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## Artificiality of multifractal phase transitions

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### Abstract

A multifractal phase transition is associated to a nonanalyticity in the generalised dimensions. We show that its occurrence is an artifact of the asymptotic scaling behaviour of integral moments and that it is not observed in an analysis based on differential  $n$ -point correlation densities. © 2000 Published by Elsevier Science B.V. All rights reserved.

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Multifractal measures appear in a number of non-linear physical phenomena like turbulence [1,2], chaotic dynamical systems [3] and high-energetic multiparticle dynamics [4], to name but a few. Due to the close analogy between the multifractal formalism and statistical thermodynamics [5–9], any non-analyticity in the generalised dimensions is interpreted as a multifractal phase transition [10–12]. This behaviour has been discussed in the context of the above mentioned phenomena [10,13–15] and is also denoted as the occurrence of strong intermittency. Note that conventionally generalised dimen-

sions are extracted from the asymptotic scaling behaviour of moments, where the latter represent integrals over the fundamental correlation densities. We will now demonstrate that a multifractal phase transition is an artefact of the integral moment analysis and is not observed in an analysis based on differential correlation densities, where the true generalised dimensions of any order, characterising the underlying multiscale process, are revealed.

As the textbook example of a multifractal process we consider a one-dimensional discrete random multiplicative cascade. It is associated with a binary tree structure, obtained by hierarchically partitioning the original interval of length  $l_0 = 1$  into subintervals of size  $l_j = 2^{-j}$ . The density  $\varepsilon_\kappa^{(j+1)}$ , associated with a  $(j+1)$ -generation interval characterised by the binary index  $\kappa = (k_1 \cdots k_{j+1})$  with each  $k$  taking on

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possible values 0 or 1, is multiplicatively linked to the density at the larger scale by

$$\varepsilon_{k_1 \dots k_j k_{j+1}}^{(j+1)} = q_{k_1 \dots k_j k_{j+1}}^{(j+1)} \varepsilon_{k_1 \dots k_j}^{(j)}. \quad (1)$$

Independently for each branching, the left and right weights,  $q_L = q_{k_1 \dots k_j 0}^{(j+1)}$  and  $q_R = q_{k_1 \dots k_j 1}^{(j+1)}$ , are drawn from a probabilistic splitting function  $p(q_L, q_R)$  with support  $0 \leq q_L, q_R < \infty$ . Without loss of generality we assume  $p(q_L, q_R) = p(q_R, q_L)$  and set  $\langle q_L \rangle = 1$  as well as  $\varepsilon^{(0)} = 1$ .

Nature does not allow an infinitely long multifractal scaling range; in fully developed turbulence, for example, it is restricted to  $\eta \ll l \ll L$ , where  $\eta$  and  $L$  represent the Kolmogorov and the integral length scales, respectively. Consequently, we restrict the random multiplicative cascade to a finite number  $J$  of cascade steps. The complete statistical information of the ensemble of generated cascade fields is then contained in the multivariate characteristic function

$$Z[\lambda^{(J)}] = \left\langle \exp \left( \sum_{k_1, \dots, k_j=0}^1 \lambda_{k_1 \dots k_j}^{(J)} \varepsilon_{k_1 \dots k_j}^{(J)} \right) \right\rangle, \quad (2)$$

from which the  $n$ -point correlation densities are derived

$$\rho_{\kappa_1 \dots \kappa_n}^{[n]} = \langle \varepsilon_{\kappa_1}^{(J)} \dots \varepsilon_{\kappa_n}^{(J)} \rangle = \frac{\partial^n Z[\lambda^{(J)}]}{\partial \lambda_{\kappa_1}^{(J)} \dots \partial \lambda_{\kappa_n}^{(J)}} \Big|_{\lambda^{(j)}=0} \quad (3)$$

by taking appropriate derivatives with respect to the conjugate field variables  $\lambda_{\kappa}^{(J)}$ . The multivariate characteristic function and the resulting  $n$ -point correlation densities have been calculated analytically in Refs. [16,17]; see also Refs. [18,19].

