s) \leq 0, \tag{A4}$$

where $o(t-s)$ is uniform, i.e. $\|o(t-s)\|/(t-s) \to 0$ as $t \to s$. Notice first that if $A_t = B_t^+ \cup B_t^- \cup B_t^0$ is the Hahn decomposition of A_t on the positive, negative and zero parts of the measure ν_s, then $\mu_s(B_t^+ \cup B_t^0) = o(t-s)$ uniformly (as it follows directly from (A4)), and consequently we can and will reduce our discussion to the case when ν_s is a negative measure on A. In this case (A4) implies that

$$\mu_s(A_t) \leq (t-s)(-\nu_s)(A_t) + o(t-s)$$

and it remains to show that $\nu_s(A_t) = o(1)_{t \to s}$. To see this we observe that for any s, Y has the representation $Y = \cup_{n=0}^{\infty} Y_n$, where $|\mu_s|(Y_0) = 0$ and

$$Y_n = \{y \in Y : f_t = f_s \quad for \quad |t-s| \leq 1/n\}.$$

Clearly $Y_n \subset Y_{n+1}$ for any $n \neq 0$ and $A_t \subset Y \setminus Y_n$ whenever $t - s \leq 1/n$. Hence A_t are subsets of a decreasing family of sets with an intersection of μ_s-measure zero. As ν_s is absolutely continuous with respect to μ_s the same holds for ν_s and hence $\nu_s(A_t) = o(1)_{t \to s}$, which completes the proof of the Lemma.

Remark. Suppose the assumptions of Lemma A hold and $L(y)$ is a measurable, non-negative and everywhere finite function on Y such that $\|L\mu_s\|$ and $\|L\nu_s\|$ are uniformly bounded for $s \in [0,t]$. Then (A1) holds with $L\mu_t$ and $L\nu_t$ instead of μ_t and ν_t respectively. In fact, though $s \mapsto L\nu_s$ may be discontinuous in the sense of norm, one can write the required identity first with the space Y_m instead of Y, where $Y_m = \{y : L(y) \leq m\}$, and then pass to the limit as $m \to \infty$.

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