ALMOST PERIODIC MOTIONS AND GLOBAL ATTRACTORS OF THE NONAUTONOMOUS NAVIER-STOKES EQUATIONS

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Abstract. The paper is about the study of the nonautonomous Navier-Stokes system. This problem is formulated and investigated in the context of general nonautonomous dynamical systems. We first prove that this system admits compact global attractors. Then we provide conditions for convergence of this system, which means the compact global attractor is a point attractor. Third, we obtain conditions for the existence of almost periodic (periodic, quasi-periodic, recurrent, pseudo-recurrent) solutions of the nonautonomous Navier-Stokes equations. Finally, we prove a global averaging principle for the nonautonomous Navier-Stokes equations.

1. Introduction

Dynamical models are essential for understanding time-evolving processes in engineering and science. Such models in the form of evolution equations may contain time-dependent coefficients and time-dependent external forcing terms. Nonautonomous dynamical systems techniques are needed for investigating these kind of mathematical models.

In this paper, we consider the two-dimensional Navier-Stokes type system in a bounded domain D with smooth boundary

$$u' + Au + B(t)(u,u) = f(t), \tag{1}$$

where velocity u is in an appropriate Sobolev space E, $-A$ is the Stokes operator, $B(t)$ is a bilinear form satisfying the identity

$$Re\langle B(t)(u,v), w\rangle = -Re\langle B(t)(u,w), u\rangle \tag{2}$$

for all $t \in \mathbb{R}$ and $u, v, w \in E$, and f is an external forcing term. The explicit time dependence of the bilinear form $B(t)$ may arise, say, when homogenizing a time-dependent boundary condition or when reformulating the momentum equations along a known unsteady flow (moving the known unsteady flow to the zero flow). We also treat the evolution equation (1) as a model for developing nonautonomous dynamical systems ideas about almost periodic solutions, attractors and global averaging principle. When the bilinear form B does not depend on time explicitly, (1) is the usual Navier-Stokes system.

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A few authors ([11, 15, 19, 20]) have studied the non-stationary equation (1) when the bilinear form B does not depend on time explicitly and when f is a function of time $t \in \mathbb{R}$. It was shown that the usual Navier-Stokes system with compact f (in particular, when f is almost periodic) admits a compact global attractor. It was also proved that when the forcing f is sufficiently small and/or when the the viscosity is sufficiently large, there exists a unique almost periodic (quasi-periodic, periodic) solution of the equation (1) if the forcing term f is almost periodic (quasi-periodic, periodic).

The aim of the present paper is to study dynamical behavior of the nonautonomous Navier-Stokes system (1) in the case when the bilinear form B, and the forcing function f are both non-stationary. We obtain conditions for the existence of a compact global attractor and almost periodic (periodic, quasi-periodic, recurrent, pseudo-recurrent) solutions. For nonautonomous dynamical systems, an averaged (and thus autonomous) approximation is sometimes desirable. A theorem on "partial" averaging on finite intervals for ordinary differential equations was proved in [17]. The works [15],[19] and [20] are devoted to generalization of the method of averaging for dissipative partial differential equations. Here we prove a theorem of "partial" averaging for nonautonomous Navier-Stokes equation (1).

Our paper is organized as follows:

In Section 2, we introduce a class of nonautonomous Navier-Stokes equations and establish its dissipativity (Theorem 2.8).

In Section 3, we prove that the nonautonomous Navier-Stokes equations admit a compact global attractor (Theorem 3.8).

Section 4 is devoted to the study of existence of almost periodic (quasi-periodic, periodic, recurrent, pseudo-recurrent) solutions of the nonautonomous Navier-Stokes equations (Corollary 4.8) and we also provide the conditions of convergence for these equations, which implies that the compact global attractor is a point attractor (Theorem 4.5 and Corollary 4.7).

In Section 5 we prove a uniform averaging principle for the nonautonomous Navier-Stokes equations on the finite time interval (Theorem 5.2).

Section 6 is devoted to the proof of a global averaging principle for nonautonomous Navier-Stokes equations on semi-axis (Theorems 6.1,6.3 and 6.7).

2. Nonautonomous Navier-Stokes equations.

We recall some results from the theory of semigroups of linear operators [18] and PDEs [19], [27],[29].

A closed operator A with domain $D(A)$ that is dense in a Banach space X is called a sectorial operator if for some $a \in \mathbb{R}$ and $\varphi \in (0, \frac{\pi}{2})$ the sector

$$S_{a,\varphi} := \{\lambda \in \mathbb{C}, \pi \geq |arg(\lambda - a)| \geq \varphi\} \tag{3}$$

is contained in the resolvent set and for $\lambda \in S_{a,\varphi}$

$$\|(\lambda I - A)^{-1}\|_{X \to X} \leq \frac{c}{|\lambda - a| + 1}. \tag{4}$$

For the sectorial operator A the analytic semigroup of linear bounded operators in X is defined and denoted by $e^{-At},\ t \geq 0$.

Let A be a sectorial operator with $Re\sigma(A) > 0$. For $\alpha \in (0,1)$ we define fractional powers of A as follows:

$$A^{\alpha} := (A^{-\alpha})^{-1}, \text{ where } A^{-\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} e^{-At} dt.$$

The corresponding domains $D(A^{\alpha})$ are Banach spaces with norm given by

$$|\cdot|_{\alpha} := |\cdot|_{D(A^{\alpha})} = |A^{\alpha} \cdot |.$$

Theorem 2.1. *The following estimates are valid:*

(i)

(5) $$\|e^{-At}\|_{X \to X} \leq Ce^{-at}, \quad t \geq 0,$$

(ii)

(6) $$\|A^{\alpha} e^{-At}\|_{X \to X} \leq C_{\alpha} t^{-\alpha} e^{-at}, \quad t > 0.$$

Let Ω be a compact metric space, $\mathbb{R} = (-\infty, +\infty), (\Omega, \mathbb{R}, \sigma)$ be a dynamical system on Ω, $\mathcal{E}$ be a real or complex Hilbert space, $L(\mathcal{E})$ be the space of all linear forms on $\mathcal{E}$, $L^2(\mathcal{E})$ be the space of all bilinear continuous forms $B : \mathcal{E} \times \mathcal{E} \to \mathcal{F}$ and $C(\Omega, W)$ be a space of all continuous functions $f : \Omega \to W$ (W is some metric space), endowed with the topology of uniform convergence.

Let us consider the equation

(7) $$u' + Au + B(\omega t)(u,u) = f(\omega t),$$

$(\omega \in \Omega)$ where $\omega t := \sigma(t, \omega), B \in C(\Omega, L^2(\mathcal{E})),\ f \in C(\Omega, \mathcal{E})$ and A is a linear operator.

Below we will use some concepts, notations and results from [20]. Let Hilbert spaces E, F, X satisfy $E \subset F;\ E, F, X \subset \mathcal{E}$, each embedding being dense and continuous.

Furthermore we suppose that the linear operator A is densely defined in $\mathcal{E}$ and is such that the linear equation

(8) $$u' + Au = 0$$

generates the c_0-semigroup of linear bounded operators

$$e^{-At} : \mathcal{E} \to \mathcal{E},\ \varphi(t,x) := e^{-At}x,$$

which for $t > 0$ can be extended to the linear bounded operators from F to E satisfying the following estimates

(9) $$\|e^{-At}\|_{E \to E} \leq Ke^{-at},$$

(10) $$\|e^{-At}\|_{F \to E} \leq Kt^{-\alpha_1} e^{-at}, \quad 0 \leq \alpha_1 < 1,$$

(11) $$\|Ae^{-At}\|_{F \to E} \leq Kt^{-\alpha_2} e^{-at}, \quad 0 \leq \alpha_2 < 2.$$

We also suppose that the following condition is satisfied

(12) $$Ae^{At} = e^{At}A,$$

in the sense of $L(F,E) := \{A : F \to E \;|A \text{ is linear and bounded}\}$ equipped with the operational norm.

Denote by $L^2(E,F)$ the space of all bilinear bounded forms $B : E \times E \to F$ with the norm

$$\|B\| := \sup\{|B(u,v)|_F \;:\; |u| \le 1, |v| \le 1\}.$$

Let $C(\Omega, L^2(E,F))$ be a space of all continuous mappings $B : \Omega \to L^2(E,F)$ and

$$C_B := \sup\{|B(\omega)(u,v)|_F \;:\; \omega \in \Omega,\; |u| \le 1, |v| \le 1\}.$$

Then the mapping $F : \Omega \times E \to F$ ($F(\omega, u) := B(\omega)(u,u)$) satisfies the following inequality

$$|B(\omega)(u_1,u_1) - B(\omega)(u_2,u_2)|_F \le C_B(|u_1|_E + |u_2|_E)|u_1 - u_2|_E \tag{13}$$

for all $u_1, u_2 \in E$.