For the extraction of generalised dimensions exponents so-called (box-) moments are considered, defined as

$$M_n(J, j) = \frac{1}{2^j} \sum_{k_1, \dots, k_j=0}^1 \left\langle \left( \bar{\varepsilon}_{k_1 \dots k_j}^{(J, j)} \right)^n \right\rangle, \quad (4)$$

where the backward density

$$\bar{\varepsilon}_{k_1 \dots k_j}^{(J, j)} = \frac{1}{2^{J-j}} \sum_{k_{j+1}, \dots, k_J=0}^1 \varepsilon_{k_1 \dots k_j}^{(J)} \quad (5)$$

has been resummed over the smallest scales from  $J$  to  $j$ . With (5) the (box-) moment (4) can be under-

stood as a (box-) integration over the  $n$ -point correlation density (3):

$$\begin{aligned} & \left\langle \left( \bar{\varepsilon}_{k_1 \dots k_j}^{(J, j)} \right)^n \right\rangle \\ &= \frac{1}{2^{n(J-j)}} \sum_{k_{j+1}^{(1)}, \dots, k_j^{(1)}=0}^1 \dots \sum_{k_{j+1}^{(n)}, \dots, k_j^{(n)}=0}^1 \\ & \times \left\langle \varepsilon_{k_1 \dots k_j k_{j+1}^{(1)} \dots k_j^{(1)}}^{(J)} \dots \varepsilon_{k_1 \dots k_j k_{j+1}^{(n)} \dots k_j^{(n)}}^{(J)} \right\rangle, \end{aligned} \quad (6)$$

yielding the explicit expressions:

$$M_1(J, j) = 1,$$

$$\begin{aligned} M_2(J, j) = \langle q^2 \rangle^j & \left[ \frac{\langle q_L q_R \rangle}{2 - \langle q^2 \rangle} \right. \\ & \left. + \frac{2 - \langle q^2 \rangle - \langle q_L q_R \rangle}{2 - \langle q^2 \rangle} \left( \frac{\langle q^2 \rangle}{2} \right)^{J-j} \right], \end{aligned} \quad (7)$$

and, for arbitrary order,

$$M_n(J, j) = \langle q^n \rangle^j \sum_{\{p\}} a_{\{p\}}^{(n)} \prod_{\sum n_i=n} \left( \frac{\langle q^{n_i} \rangle}{2^{n_i-1}} \right)^{J-j}, \quad (8)$$

where  $\{p\}$  stands for all possible partitions of  $\sum_i n_i = n$  with  $n_i \in \{1, 2, \dots, n\}$ ; the coefficients  $a_{\{p\}}^{(n)}$  are simple scale-independent functionals of the splitting-function moments  $\langle q_L^{m_1} q_R^{m_2} \rangle$  with  $0 \leq m_1 + m_2 \leq n$ . While we will present a full technical understanding of the structure reflected in expression (8) at a later stage of this presentation, some intuitive understanding can already be derived: the backward density (5) can be rewritten as

$$\begin{aligned} \bar{\varepsilon}_{k_1 \dots k_j}^{(J, j)} &= q_{k_1}^{(1)} \dots q_{k_j}^{(j)} \frac{1}{2^{J-j}} \sum_{k_{j+1}, \dots, k_J=0}^1 \varepsilon_{k_{j+1} \dots k_J}^{(J-j)} \\ &= \varepsilon_{k_1 \dots k_j}^{(j)} \left( 1 + \Delta_{k_1 \dots k_j}^{(J-j)} \right), \end{aligned} \quad (9)$$

where  $(1 + \Delta^{(J-j)})$  represents the resummed density of the subcascade with length  $J-j$  following the branching point  $(k_1 \dots k_j)$ . This resummed density need not be strictly equal to 1 and in general will fluctuate around 1. Hence, the backward density (5) need not be identical to the forward density  $\varepsilon_{k_1 \dots k_j}^{(j)}$ .

In view of (9), the first factor  $\langle q^n \rangle^j$  of the expression (8) then originates from  $\langle (\varepsilon_{k_1 \dots k_j}^{(j)})^n \rangle$  while the remainder of the expression (8) is equal to  $\langle (1 + \Delta^{(J-j)})^n \rangle$ .

For the very special case of a conservative cascade, where with  $p(q_L, q_R) = p(q_L)\delta(q_L + q_R - 2)$  the sum of the left and right weight is conserved at every branching, the two coefficients  $a_{\{p\}}^{(2)}$  of the second-order moment (7) become

$$a_{\{1,1\}} = \langle q_L q_R \rangle / (2 - \langle q^2 \rangle) = 1$$

and

$$a_{\{2,0\}} = (2 - \langle q^2 \rangle - \langle q_L q_R \rangle) / (2 - \langle q^2 \rangle) = 0,$$

since  $\langle q_L q_R \rangle = \langle q(2 - q) \rangle$ . Similarly, all but one coefficients  $a_{\{p\}}^{(n)}$  of expression (8) vanish and the moment of order  $n$  becomes exactly  $M_n(J, j) = \langle q^n \rangle^j$ . Also in view of (9) this outcome is intuitively clear since due to the conservative nature of the splitting function the resummed density  $1 + \Delta^{(J-j)}$  is strictly equal to one. The moment  $M_n(J, j)$  does not depend on the length  $J$  of the cascade and exhibits perfect scaling. The multifractal scaling exponents

$$\tau(n) = \frac{\ln \langle q^n \rangle}{\ln 2} \quad (10)$$

are deduced by setting  $M_n(J, j) = (l_0/l_j)^{\tau(n)}$  and are related to the generalised dimensions  $D_n$  by  $\tau(n) = (n-1)(1 - D_n)$ .

A factorised splitting function  $p(q_L, q_R) = p(q_L)p(q_R)$  is a representative of non-conservative cascades, where the sum of the left and right weight is only globally conserved ( $\langle q_L + q_R \rangle = 2$ ), but not locally ( $q_L + q_R \neq 2$ ). As a consequence, the mixed splitting-function moments  $\langle q_L^{n_1} q_R^{n_2} \rangle \neq \langle q^{n_1} (2 - q)^{n_2} \rangle$  do not show the anticorrelation typical of conservative cascades. For this case coefficients  $a_{\{p\}}^{(n)}$  are generally nonzero and the moments (8) do not show rigorous scaling. The resummed density  $1 + \Delta^{(J-j)}$  of the subcascade with length  $J - j$  now fluctuates around one and causes the deviations from rigorous scaling behaviour.

For nonconservative cascades two subclasses of splitting functions have to be distinguished: in the case of so-called weak intermittency, the support of  $p(q_L, q_R)$  is restricted to  $0 \leq q_L, q_R < 2$ , so that the

respective moments are restricted to  $\langle q^n \rangle < 2^{n-1}$ , where  $n > 1$  and where the extra  $1/2$  on the right hand side of this inequality comes from the requirement  $\langle q \rangle = 1$ . This implies that, in the limit of an infinitely long cascade,  $J \rightarrow \infty$ , only the true scaling term  $M_n(J, j) \sim \langle q^n \rangle^j$  survives in expression (8), so that asymptotically for  $j \ll J$  the same multifractal scaling exponents (10) are extracted as from the corresponding conservative cascade. For the resummed density  $1 + \Delta^{(J-j)}$  of the subcascade, this implies that in this asymptotic scaling range its probability distribution has converged to a scale-independent fixed point; see also Refs. [20,21].

The second subclass of nonconservative splitting functions exhibits the phenomenon that has become known as multifractal phase transition or strong intermittency: once the support of  $p(q_L, q_R)$  allows values  $q_L$  and/or  $q_R$  to exceed 2, a critical order  $n_{\text{crit}}$  exists, so that  $\langle q^n \rangle / 2^{n-1} < 1$  for  $n < n_{\text{crit}}$  and  $\langle q^n \rangle / 2^{n-1} > 1$  for  $n > n_{\text{crit}}$ . Then, again in the limit of a very long, but finite cascade and given that  $n > n_{\text{crit}}$ , the term corresponding to the partition  $\{n, 0, \dots, 0\}$  dominates the moment (8) of order  $n$ , which hence for  $j \ll J$  scales as

$$M_{n > n_{\text{crit}}}(J, j \ll J) \approx a_{\{n,0,\dots,0\}}^{(n)} \left( \frac{\langle q^n \rangle}{2^{n-1}} \right)^j 2^{j(n-1)} \sim \left( \frac{l_0}{l_j} \right)^{n-1}. \quad (11)$$