From the inequality (13) it follows that on every ball $B[0,R] := \{u \in E \;:\; |u| \le R\}$ we have

$$|B(\omega)(u_1,u_1) - B(\omega)(u_2,u_2)|_F \le 2C_B R|u_1 - u_2|_E \tag{14}$$

for all $u_1, u_2 \in E$.

Remark 2.2. *The space of all the bilinear forms $C(\Omega, L^2(E,F))$ is a Banach space with the norm $\|B\| := C_B$.*

The external forcing $f : \Omega \to X$ is continuous, i.e. $f \in C(\Omega, X)$.

The operators e^{-At} $(t > 0)$ can be extended to the linear bounded operators from X to E satisfying the estimates

$$\|e^{-At}\|_{X\to E} \le Kt^{-\beta_1}e^{-at},\; 0 \le \beta_1 < 1, \tag{15}$$

$$\|Ae^{-At}\|_{X\to E} \le Kt^{-\beta_2}e^{-at},\; 0 \le \beta_2 < 2, \tag{16}$$

and the equation (12), this time in the sense of $L(X,E)$.

We suppose that the following conditions are fulfilled:

(i) there exists $\alpha > 0$ such that

$$Re\langle Au, u\rangle \ge \alpha |u|^2 \tag{17}$$

for all $u \in E$, where $|\cdot|$ is a norm in E ;

(ii)

$$Re\langle B(\omega)(u,v), w\rangle = -Re\langle B(\omega)(u,w), v\rangle \tag{18}$$

for every $u, v, w \in E$ and $\omega \in \Omega$.

Remark 2.3. *a. From (18) it follows that*

$$Re\langle B(\omega)(u,v), v)\rangle = 0 \tag{19}$$

for every $u, v \in E$ and $\omega \in \Omega$.

b.

$$|B(\omega)(u,v)|_F \le C_B|u|_E|v|_E \tag{20}$$

for all $u, v \in E$ *and* $\omega \in \Omega$*, where* $C_B = \sup\{|B(\omega)(u,v)|_F \ : \ \omega \in \Omega, \ u,v \in E, \ |u|_E \le 1, \text{ and } |v|_E \le 1\}$*.*

The equation (7) with conditions (17) and (18) is called a nonautonomous Navier-Stokes equation. We will consider the mild solutions of the equation (7), i.e. $u \in C([0,T],E)$ and satisfy the following integral equation

$$u(t) = e^{-At}x + \int_0^t e^{-A(t-s)}(-B(\omega s)(u(s),u(s)) + f(\omega s))ds. \tag{21}$$

Theorem 2.4. *Let* $x_0 \in E$*,* $r > 0$ *and the conditions (9),(10) and (17) be fulfilled. Then there exist positive numbers* $\delta = \delta(x_0, r)$ *and* $T = T(x_0, r)$ *such that the equation (21) admits a unique solution* $\varphi(t, x, \omega)$ $(x \in B[x_0, \delta] = \{x \in E \mid |x - x_0| \le \delta\})$ *defined on the interval* $[0, T]$ *with the conditions:* $\varphi(0, x, \omega) = x$*,* $|\varphi(t, x, \omega) - x_0| \le r$ *for all* $t \in [0, T]$ *and the mapping* $\varphi : [0, T] \times B[x_0, \delta] \times \Omega \to E$ $((t, x, \omega) \to \varphi(t, x, \omega))$ *is continuous.*

Proof. Let $x_0 \in E, \ r > 0, \ \delta > 0$ and $T > 0$. We consider the space $C_{x_0,r,\delta,T}$ of all continuous functions $\psi : [0,T] \times B[x_0,\delta] \times \Omega \to B[x_0,r]$ equipped with the distance

$$d(\psi_1,\psi_2) := \sup\{|\psi_1(t,x,\omega) - \psi(t,x,\omega)|_E \ : \ 0 \le t \le T, x \in B[x_0,\delta], \omega \in \Omega\}$$

which is a complete metric space.

We define the operator Φ acting onto $C_{x_0,r,\delta,T}$ by the equality

$$(\Phi\psi)(t,x,\omega) = e^{-At}x + \int_0^t e^{-A(t-s)}(-B(\omega s)(\psi(s,x,\omega),\psi(s,x,\omega)) + f(\omega s))ds.$$

There exist $\delta_1 = \delta_1(x_0,r) > 0$ and $T_1 = T_1(x_0,r) > 0$ such that $\Phi C_{x_0,r,\delta,T} \subseteq C_{x_0,r,\delta,T}$ for all $\delta \in (0,\delta_1]$ and $T \in (0,T_1]$. In fact,

$$|(\Phi\psi)(t,x,\omega) - x_0|_E \le |e^{-At}x - x_0|_E +$$
$$|\int_0^t e^{-A(t-s)}B(\omega s)(\psi(s,x,\omega),\psi(s,x,\omega)))ds|_E + |\int_0^t e^{-A(t-s)}f(\omega s)ds|_E \le$$
$$m(\delta,T) + \int_0^t Ke^{-a(t-s)}(t-s)^{-\alpha_1}|\psi(s,x,\omega)|_E^2 ds +$$
$$\int_0^t Ke^{-a(t-s)}(t-s)^{-\beta_1}\|f\|ds \le m(\delta,T) + K(|x_0|_E + r)^2\frac{T^{1-\alpha_1}}{1-\alpha_1}$$
$$+K\|f\|\frac{T^{1-\beta_1}}{1-\beta_1} := d_1(x_0,r,\delta,T) \to 0$$

as $\delta + T \to 0$, where $m(\delta,T) := \sup\{|e^{-tA}x - x_0|_E \ : \ t \in [0,T], x \in B[x_0,r])$ and $\|f\| := \sup\{|f(\omega)|_X \ : \omega \in \Omega\}$. Thus there exist $\delta_1 = \delta_1(x_0,r) > 0$ and $T_1 = T_1(x_0,r) > 0$ such that $d_1(x_0,r,\delta,T) \le r$ for all $\delta \in (0,\delta_1]$ and $T \in (0,T_1]$.

Let now $\psi_1,\psi_2 \in C_{x_0,r,\delta,T}$, then

$$|(\Phi\psi_1)(t,x,\omega)) - (\Phi\psi_2)(t,x,\omega))|_E =$$
$$|\int_0^t [B(\omega s)(\psi_1(s,x,\omega),\psi_1(s,x,\omega)) - B(\omega s)(\psi_2(s,x,\omega),\psi_2(s,x,\omega))]|_E \le$$
$$2C_B(|x_0|_E + r)Td(\psi_1,\psi_2)$$

and, consequently, $d(\Phi\psi_1,\Phi\psi_2)\le L(x_0,r,T)d(\psi_1,\psi_2)$, where $L(x_0,r,T)=2C_B(|x_0|+r)T\to 0$ as $T\to 0$. Thus there exists $T_2=T_2(x_0,r)>0$ such that $L(x_0,r,T)<1$ for all $T\in(0,T_2]$. Denote by $\delta(x_0,r):=\delta_1(x_0,r)$ and $T(x_0,r):=min(T_1(x_0,r),T_2(x_0,r))$, then the mapping $\Phi: C_{x_0,r,\delta,T}\to C_{x_0,r,\delta,T}$ is a contraction and, consequently, there exists a unique function $\varphi\in C_{x_0,r,\delta,T}$ satisfying the equation (21) on the interval $[0,T]$. The theorem is proved. □

Remark 2.5. *The theorem 2.4 is true and for the equation*

$$u'+Au=\mathcal{F}(\omega t,u)$$

if the continuous function $\mathcal{F}:\Omega\times E\to F$ satisfies the following conditions:

(i)
$$\sup\{|\mathcal{F}(\omega,0)|_E\ :\ \omega\in\Omega\}<\infty$$
(Ω ,generally speaking, is not compact);

(ii) *F is locally Lipschitz, i.e. for every $r>0$ there exists $L(r)>0$ such that*
$$|\mathcal{F}(\omega,u_1)-\mathcal{F}(\omega,u_2)|_F\le L(r)|u_1-u_2|_E$$
for all $u_1,u_2\in E$ with the condition that $|u_i|_E\le r$ $(i=1,2)$.

Theorem 2.6. *Let $\mathcal{K}$ be a family of solutions of the equation (21) satisfying the following condition: there exists a positive constant M such that $|x(t)|_{\mathcal{D}(A)}\le M$ for all $t\in\mathbb{R}_+$ ($|x|_{\mathcal{D}(A)}:=|Ax|_E$). If there exists $C_B>0$ such that*

$$|B(\omega)(u,v)|_F\le C_B|u|_{\mathcal{D}(A)}|v|_{\mathcal{D}(A)}$$

for all $u,v\in\mathcal{D}(A)$, then this family of functions is uniformly equicontinuous on $\mathbb{R}_+$, i.e. for every $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that $|t_1-t_2|<\delta$ implies $|x(t_1)-x(t_2)|<\varepsilon$ for all $t_1,t_2\in\mathbb{R}_+$ and $x\in\mathcal{K}$.

Proof. Let $\psi\in\mathcal{K}$ and $x:=\psi(0)$, then $\psi(t)=\varphi(t,x,\omega)$ for all $t\in\mathbb{R}_+$ and we have

$$|\varphi(t,x,\omega)-x|_E\le|e^{-At}x-x|_E+|\int_0^t e^{-A(t-\tau)}(-B(\omega s)(\varphi(s,x,\omega),\varphi(s,x,\omega))+$$
$$f(\omega s))ds|_E\le\int_0^t e^{-as}s^{-\alpha_1}|x|_{\mathcal{D}(A)}ds+\int_0^t e^{-as}(t-s)^{-\alpha_1}C_B|\varphi(s,x,\omega)|^2_{\mathcal{D}(A)}ds+$$
$$\int_0^t e^{-a(t-s)}(t-s)^{-\beta_1}\|f\|ds\le\frac{t^{1-\alpha_1}}{1-\alpha_1}M+C_BM^2\frac{t^{1-\alpha_1}}{1-\alpha_1}+\|f\|\frac{t^{1-\beta_1}}{1-\beta_1}. \tag{22}$$

From (22) we obtain

$$\sup\{|\varphi(t,x,\omega)-x|_E\ :\ |x|_{\mathcal{D}(A)},\ \omega\in\Omega\}\to 0$$

as $t\to 0$ and, consequently,

$$|\varphi(t_2,x,\omega)-\varphi(t_1,x,\omega)|_E=|\varphi(t_2-t_1,\varphi(t_1,x,\omega),\omega t_1)-\varphi(t_1,x,\omega)|_E\le$$
$$\sup\{|\varphi(t_2-t_1,x,\omega)-x|_E\ :\ |x|_{\mathcal{D}(A)},\ \omega\in\Omega\}\to 0$$

as $t_2-t_1\to 0$. The theorem is proved. □

Example 2.7. *Navier-Stokes equations. We consider the two-dimensional Navier-Stokes system*

$$u' + q(t)\sum_{i=1}^{2} u_i \partial_i u = \nu \Delta u - \nabla p + \phi(t) \tag{23}$$
$$div\ u = 0, \quad u|_{\partial D} = 0,$$

where D is an open bounded set with smooth boundary $\partial D \in C^2$.

The functional setting of the problem is well known [22],[28] *. We denote by H and V the closure of the linear space $\{u \in C_0^\infty(D)^2,\ div\ u = 0\}$ in $L^2(D)^2$ and $H_0^1(D)^2$, respectively. Denote by P the corresponding orthogonal projection $P : L_2(D)^2 \to H$. We further set*

$$A := -\nu P \Delta,\ B(t)(u,v) := q(t)P(\sum_{i=1}^{2} u_i \partial_i v).$$

The Stokes operator A is self-adjoint and positive with domain $\mathcal{D}(A)$ dense in H. The inverse operator is compact. We define the Hilbert spaces $\mathcal{D}(A^\alpha),\ \alpha \in (0,1]$ as the domains of the powers of A in the standard way. Furthermore, $V := \mathcal{D}(A^{1/2})$, and $|u|_{\mathcal{D}(A^{1/2})} = |\nabla u|$.

Applying P we write (23) as the evolutionary equation of the following form

$$u' + Au + \mathcal{B}(t)(u,u) = \mathcal{F}(t),\ \mathcal{F}(t) := P\phi(t). \tag{24}$$

We suppose that $\mathcal{F} \in C(\mathbb{R}, H)$ $(X = H)$ and $\mathcal{B} \in C(\mathbb{R}, L^2(H, \mathcal{D}(A^{-\delta}))$ $(F = \mathcal{D}(A^{-\delta}))$. Denote by $Y := C(\mathbb{R}, H) \times C(\mathbb{R}, L^2(H, \mathcal{D}(A^{-\delta}))$ and by $(Y, \mathbb{R}, \sigma)$ a dynamical system of translations (Bebutov's dynamical system, see for example, [24],[25] *and* [26]*). Let $\Omega := H(\mathcal{B}, \mathcal{F}) = \overline{\{(\mathcal{B}_\tau, \mathcal{F}_\tau)|\ \tau \in \mathbb{R}\}}$, where $\mathcal{B}_\tau(t) := \mathcal{B}(t+\tau)$ (respectively $\mathcal{F}_\tau(t) := \mathcal{F}(t+\tau)$) for all $t \in \mathbb{R}$, by bar we denote a closure in the compact-open topology and let $(\Omega, \mathbb{R}, \sigma)$ be a dynamical system of translations on Ω.*

Along with the equation (24) we consider its H-class

$$u' + Au + \tilde{\mathcal{B}}(t)(u,u) = \tilde{\mathcal{F}}(t), \tag{25}$$

where $(\tilde{\mathcal{B}}, \tilde{\mathcal{F}}) \in H(\mathcal{B}, \mathcal{F})$. Let $B : \Omega \to L^2(H, \mathcal{D}(A^{-\delta}))$ (respectively $f : \Omega \to H$) be a mapping defined by the equality

$$B(\omega) = B(\tilde{\mathcal{B}}, \tilde{\mathcal{F}}) := \tilde{\mathcal{B}(0)}\ (f(\omega) = f(\tilde{\mathcal{B}}, \tilde{\mathcal{F}}) := \tilde{\mathcal{F}(0)}),$$

where $\omega = (\tilde{\mathcal{B}}, \tilde{\mathcal{F}}) \in \Omega$, then the equation (24) and its H-class can be written in the form (20).

We now use the notations $E = \mathcal{D}(A^{1/2}),\ X = H,\ F = \mathcal{D}(A^{-\delta})$ and see that (9)-(11), (17) and (15)-(16) are valid with $\alpha_1 = 1/2 + \delta,\ \beta_1 = 1/2,\ \beta_2 = 3/2$.

We note that from the conditions (18) -(20), it follows that

$$|B(\omega)(x_1, x_1) - B(\omega)(x_2, x_2)|_F \le C_B(|x_1|_E + |x_2|_E)|x_1 - x_2|_E \tag{26}$$

for all $x_1, x_2 \in E$ and $\omega \in \Omega$.

According to Theorem 2.4 above, through every point $x \in H$ passes a unique solution $\varphi(t,x,\omega)$ of the equation (7) at the initial moment $t=0$. And this solution is defined on some interval $[0, t_{(x,\omega)})$. Let us note that

$$\begin{aligned} w'(t) &= 2Re\langle \varphi'(t,x,\omega), \varphi(t,x,\omega)\rangle = 2Re\langle A(\omega t)\varphi(t,x,\omega), \varphi(t,x,\omega)\rangle + \\ &2Re\langle B(\omega t)(\varphi(t,x,\omega), \varphi(t,x,\omega)), \varphi(t,x,y)\rangle + 2Re\langle f(\omega t), \varphi(t,x,\omega)\rangle \\ &= 2Re\langle A(\omega t)\varphi(t,x,\omega), \varphi(t,x,\omega)\rangle + 2Re\langle f(\omega t), \varphi(t,x,\omega)\rangle \\ &\le -2\alpha|\varphi(t,x,\omega)|_E^2 + 2\|f\||\varphi(t,x,\omega)|_E, \end{aligned} \tag{27}$$

where $\|f\| := \max\{|f(\omega)|_X : \omega \in \Omega\}$ and $w(t) = |\varphi(t,x,\omega)|_E^2$. Then

$$w' \le -2\alpha w + 2\|f\| w^{\frac{1}{2}} \tag{28}$$

and consequently

$$w(t) \le v(t) \tag{29}$$

for all $t \in [0, t_{(x,\omega)})$, where $v(t)$ is an upper solution of the equation

$$v' = -2\alpha v + 2||f|| v^{\frac{1}{2}}, \tag{30}$$

satisfying the condition $v(0) = w(0) = |x|^2$. Thus

$$v(t) = [(|x|_E - \frac{\|f\|}{\alpha})e^{-\alpha t} + \frac{\|f\|}{\alpha}]^2 \tag{31}$$

and consequently

$$|\varphi(t,x,\omega)|_E \le (|x|_E - \frac{\|f\|}{\alpha})e^{-\alpha t} + \frac{\|f\|}{\alpha} \tag{32}$$

for all $t \in [0, t_{(x,\omega)})$. From the inequality (28) it follows that the solution $\varphi(t,x,\omega)$ is bounded and therefore it may be prolonged on $\mathbb{R}_+ = [0, +\infty)$.

Thus we have proved the following theorem.