For the multifractal scaling exponents this implies

$$\tau(n) = \begin{cases} \ln \langle q^n \rangle / \ln 2 & (n < n_{\text{crit}}) \\ n - 1 & (n > n_{\text{crit}}), \end{cases} \quad (12)$$

so that there is a discontinuity in the first derivative of  $\tau(n)$  with respect to the moment-order  $n$  at  $n_{\text{crit}}$ . For  $n < n_{\text{crit}}$  the multifractal scaling exponents  $\tau(n) = \ln \langle \exp(n \ln q) \rangle / \ln 2$  may be interpreted as a free-energy-like function with moment order  $n$  as inverse temperature. – According to (11), note that in the limit  $J \rightarrow \infty$  of a non-physical, infinitely long cascade, moments with order larger than  $n_{\text{crit}}$  would diverge. In view of (9) this implies that the probability distribution of the subcascade-resummed density  $1 + \Delta^{(J-j)}$  comes with an algebraic tail.

For a nonconservative random multiplicative cascade with a factorised splitting function  $p(q_L, q_R) = p(q_L)p(q_R)$ , where, for example,

$$p(q) = \frac{1}{\sqrt{2\pi}\sigma q} \exp\left(-\frac{1}{2\sigma^2}\left(\ln q + \frac{\sigma^2}{2}\right)^2\right) \quad (13)$$

is of log-normal type, the multifractal scaling exponents (12) are found to be  $\tau(n < n_{\text{crit}}) = \sigma^2 n(n-1)/(2\ln 2)$  below the critical order  $n_{\text{crit}} = 2\ln 2/\sigma^2$ , which defines the multifractal phase transition at  $\tau(n_{\text{crit}}) = n_{\text{crit}} - 1$ . In fully developed turbulence a good qualitative description of observed multiplier distributions in the surrogate energy dissipation field [21] and an acceptable intermittency exponent  $\tau(2) = 0.25$  has been found for the parameter choice  $\sigma = 0.42$ ; for this value we find  $n_{\text{crit}} = 7.86$ . Note, that for the log-normal weight distribution the so-called Novikov rules [22,23] do not apply since weights may exceed a value of 2 with nonzero probability. – The scale-dependence of the exact second-order moment expression (7) is illustrated in Fig. 1. As expected for  $j \ll J$ ,  $M_2(J, j)$  scales as  $\langle q^2 \rangle^j$  for  $\sigma < \sigma_{\text{crit}}$  and as  $2^j$  for  $\sigma > \sigma_{\text{crit}}$ , where  $\sigma_{\text{crit}} = \sqrt{\ln 2}$  corresponds to  $n_{\text{crit}} = 2$ . As  $j \rightarrow J$  noticeable deviations from this scaling behaviour occur. Directly at  $\sigma = \sigma_{\text{crit}}$ , where  $\langle q^2 \rangle = 2$ , these finite size effects become so large that it becomes difficult

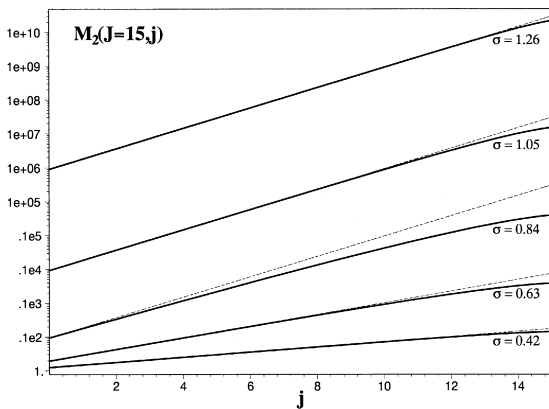


Fig. 1. Scale-dependence of the second-order moment  $M_2(J, j)$  resulting from a finite discrete random multiplicative cascade with a factorised log-normal splitting function. The dashed straight lines represent the asymptotic scaling  $M_2(J, j) \sim \langle q^2 \rangle^j$  and  $M_2(J, j) \sim 2^j$  for  $\sigma < \sigma_{\text{crit}} = \sqrt{\ln 2} = 0.833$  and  $\sigma > \sigma_{\text{crit}}$ , respectively.