Theorem 2.8. *Let the conditions (17) and (18) be fulfilled. Then the following statements hold:*

(i) *Every solution $\varphi(t,x,\omega)$ of nonautonomous Navier-Stokes equation (7) is bounded and therefore it may be prolonged on $\mathbb{R}_+$.*

(ii)

$$|\varphi(t,x,\omega)|_E \le C(|x|_E), \tag{33}$$

for all $t \ge 0$, $\omega \in \Omega$ and $x \in E$, where $C(r) = r$ if $r \ge r_0 := \frac{\|f\|}{\alpha}$ and $C(r) = r_0$ if $r \le r_0$;

(iii)

$$\limsup_{t\to+\infty} \sup\{|\varphi(t,x,\omega)|_E \ : |x|_E \le r, \omega \in \Omega\} \le \frac{||f||}{\alpha} \tag{34}$$

for every $r > 0$.

Lemma 2.9. *Under the conditions of Theorem 2.8 we have*

$$\int_t^{t+l} |\varphi(\tau,x,\omega)|_E^2 d\tau \le \frac{r^2}{2\alpha} + \frac{r}{\alpha} l\|f\| := M(r) \tag{35}$$

for all $t \ge 0$ and $r \ge r_0$.

Proof. From the equality (28) after integration in t between t and $t+l$ we obtain

$$2\alpha \int_t^{t+l} |\varphi(\tau, x, \omega)|_E^2 d\tau \leq |\varphi(t, x, \omega)|_E^2 + 2rl\|f\| \tag{36}$$

and,consequently,

$$\int_t^{t+l} |\varphi(\tau, x, \omega)|_E^2 d\tau \leq \frac{r^2}{2\alpha} + \frac{r}{\alpha} l\|f\| := M(r). \tag{37}$$

□

Lemma 2.10. *([29, Ch.3]) (The Uniform Gronwal Lemma). Let g, h, y be three positive locally integrable functions on $]t_0, \infty[$ such that y' is locally integrable on $]t_0, \infty[$, and let them satisfy*

$$y' \leq gy + h \text{ for } t \geq t_0,$$
$$\int_t^{t+l} g(s)ds \leq a_1, \ \int_t^{t+l} h(s)ds \leq a_2, \ \int_t^{t+l} y(s)ds \leq a_3 \text{ for } t \geq t_0,$$

where $l, a_1, a_2, a_3,$ are positive constants. Then

$$y(t+l) \leq (\frac{a_3}{l} + a_2)e^{a_1} \ \forall \ t \geq t_0.$$

Theorem 2.11. *Assume the conditions of Theorem 2.8. If*

$$|\langle B(\omega)(u, v), w\rangle| \leq C|u|^{1/2}|Au|^{1/2}|v|_{1/2}|w| \tag{38}$$
$$\forall \ u \in \mathcal{D}(A), \ v \in V, \ w \in H, \tag{39}$$

then

$$|\varphi(t, x, \omega)|_{\mathcal{D}(A)} \leq K(r) \ \forall |x| \leq r \ (r \geq r_0), \tag{40}$$

where $K(r)$ is some positive constant depending only on r.

Proof. Since

$$\langle A\varphi(t, x, \omega), \varphi(t, x, \omega)\rangle = \frac{1}{2}\frac{d}{dt}|\varphi(t, x, \omega)|_E^2 \tag{41}$$

by taking the scalar product of (7) with Au we find

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|\varphi(t, x, \omega)|_E^2 + |A\varphi(t, x, \omega)|_E^2 + \\ &\langle B(\varphi(t, x, \omega), \varphi(t, x, \omega)), A\varphi(t, x, \omega)\rangle = \langle f(\omega t), A\varphi(t, x, \omega)\rangle. \end{aligned} \tag{42}$$

Taking into account the inequality

$$|\langle f(\omega t), A\varphi(t, x, \omega)\rangle| \leq |f(\omega t)|_E |A\varphi(t, x, \omega)|_E \leq \frac{1}{4}|A\varphi(t, x, \omega)|^2 + \|f\|^2 \tag{43}$$

and using (41) and the Young inequality we obtain

$$\begin{aligned} &|\langle B(\omega)(\varphi(t, x, \omega), \varphi(t, x, \omega)), A\varphi(t, x, \omega)\rangle| \leq \\ &c_1|\varphi(t, x, \omega)|^{1/2}\|\varphi(t, x, \omega)\||A\varphi(t, x, \omega)|^{3/2} \leq \\ &\frac{1}{4}|A\varphi(t, x, \omega)|^2 + c_1'|\varphi(t, x, \omega)|^2\|\varphi(t, x, \omega)\|^4. \end{aligned} \tag{44}$$

Hence

$$\frac{1}{2}\frac{d}{dt}\|\varphi(t,x,\omega)\|^2 + |A\varphi(t,x,\omega)|^2 \le \|f\|^2 + \frac{1}{2}|A\varphi(t,x,\omega)|^2 + c_1'|\varphi(t,x,\omega)|^2\|\varphi(t,x,\omega)\|^4. \tag{45}$$

From this inequality according to Gronwall lemma we can prove that $|\varphi(t,x,\omega)|_{\mathcal{D}(A)}$ is uniformly bounded (w.r.t. x and ω) on interval $[0,l]$. Applying the uniform Gronwall lemma with g,h,y replaced by

$$c_1'|\varphi(t,x,\omega)|^2\|\varphi(t,x,\omega)\|^2\ ,\ \|f\|^2\ ,\ \|\varphi(t,x,\omega)\|^2 \tag{46}$$

we obtain that $\|\varphi(t,x,\omega)\|^2$ is bounded on $[l,\infty[$ and, consequently, it is uniformly bounded on $[0,\infty[$ w.r.t. $\|x\|\le r$ and $\omega\in\Omega$. The theorem is proved. $\square$

3. Nonautonomous dissipative systems and their attractors.

Let Ω and W be two metric spaces, $(\Omega,\mathbb{R},\sigma)$ be a dynamical system on Ω. Let us consider a continuous mapping $\varphi:\mathbb{R}^+\times\Omega\times W\to W$ satisfying the following conditions:

$$\varphi(0,\cdot,\omega)=id_W\quad \varphi(t+\tau,x,\omega)=\varphi(t,\varphi(\tau,x,\omega),\omega\tau)$$

for all $t,\tau\in\mathbb{R}^+$, $\omega\in\Omega$ and $x\in W$. Such mapping φ (or more explicit $\langle W,\varphi,(\Omega,\mathbb{R},\sigma)\rangle$) is called ([1], [26]) a cocycle on $(\Omega,\mathbb{R},\sigma)$ with fiber W.

Remark 3.1. *The nonautonomous Navier-Stokes equation (7) generates a cocycle φ (or more explicit $\langle E,\varphi,(\Omega,\mathbb{R},\sigma)\rangle$), where $\varphi(t,x,\omega)$ is a unique solution of the equation (7) defined on $\mathbb{R}_+$ with the initial condition $\varphi(0,x,\omega)=x$.*

In fact, according to Theorems 2.4 the mapping $\varphi:\times E\times\Omega\to E$ $((t,x,\omega)\to\varphi(t,x,\omega))$ is continuous and in view of uniqueness of solution $\varphi(t,x,\omega)$ we have the following identity: $\varphi(t+\tau,x,\omega)=\varphi(t,\varphi(\tau,x,\omega),\omega\tau)$ for all $t,\tau\in\mathbb{R}_+,x\in E$ and $\omega\in\Omega$, where $\omega\tau:=\sigma(\tau,\omega)$.

Let us look at a relevant example.

Example 3.2. *Let E be a Banach space and $C(\mathbb{R}\times E,E)$ be a space of all continuous functions $F:\mathbb{R}\times E\to E$ equipped with the compact-open topology. Let us consider a parameterized differential equation*

$$\frac{dx}{dt}+Ax=F(\sigma_t\omega,x)\quad(\omega\in\Omega)$$

in Banach space E with $\Omega=C(\mathbb{R}\times E,E)$, where $\sigma_t\omega:=\sigma(t,\omega)$ and the linear operator A is densely defined in E and is such that the linear equation

$$u'+Au=0$$

generates the c_0-semigroup of linear bounded operators

$$e^{-At}:E\to E,\ \varphi(t,x):=e^{-At}x.$$

We will define $\sigma_t:\Omega\to\Omega$ by $\sigma_t\omega(\cdot,\cdot)=\omega(t+\cdot,\cdot)$ for each $t\in\mathbb{R}$ and interpret $\varphi(t,x,\omega)$ as mild solution of the initial value problem

$$\frac{d}{dt}x(t)+Ax=F(\sigma_t\omega,x(t)),\quad x(0)=x. \tag{47}$$

Under appropriate assumptions on $F : \Omega\times E \to E$ (or even $F : \mathbb{R} \times E \to E$ with $\omega(t)$ instead of $\sigma_t\omega$ in (47)) to ensure forward existence and uniqueness, then the solution mapping φ is a cocycle on $(C(\mathbb{R}\times E, E), \mathbb{R}, \sigma)$ with fiber E, where $(C(\mathbb{R}\times E, E), \mathbb{R}, \sigma)$ is a Bebutov's dynamical system (see for example [3],[7]*,* [24],[26]*).*

The triplet $\langle (X, \mathbb{R}_+, \pi), (\Omega, \mathbb{R}, \sigma), h\rangle$, where $h : X \to \Omega$ is a homomorphism from the dynamical system $(X, \mathbb{R}_+, \pi)$ onto $(\Omega, \mathbb{R}, \sigma)$, and is called (see [4],[7]) a nonautonomous dynamical system.

Let φ be a cocycle on $(\Omega, \mathbb{R}, \sigma)$ with the fiber E. Then the mapping $\pi : \mathbb{R}^+ \times\Omega \times E \to \Omega \times E$ defined by

$$\pi(t, x, \omega) := (\varphi(t, x, \omega), \sigma_t\omega)$$

for all $t \in \mathbb{R}^+$ and $(x, \omega)\in E \times \Omega$ forms a semi-group $\{\pi(t, \cdot, \cdot)\}_{t\in\mathbb{R}^+}$ of mappings of $X := \Omega \times E$ into itself, thus a semi-dynamical system on the state space X which is called a skew-product flow [26] and the triplet $\langle (X, \mathbb{R}_+, \pi), (\Omega, \mathbb{R}, \sigma), h\rangle$ (where $h := pr_2 : X \to \Omega$) is a nonautonomous dynamical system.

A cocycle φ over $(\Omega, \mathbb{R}, \sigma)$ with the fiber W is called a compact (bounded) dissipative cocycle, if there is a nonempty compact $K \subseteq W$ such that

$$\lim_{t\to+\infty} \sup\{\beta(U(t, \omega)M, K)|\omega \in \Omega\} = 0 \tag{48}$$

for any $M \in C(W)$ (respectively $M \in \mathcal{B}(W)$), where $C(W)(\ \mathcal{B}(W))$ is the family of all compact (bounded) subsets of W, β a semidistance of Hausdorff and $U(t, \omega) := \varphi(t, \cdot, \omega)$.

Lemma 3.3. *Let Ω be a compact metric space and $\langle W, \varphi, (\Omega, \mathbb{R}, \sigma)\rangle$ be a cocycle over $(\Omega, \mathbb{R}, \sigma)$ with the fiber W. In order for $\langle W, \varphi, (\Omega, \mathbb{R}, \sigma)\rangle$ to be compact (bounded) dissipative, it is necessary and sufficient that the skew-product dynamical system $(X, \mathbb{R}_+, \pi)$ should be a compact (bounded) dissipative one.*

This assertion directly follows from the corresponding definitions (see for example [16],[7]).

A whole trajectory of the semi-group dynamical system $(X, \mathbb{R}_+, \pi)$ (of the cocycle $\langle W, \varphi, (\Omega, \mathbb{R}, \sigma)\rangle$ over $(\Omega, \mathbb{R}, \sigma)$ with the fiber W), which passes through the point $x \in X((u, y) \in W \times \Omega)$, is a continuous mapping $\gamma : \mathbb{R} \to X(\nu : \mathbb{R} \to W)$ which satisfies the conditions : $\gamma(0) = x$ $(\nu(0) = u)$ and $\pi^t\gamma(\tau) = \gamma(t + \tau)$ $(\nu(t + \tau) = \varphi(t, \nu(\tau), \omega t))$ for all $t \in \mathbb{R}_+$ and $\tau \in \mathbb{R}$. If $M \subseteq W$, then we denote by

$$\Omega_\omega(M) = \bigcap_{t\geq 0} \overline{\bigcup_{\tau\geq t} \varphi(\tau, M, \omega^{-\tau})}$$

for every $\omega \in \Omega$, where $\omega^{-\tau} := \sigma(-\tau, \omega)$.

Theorem 3.4. *(*[6],[7]*) Let Ω be a compact metric space, $\langle W, \varphi, (\Omega, \mathbb{R}, \sigma)\rangle$ be a compactly (boundedly) dissipative cocycle and K be a nonempty compact, arising in the equality (84). Then the following assertions hold:*

(i) *$I_\omega := \Omega_\omega(K) \neq \emptyset$, is compact, $I_\omega \subseteq K$ and*

$$\lim_{t\to+\infty} \beta(U(t, \omega^{-t})K, I_\omega) = 0$$

for every $\omega \in \Omega$;

(ii) *$U(t, \omega)I_\omega = I_{\omega t}$ for all $\omega \in \Omega$ and $t \in \mathbb{R}_+$;*

(iii)
$$\lim_{t\to+\infty}\beta(U(t,\omega^{-t})M, I_\omega) = 0$$
for all $M \in C(W)$ *(respectively* $M \in \mathcal{B}(X)$*) and* $\omega \in \Omega$ *;*

(iv)
$$\lim_{t\to+\infty}\sup\{\beta(U(t,\omega^{-t})M, I)|\omega \in \Omega\} = 0$$
for any $M \in C(W)$ *(respectively* $M \in \mathcal{B}(X)$*) , where* $I = \cup\{I_\omega|\ \omega \in \Omega\}$*;*

(v) $I_\omega := pr_1 J_\omega$ *for all* $\omega \in \Omega$*, where* J *is a Levinson's centre of* $(X, \mathbb{R}_+, \pi)$*, and, hence,* $I = pr_1 J$*;*

(vi) *the set* I *is compact;*

(vii) *the set* I *is connected if the spaces* W *and* Y *are connected.*

The family of compact sets $\{I_\omega|\omega \in \Omega\}$ ($I_\omega \subset W$ is nonempty compact for every $\omega \in \Omega$) is called (see, for example, [6] or [7]) the compact global attractor of cocycle φ if the following conditions are fulfilled:

(i) The set $I := \bigcup\{I_\omega|\quad \omega \in \Omega\}$ is precompact.

(ii) $\{I_\omega|\quad \omega \in \Omega\}$ is invariant w.r.t. the cocycle φ, i.e. $\varphi(t,\omega,I_\omega) = I_{\sigma_t\omega}$ for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$.

(iii) The equality $\lim\limits_{t\to+\infty}\sup\limits_{\omega\in\ \Omega}\beta(\varphi(t,K,\omega), I) = 0$ holds for every bounded set $K \subset W$.

Corollary 3.5. *Under the conditions of Theorem 3.4 the cocycle* φ *admits a compact global attractor.*

Dynamical system $(X, \mathbb{R}_+, \pi)$ is called asymptotically compact (see [16],[21] and also [6],[7]) if for any positive invariant bounded set $A \subset X$ there is a compact $K_A \subset X$ such that

$$\lim_{t\to+\infty}\beta(\pi^t A, K_A) = 0. \tag{49}$$

Dynamical system $(X, \mathbb{R}_+, \pi)$ is called compact (completely continuous) if for every bounded set $A \subset X$ there exists a positive number $l = l(A)$ such that the set $\pi^l A$ will be precompact.

It is easy to verify (see for example [7]) that every compact dynamical system $(X, \mathbb{R}_+, \pi)$ is asymptotically compact.

The cocycle $\langle W, \varphi, (Y, \mathbb{R}, \sigma)\rangle$ is called compact (asymptotically compact) if the skew-product dynamical system $(X, \mathbb{R}_+, \pi)$ $(X = W \times Y, \pi = (\varphi, \sigma))$ is compact (respectively asymptotically compact).

Let $(X, \mathbb{R}_+, \pi)$ be compact dissipative and K be a compact set, which attracts all compact subsets of X. Suppose

$$J = \Omega(K), \tag{50}$$

where $\Omega(K) = \bigcap_{t\ge 0}\overline{\bigcup_{\tau\ge t}\pi^\tau K}$. The set J defined by the equality (50) does not depend on selection of the attractor K, and is characterized only by the properties of the dynamical system $(X, \mathbb{R}_+, \pi)$ itself. The set J is called the Levinson's centre of the compact dissipative system $(X, \mathbb{R}_+, \pi)$.

Theorem 3.6. *([6],[7]) Let (E,Ω,h) be a local-trivial Banach fibering, $\langle(E,\mathbb{R}_+,\pi),(\Omega,\mathbb{R},\sigma),h\rangle$ be a nonautonomous dynamical system and the dynamical system $(E,\mathbb{R}_+,\pi)$ be completely continuous. Then the following conditions are equivalent :*

(i) *there is a positive number r such that for any $x\in X$ there will be $\tau=\tau(x)\geq 0$ for which $|x\tau|<r$;*
(ii) *the dynamical system $(E,\mathbb{R}_+,\pi)$ is compact dissipative and*

$$\lim_{t\to+\infty}\sup_{|x|\leq R}\rho(xt,J)=0$$

for any $R>0$, where J is a Levinson's centre of dynamical system $(E,\mathbb{R}_+,\pi)$, that is the nonautonomous system $\langle(E,\mathbb{R}_+,\pi),(\Omega,\mathbb{R},\sigma),h\rangle$ admits a compact global attractor J.

A dynamical system $(X,\mathbb{R}_+,\pi)$ satisfies the condition of Ladyzhenskaya (see [21] and also [7]) if for any bounded set $A\subset X$ there is a compact $K_A\subset X$ such that the equality (86) holds.