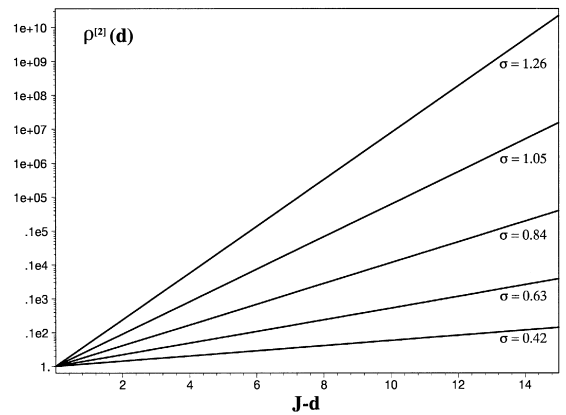


Fig. 2. Dependence of the two-point correlation density  $\rho^{[2]}(d)$ , resulting from a finite discrete random multiplicative cascade with  $J = 15$  steps and a factorised log-normal splitting function, on the ultrametric distance.

to extract a scaling exponent asymptotically. Analogous findings hold for higher order moments  $M_n(J, j)$ .

In the presence of a multifractal phase transition the information on the true multifractal scaling exponents (10) with  $n > n_{\text{crit}}$  appears to become lost. But this is not the case! Note in this respect that, according to (4)-(6), the moments  $M_n(J, j)$  are box-integrals over the  $n$ -point correlation density (3), the latter thus being more fundamental than the former. For demonstration, we pick the two-point correlation density

$$\begin{aligned} \rho^{[2]}(d) &= \langle \varepsilon_{\kappa_1}^{(J)} \varepsilon_{\kappa_2}^{(J)} \rangle \\ &= \langle q^2 \rangle^{J-d} (\langle q_L q_R \rangle \\ &\quad + (1 - \langle q_L q_R \rangle) \delta_{d,0}), \end{aligned} \quad (14)$$

which is a function of the ultrametric distance  $d = J - j$  between the two bins  $\kappa_1 = (k_1 \cdots k_j k_{j+1} \cdots k_j)$  and  $\kappa_2 = (k_1 \cdots k_j k'_{j+1} \cdots k'_j)$ , where  $k_{j+1} \neq k'_{j+1}$ . It is depicted in Fig. 2 for a random multiplicative cascade with a factorised splitting function of log-normal type and reveals perfect scaling with the true multifractal scaling exponents. For the two-point correlation density and, in general, for all  $n$ -point correlation densities no multifractal phase transition occurs. Why, then, does a multifractal phase transition occur once based on integral moments and not on correlation densities?

The answer to this question will be found in an inconspicuous property of the correlation densities. For demonstration we discuss this again only for second order. First, we realize that the second-order moment (4) can be cast in the form

$$M_2(J, j) = \frac{1}{2^{J-j}} \left( \rho^{[2]}(d=0) + \sum_{d=1}^{J-j} 2^{d-1} \rho^{[2]}(d) \right). \quad (15)$$

Next, by adding and again subtracting something for  $d=0$ , the expression (14) is rewritten as

$$\begin{aligned} \rho^{[2]}(d) &= \langle q^2 \rangle^{J-d} \left[ \langle q_L q_R \rangle \right. \\ &\quad \left. + \left( \frac{\langle q_L q_R \rangle}{2 - \langle q^2 \rangle} - \langle q_L q_R \rangle \right) \delta_{d0} \right] \\ &\quad + \frac{2 - \langle q^2 \rangle - \langle q_L q_R \rangle}{2 - \langle q^2 \rangle} \langle q^2 \rangle^J \delta_{d0} \\ &\equiv \tilde{\rho}^{[2]}(d) + \frac{2 - \langle q^2 \rangle - \langle q_L q_R \rangle}{2 - \langle q^2 \rangle} \langle q^2 \rangle^J \delta_{d0}. \end{aligned} \quad (16)$$

For  $d \neq 0$  the modified two-point correlation density  $\tilde{\rho}^{[2]}(d)$  is identical to the original two-point correlation density  $\rho^{[2]}(d)$ . The difference comes for  $d=0$ , where

$$\frac{\tilde{\rho}^{[2]}(d=0)}{\rho^{[2]}(d=1)} = \frac{\langle q^2 \rangle}{2 - \langle q^2 \rangle} \quad (17)$$

contrary to  $\rho^{[2]}(d=0)/\rho^{[2]}(d=1) = \langle q^2 \rangle / \langle q_L q_R \rangle$ . If we were only to substitute the modified two-point correlation density into (15), then this difference insures perfect moment scaling with the true multifractal scaling exponents as we arrive at the first term of the expression (7). Substitution of only the second term appearing in (16), which is proportional to  $\delta_{d0}$ , produces the second term in (7), which is proportional to  $(\langle q^2 \rangle / 2)^J 2^j$  and reflects a trivial scaling with exponent  $\tau^{[2]}|_{\text{trivial}} = 2 - 1$ . The appearance of the second term in (7) and (16), respectively, is a consequence of the missing anticorrelation between weights  $q_L$  and  $q_R$  in the splitting

function; it vanishes only for a conservative splitting function  $p(q_L, q_R) = p(q_L) \delta(q_L + q_R - 2)$  because then  $\langle q_L q_R \rangle = 2 - \langle q^2 \rangle$ .

It is worthwhile elaborating in more detail on the occurrence of a multifractal phase transition from the viewpoint of Eq. (16). For demonstration we pick again a factorised splitting function of log-normal type (13), where  $\langle q^2 \rangle = \exp(\sigma^2)$  and  $\langle q_L q_R \rangle = 1$ . As  $\sigma$  increases monotonically from 0 to  $\sigma_{\text{crit}} = \sqrt{\ln 2}$ , the ratio  $\langle q^2 \rangle / (2 - \langle q^2 \rangle)$  appearing in (17) increases from 1 to  $+\infty$ ; then, as  $\sigma$  further increases, it changes sign and increases from  $-\infty$  at  $\sigma = \sigma_{\text{crit}}^+$  monotonically to  $-1$  as  $\sigma \rightarrow \infty$ . Consequently, as the ( $d=0$ ) and ( $d=1$ ) elements of the modified two-point correlation density contribute as the sum  $\rho^{[2]}(d=0) + \rho^{[2]}(d=1)$  to the second-order moment (15), they enhance each other for  $\sigma < \sigma_{\text{crit}}$ , but more or less cancel each other for  $\sigma > \sigma_{\text{crit}}$ . Hence, for the former case the modified two-point correlation density dominates the moment scaling whereas for the latter case the  $\delta$ -function-like correction in (16) becomes dominant.

We conclude: when generalised dimensions are determined via integral moments, one is likely to encounter artificial multifractal phase transitions, which are not a property of the underlying strongly intermittent nonconservative cascade process, but artifacts of small-scale resummation. The fundamental  $n$ -point correlation densities and, should the underlying process only be resolvable at an intermediate scale, so-called density correlators

$$\mathcal{E}_{\kappa_1 \dots \kappa_n}^{[m_1, \dots, m_n]} = \frac{\left\langle \left( \bar{\varepsilon}_{\kappa_1}^{(J, j)} \right)^{m_1} \dots \left( \bar{\varepsilon}_{\kappa_n}^{(J, j)} \right)^{m_n} \right\rangle}{\left\langle \left( \bar{\varepsilon}_{\kappa_1}^{(J, j)} \right)^{m_1} \right\rangle \dots \left\langle \left( \bar{\varepsilon}_{\kappa_n}^{(J, j)} \right)^{m_n} \right\rangle}, \quad (18)$$

avoid such contributions and are therefore a better choice in estimating generalised dimensions.

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