Theorem 3.7. *([6],[7]) Let $\langle(E,\mathbb{R}_+,\pi),(\Omega,\mathbb{R},\sigma),h\rangle$ be a nonautonomous dynamical system and let $(E,\mathbb{R}_+,\pi)$ satisfy the condition of Ladyzhenskaya. Then the conditions 1. and 2. of the theorem 3.6 are equivalent.*

Applying general theorems about nonautonomous dissipative systems to nonautonomous system constructed in example 3.2, we will obtain series of facts concerning the equation (7). In particular, from Theorems 2.8, 3.4 and 3.7 the following theorem holds.

Theorem 3.8. *Let Ω be a compact metric space, $(\Omega,\mathbb{R},\sigma)$ be a dynamical system on Ω and let the conditions (17) and (18) be fulfilled. If the cocycle φ generated by the nonautonomous Navier-Stokes equation is asymptotically compact, then for every $\omega\in\Omega$ there exists a non-empty compact and connected $I_\omega\subset E$ such that the following conditions hold:*

(i) *the set $I=\cup\{I_\omega\ :\ \omega\in\Omega\}$ is compact and connected in E;*
(ii) *I is connected if Ω is connected;*
(iii)

$$\lim_{t\to+\infty}\sup_{\omega\in\Omega}\beta(U(t,\omega^{-t})M,I)=0$$

for any bounded set $M\subset E$, where $U(t,\omega)=\varphi(t,\cdot,\omega)$ and β is the semi-distance of Hausdorff;
(iv) *$U(t,\omega)I_\omega=I_{\omega t}$ for all $t\in\mathbb{R}_+$ and $\omega\in\Omega$;*
(v) *I_ω consists of those and only those points $x\in E$ through which passes the solution of the equation (7) bounded on $\mathbb{R}$.*

Theorem 3.9. *Under the conditions of Theorem 3.8*

$$|\varphi(t,x,\omega)|\leq\frac{\|f\|}{\alpha}$$

for all $t\in\mathbb{R}$, $\omega\in\Omega$ and $x\in I_\omega$, where φ is a cocycle generated by the nonautonomous Navier-Stokes equation.

Proof. According to Theorem 3.4 the set $J=\bigcup\{I_\omega\times\{\omega\}:\omega\in\Omega\}$ is a Levinson's centre of dynamical system $(X,\mathbb{R}_+,\pi)$ and according to (50) for any point $(u_0,y_0)=$

$z \in J$ there exists $t_n \to +\infty, u_n \in E$ and $\omega_n \in \Omega$ such that the sequence $\{u_n\}$ is bounded, $u_0 = \lim_{n\to+\infty} \varphi(t_n, u_n, \omega_n)$ and $\omega_0 = \lim_{n\to+\infty} \omega_n t_n$. From the inequality (28) follows that $|u_0| \le \frac{\|f\|}{\alpha}$, i.e. $\varphi(t, x, \omega) \in I_{\omega t}$ for all $\omega \in \Omega$ and $t \ge 0$, hence $|\varphi(t, x, \omega)| \le \frac{\|f\|}{\alpha}$ for any $t \in \mathbb{R}, x \in I_\omega$ and $\omega \in \Omega$. The theorem is proved. □

4. Almost periodic and recurrent solutions of nonautonomous Navier-Stokes equations

Let $\mathbb{T} = \mathbb{R}$ or $\mathbb{R}_+$ and $(X, \mathbb{T}, \pi)$ be a dynamical system. The point $x \in X$ is called a stationary (τ-periodic, $\tau > 0, \tau \in \mathbb{T}$) point, if $xt = x$ ($x\tau = x$ respectively) for all $t \in \mathbb{T}$, where $xt := \pi(t, x)$.

The number $\tau \in \mathbb{T}$ is called $\varepsilon > 0$ shift (almost period) of point $x \in X$ if $\rho(x\tau, x) < \varepsilon$ (respectively $\rho(x(\tau + t), xt) < \varepsilon$ for all $t \in \mathbb{T}$).

The point $x \in X$ is called almost recurrent (almost periodic) if for any ε there exists a positive number l such that on any segment of length l, there will be found a ε shift (almost period) of point $x \in X$.

If point $x \in X$ is almost recurrent and the set $H(x) = \overline{\{xt \mid t \in \mathbb{T}\}}$ is compact, then x is called recurrent.

An autonomous dynamical system $(\Omega, \mathbb{T}, \sigma)$ is said to be pseudo-recurrent if the following conditions are fulfilled:

a) Ω is compact;
b) $(\Omega, \mathbb{T}, \sigma)$ is transitive, i.e. there exists a point $\omega_0 \in \Omega$ such that $\Omega = \overline{\{\omega_0 t \mid t \in \mathbb{T}\}}$;
c) every point $\omega \in \Omega$ is stable in the sense of Poisson, i.e.

$$\mathfrak{N}_\omega = \{\{t_n\} \mid \omega t_n \to \omega \text{ and } |t_n| \to +\infty\} \ne \emptyset.$$

Lemma 4.1. *([10]) Let $< (X, \mathbb{T}_1, \pi), (\Omega, \mathbb{T}_2, \sigma), h >$ be a nonautonomous dynamical system and the following conditions are fulfilled:*

1) *$(\Omega, \mathbb{T}_2, \sigma)$ is pseudo recurrent;*
2) *$\gamma \in C(\Omega, X)$ is an invariant section of the homomorphism $h : X \to \Omega$, i.e. $h(\gamma(\omega)) = \omega$ and $\gamma(\sigma(t, \omega)) = \pi(t, \gamma(\omega))$ for all $\omega \in \Omega$ and $t \in \mathbb{T}_2$.*

Then the autonomous dynamical system $(\gamma(\Omega), \mathbb{T}_2, \pi)$ is pseudo-recurrent.

The solution $\varphi(t, x, \omega)$ of the nonautonomous Navier-Stokes equation (7) is called recurrent (pseudo-recurrent, almost periodic, quasi-periodic), if the point $(x, \omega) \in H \times \Omega$ is a recurrent (pseudo-recurrent, almost periodic, quasi-periodic) point of skew-product dynamical system $(X, \mathbb{R}_+, \pi)$ $(X = H \times \Omega$ and $\pi = (\varphi, \sigma))$.

We note (see, for example, [24],[25] and [23]) that if $\omega \in \Omega$ is a stationary (τ-periodic, almost periodic, quasi-periodic, recurrent) point of dynamical system $(\Omega, \mathbb{R}, \sigma)$ and $h : \Omega \to X$ is a homomorphism of dynamical system $(\Omega, \mathbb{R}, \sigma)$ onto $(X, \mathbb{R}_+, \pi)$, then the point $x = h(\omega)$ is a stationary (τ-periodic, almost periodic, quasi-periodic, recurrent) point of the system $(X, \mathbb{R}_+, \pi)$.

Let $X = H \times \Omega$ and $\pi = (\varphi, \sigma)$, then mapping $h : \Omega \to X$ is a homomorphism of dynamical system $(\Omega, \mathbb{R}, \sigma)$ onto $(X, \mathbb{R}_+, \pi)$ if and only if $h(\omega) = (u(\omega), \omega)$ for all $\omega \in \Omega$, where $u : \Omega \to H$ is a continuous mapping with the condition that $u(\omega t) = \varphi(t, u(\omega), \omega)$ for all $\omega \in \Omega$ and $t \in \mathbb{R}_+$.

The following affirmations hold:

Lemma 4.2. *Let Ω be a compact metric space, A be a linear operator densely defined in E such that the equation*

$$x' + Ax = 0$$

generates a c_0-semigroup $\{U(t)\}_{t\geq 0}$. If the condition (17) is fulfilled, then

$$\|U(t)\| \leq \exp(-\alpha t)$$

for all $t \in \mathbb{R}_+$, where $U(t)$ is an operator of Cauchy of the equation (51).

Proof. Let $\varphi(t, x) := U(t)x$, then according to inequality 17 we have

$$\frac{d}{dt}|\varphi(t, x)|^2 \leq -2\alpha|\varphi(t, x)|^2$$

and, consequently,$|\varphi(t, x)| \leq \exp(-\alpha t)|x|$ for all $x \in H$ and $t \in \mathbb{R}_+$. Thus we have $|U(t)x| \leq \exp(-\alpha t)|x|$, therefore $\|U(t)\| \leq \exp(-\alpha t)$ for all $t \in \mathbb{R}_+$. □

Lemma 4.3. *Suppose that condition (17) is fulfilled. Then for every function $f \in C(\Omega, H)$ there exists a unique function $\gamma \in C(\Omega, H)$ defined by equality*

$$\gamma(\omega) = \int_{-\infty}^{0} U(-\tau) f(\omega\tau) d\tau$$

such that

$$\gamma(\omega t) = \varphi(t, \gamma(\omega), \omega) \tag{51}$$

for every $\omega \in \Omega$ and $t \in \mathbb{R}_+$, where $\varphi(t, x, \omega)$ is a solution of the equation

$$u' = Au + f(\omega t)$$

with the initial condition $\varphi(0, x, \omega) = x$ and the following inequality

$$\|\gamma\| \leq \frac{1}{\alpha}\|f\|$$

takes place.

Proof. The formulated statement results from Lemma 4.2 and Proposition 7.33 from [12]. □

Lemma 4.4. *Let Ω be a compact metric space, φ be a cocycle generated by the nonautonomous Navier-Stokes equation (7) and $\alpha^{-2}\|f\|C_B < 1$, then the following inequality*

$$|\varphi(t, x_1, \omega) - \varphi(t, x_2, \omega)| \leq e^{-(\alpha - C_B\frac{\|f\|}{\alpha})t}|x_1 - x_2|$$
$$(x_1, x_2 \in B[0, r_0], \omega \in \Omega \text{ and } t \in \mathbb{R}_+)$$

takes place.

Proof. Let $r_0 := \frac{\|f\|}{\alpha}$ and $x_1, x_2 \in B[0, r_0] := \{x \in E \ : |x| \le r_0\}$. According to Theorem 2.8 we have $|\varphi(t, x_i, \omega)| \le r_0$ for all $t \ge 0, \omega \in \Omega$ and $i = 1, 2$. Denote by $\psi(t) := \varphi(t, x_1, \omega) - \varphi(t, x_2, \omega)$, then we obtain

$$\frac{d}{dt}|\psi(t)|^2 = 2Re\langle A\psi(t), \psi(t)\rangle + 2Re\langle B(\omega t)(\psi(t), \varphi(t, x_2, \omega)), \psi(t)\rangle \le$$
$$-2\alpha|\psi(t)|^2 + 2C_B|\varphi(t, x_2, \omega)||\psi(t)|^2 \le$$
$$-2\alpha|\psi(t)|^2 + 2C_B\frac{\|f\|}{\alpha}|\psi(t)|^2 = -2(\alpha - C_B\frac{\|f\|}{\alpha})|\psi(t)|^2$$

and, consequently,

$$|\psi(t)|^2 \le e^{-2(\alpha - C_B\frac{\|f\|}{\alpha})t}|\psi(0)|^2.$$

Thus we have

$$|\varphi(t, x_1, \omega) - \varphi(t, x_2, \omega)| \le e^{-(\alpha - C_B\frac{\|f\|}{\alpha})t}|x_1 - x_2|$$
$$(x_1, x_2 \in B[0, r_0], \omega \in \Omega \text{ and } t \in \mathbb{R}_+)$$

for all $x_1, x_2 \in B[0, r_0], \omega \in \Omega$ and $t \in \mathbb{R}_+$. The Lemma is proved. □

Theorem 4.5. *Let Ω be a compact metric space, φ be a cocycle generated by the nonautonomous Navier-Stokes equation (7) and $\frac{\|f\|C_B}{\alpha^2} < 1$. Then there exists a function $\gamma \in C(\Omega, B[0, r_0])$ such that:*

a.

$$\gamma(\omega t) = \varphi(t, \gamma(\omega), \omega) \tag{52}$$

for every $\omega \in \Omega$ and $t \in \mathbb{R}_+$, where $\varphi(t, x, \omega)$ is a solution of the equation (7) with the initial condition $\varphi(0, x, \omega) = x$;

b.

$$\|\gamma\| \le \frac{\|f\|}{\alpha}; \tag{53}$$

c.

$$|\varphi(t, x, \omega) - \gamma(\omega t)| \le e^{-(\alpha - C_B\frac{\|f\|}{\alpha})t}|x - \gamma(\omega)| \tag{54}$$

for all $x \in E$, $\omega \in \Omega$ and $t \in \mathbb{R}_+$, where $\|\gamma\| := \sup\{|\gamma(\omega)| \ : \omega \in \Omega\}$.

Proof. Let $\Gamma := C(\Omega, B[0, r_0])$ $(C(\Omega, E))$ be a space of all the continuous functions $f : \Omega \to B[0, r_0]$ (respectively $f : \Omega \to E$) equipped with the distance

$$d(f_1, f_2) = \max\{|f_1(\omega) - f_2(\omega)| \ : \ \omega \in \Omega\} \ .$$

Then (Γ, d) (respectively $(C(\Omega, E), d)$) is a complete metric space.

Let $t \in \mathbb{R}_+$. We define the mapping $S^t : \Gamma \to C(\Omega, E)$ by the equality

$$(S^t\nu)(\omega) := U(t, \omega^{-t})\nu(\omega^{-t})$$

for all $\omega \in \Omega$, where $\omega^{-t} := \sigma(-t, \omega)$ and $U(t, \omega) := \varphi(t, \cdot, \omega)$. According to Theorem 2.8 we have $S^t(\Gamma) \subseteq \Gamma$ for all $t \in \mathbb{R}_+$. It is easy to see that the family of mappings $\{S^t \mid t \in \mathbb{R}_+\}$ possesses the following properties:

(i)

$$S^0 = Id_\Gamma$$

and

(ii)

$$S^{t+\tau} = S^t S^\tau$$

for all $t, \tau \in \mathbb{R}_+$.

Thus $\{S^t \mid t \in \mathbb{R}_+\}$ forms a commutative semigroup with identity element. Now we will show that the mapping $S^t (t > 0)$ is a contraction. In fact, let $\nu_1, \nu_2 \in \Gamma$, then we have

$$(S^t\nu_1)(\omega) - (S^t\nu_2)(\omega) = U(t, \omega^{-t})\nu_1(\omega^{-t}) - U(t, \omega^{-t})\nu_2(\omega^{-t}). \tag{55}$$

From Lemma 4.4 and the equality (55) follows that

$$d(S^t\nu_1, S^t\nu_2) \le e^{-(\alpha - C_B \frac{\|f\|}{\alpha})t} d(\nu_1, \nu_2)$$

for all $t \in \mathbb{R}_+$ and, consequently, there exists a unique common fix point $\gamma \in \Gamma$, i.e. $S^t\gamma = \gamma$ for all $t \in \mathbb{R}_+$. In particular,

$$U(t, \omega^{-t})\gamma(\omega^{-t}) = \gamma(\omega)$$

for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$. From this equality follows that

$$\gamma(\omega t) = U(t, \omega)\gamma(\omega) = \varphi(t, \gamma(\omega), \omega)$$

for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$.

Let $x \in E, \varphi(t, x, \omega)$ be a unique solution of the equation (7) with the initial condition $\varphi(0, x, \omega) = x$ and let $\gamma \in \Gamma$ be a function with the property (52). Denote by $\psi(t) := \varphi(t, x, \omega) - \gamma(\omega t)$, then we have

$$\frac{d}{dt}|\psi(t)|^2 = 2Re\langle A\psi(t), \psi(t)\rangle + 2Re\langle B(\omega t)(\psi(t), \gamma(\omega t)), \psi(t)\rangle \le$$
$$-2\alpha|\psi(t)|^2 + 2C_B|\gamma(\omega t)||\psi(t)|^2 \le -2\alpha|\psi(t)|^2 +$$
$$2C_B\frac{\|f\|}{\alpha}|\psi(t)|^2 = -2(\alpha - C_B\frac{\|f\|}{\alpha})|\psi(t)|^2$$

and, consequently,

$$|\psi(t)|^2 \le e^{-2(\alpha - C_B \frac{\|f\|}{\alpha})t}|\psi(0)|^2.$$

Thus we have

$$|\varphi(t, x, \omega) - \gamma(\omega t)| \le e^{-(\alpha - C_B \frac{\|f\|}{\alpha})t}|x - \gamma(\omega)|$$

for all $x \in E$, $\omega \in \Omega$ and $t \in \mathbb{R}_+$. The theorem is proved. □

Corollary 4.6. *Under the conditions of Theorem 4.5 there exists a unique function* $\gamma \in C(\Omega, E)$ *such that*

$$\gamma(\omega t) = \varphi(t, \gamma(\omega), \omega) \tag{56}$$

for all $t \in \mathbb{R}_+$ *and* $\omega \in \Omega$.

Proof. Let $\overline{\gamma} \in C(\Omega, E)$ be a function satisfying the equality (56) and $\gamma \in \Gamma = C(\Omega, B[0, r_0])$ be the function from Theorem 4.5. Since $\overline{\gamma}(\omega t) = \varphi(t, \overline{\gamma}(\omega), \omega)$ for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$, then according to the inequality (54) we have

$$|\overline{\gamma}(\omega t) - \gamma(\omega t)| \le e^{-(\alpha - C_B \frac{\|f\|}{\alpha})t} |\overline{\gamma}(\omega) - \gamma(\omega)| \tag{57}$$

for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$. In particular, from (57) we obtain

$$|\overline{\gamma}(\omega) - \gamma(\omega)| \le e^{-(\alpha - C_B \frac{\|f\|}{\alpha})t} |\overline{\gamma}(\omega^{-t}) - \gamma(\omega^{-t})| \le e^{-(\alpha - C_B \frac{\|f\|}{\alpha})t} \|\overline{\gamma} - \gamma\| \tag{58}$$

for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$, where $\omega^{-t} := \sigma(-t, \omega)$ and $\|\overline{\gamma} - \gamma\| := \max\{|\overline{\gamma}(\omega) - \gamma(\omega)| \ : \ \omega \in \Omega\}$. Passing to the limit in the inequality (58) we obtain $\overline{\gamma}(\omega) = \gamma(\omega)$ for all $\omega \in \Omega$. □

Corollary 4.7. *Under the conditions of Theorem 4.5 the equation (7) admits a compact global attractor* $\{I_\omega \ : \ \omega \in \Omega\}$ *and* $I_\omega = \{\gamma(\omega)\}$ *for all* $\omega \in \Omega$, *where* $\gamma \in \Gamma$ *is a function from Theorem 4.5.*

Corollary 4.8. *Let* Ω *be a compact minimal set containing only periodic (quasi-periodic, almost periodic, recurrent, pseudo-recurrent) motions, then under the conditions of Theorem 4.5, the nonautonomous Navier-Stokes equation (7) admits a unique periodic (quasi-periodic, almost periodic, recurrent, pseudo-recurrent) solution* $\gamma(\omega t)$ *and every other solution of this equation is asymptotically periodic (asymptotically quasi-periodic, asymptotically almost periodic, asymptotically recurrent, asymptotically pseudo-recurrent).*

Proof. Let $\gamma \in \Gamma$ be a function from Theorem 4.5, then according to this theorem we have $\varphi(t, \gamma(\omega), \omega) = \gamma(\omega t)$ for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$ and, consequently, the solution $\varphi(t, \gamma(\omega), \omega)$ is periodic (quasi-periodic, almost periodic, recurrent). Let $\varphi(t, x, \omega)$ be an arbitrary solution of the equation (7). Then taking into consideration the inequality (54), we conclude that $\varphi(t, x, \omega)$ is asymptotically periodic (asymptotically quasi-periodic, asymptotically almost periodic, asymptotically recurrent, asymptotically pseudo-recurrent). □

5. Uniform averaging principle on a finite time interval

We shall be dealing with the nonautonomous Navier-Stokes equation

$$u' + \varepsilon A u + \varepsilon B(u, u) = \varepsilon f(\omega t), \tag{59}$$

where $\varepsilon \in [0, \varepsilon_0]$, A is linear and B is a bilinear operator, and f is a forsing term.

Below we will use some notions, notations and results from [20]. Let Banach spaces $E, F, X, \mathcal{E}$ satisfy

$$E \subset F; \quad E, F, X \subset \mathcal{E},$$

each embedding being dense and continuous.

We suppose that the linear equation

$$u' = Au \tag{60}$$

generates the c_0-semigroup of linear bounded operators

$$e^{At} : \mathcal{E} \to \mathcal{E}, \tag{61}$$

which for $t > 0$ can be extended to the linear bounded operators from F to E satisfying the estimates (9)-(11).

We also suppose that the following condition is satisfied

$$Ae^{At} = e^{At}A, \tag{62}$$

in the sense of $L(F,E) := \{A : F \to E \ |A \text{ is linear and bounded } \}$ equipped with the operational norm.

The external force $f : \Omega \to X$ is continuous, i.e. $f \in C(\Omega, X)$.

The operators e^{-At} $(t > 0)$ can be extended to linear bounded operators from X to E satisfying the estimates (15)- (16) and the equation (62), this time in the sense of (X, E) .

Let $f(\omega) = f_0(\omega) + f_1(\omega)$ $(f_0, f_1 \in C(\Omega, X))$ for all $\omega \in \Omega$ and the average of $f_1(\omega)$ is equal to 0, that is,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f_1(\omega\tau)d\tau = 0 \tag{63}$$

uniformly with respect to $\omega \in \Omega$.

Remark 5.1. *1. The condition* (63) *is fulfilled if a dynamical system* $(\Omega, \mathbb{R}, \sigma)$ *is strictly ergodic, i.e. on* Ω *there exists a unique invariant measure* μ *w.r.t.* $(\Omega, \mathbb{R}, \sigma)$.

2. According to Lemma 5.1 from [9] *the equality (63) takes place if and only if there exists a positive continuous on* $\mathbb{R}_+$ *function* k *with* $\lim_{t\to\infty} k(t) = 0$ *such that*

$$|\frac{1}{t} \int_0^t f_1(\omega\tau)d\tau|_X \le k(t) \tag{64}$$

for all $\omega \in \Omega$ *and* $t \in \mathbb{R}_+$.

Along with the equation (59) we consider also the partial averaged equation

$$u' + \varepsilon Ax + \varepsilon B(u,u) = \varepsilon f_0(\omega t). \tag{65}$$

If we introduce the "slow time" $\tau := \varepsilon t$ $(\varepsilon > 0)$, then the equations (59) and (65) can be written in the following way

$$u' + Au + B(u,u) = f(\omega \frac{\tau}{\varepsilon}) \tag{66}$$

and

$$\bar{u}' + A\bar{u} + B(\bar{u},\bar{u}) = f_0(\omega \frac{\tau}{\varepsilon}). \tag{67}$$

We will consider mild solutions $u(t)$ and $\bar{u}(t)$ of the equations (66) and (67), i.e. $u, \bar{u} \in C([0,T], E)$ and satisfy the following integral equations

$$u(\tau) = e^{-A\tau}x + \int_0^\tau e^{-A(\tau-s)}(-B(u(s),u(s)) + f(\omega\frac{s}{\varepsilon}))ds, \tag{68}$$

and

$$\bar{u}(\tau) = e^{-A\tau}x + \int_0^\tau e^{-A(\tau-s)}(-B(\bar{u}(s),\bar{u}(s)) + f_0(\omega\frac{s}{\varepsilon}))ds. \tag{69}$$

Denote by $\varphi(\tau,x,\omega,\varepsilon)$ ($\bar{\varphi}(\tau,x,\omega,\varepsilon)$) a unique solution of the equation (68) (respectively (69)). According to Theorem 2.8 the cocycle $\varphi(\cdot,\cdot,\cdot,\varepsilon)$ ($\bar{\varphi}(\cdot,\cdot,\cdot,\varepsilon)$), generated by the equation (68) (respectively (69)), has an absorbing ball $B[0,R_0]$ ($B[0,\bar{R}_0]$) in E, where $R_0 := \frac{\|f\|}{\alpha}$ ($\bar{R}_0 := \frac{\|f_0\|}{\alpha}$). This means that for every ball $B[0,R]$ (respectively $B[0,\bar{R}]$) there exists a positive number $L = L(R)$ (respectively $\bar{L} = \bar{L}(\bar{R})$) such that

$$U(t,\omega,\varepsilon)B[0,R] \subseteq B[0,R_0] \tag{70}$$

$$(\bar{U}(t,\omega,\varepsilon)B[0,\bar{R}] \subseteq B[0,\bar{R}_0]) \tag{71}$$

for all $t \geq L$ ($t \geq \bar{L}$), $\varepsilon \in [0,\varepsilon_0]$ and $\omega \in \Omega$, where $U(t,\omega,\varepsilon) := \varphi(t,\cdot,\omega,\varepsilon)$ ($\bar{U}(t,\omega,\varepsilon) := \bar{\varphi}(t,\cdot,\omega,\varepsilon)$.

According to Theorem 2.8 the cocycle $\varphi(\cdot,\cdot,\cdot,\varepsilon)$ ($\bar{\varphi}(\cdot,\cdot,\cdot,\varepsilon)$) is uniformly bounded for $t \geq 0$, that is, for every ball $B[0,R_1]$ ($B[0,\bar{R}_1]$) there exists a ball $B[0,R_2]$ ($B[0,\bar{R}_2]$) such that

$$U(t,\omega,\varepsilon)B[0,R_1] \subseteq B[0,R_2] \tag{72}$$

$$(\bar{U}(t,\omega,\varepsilon)B[0,\bar{R}_1] \subseteq B[0,\bar{R}_2]) \tag{73}$$

for all $t \geq 0, \varepsilon \in [0,\varepsilon_0]$ and $\omega \in \Omega$.

Theorem 5.2. *Let $L > 0$ be arbitrary but fixed. If $\varphi(0,x,\omega,\varepsilon) = \bar{\varphi}(0,x,\omega,\varepsilon) = x \in B[0,\overline{R}_0]$, that is, the initial points coincide and belong to the absorbing ball of the equation (67) and condition (38) is fulfilled, then the following relation takes place*

$$\sup\{|\varphi(t,x,\omega,\varepsilon) - \bar{\varphi}(t,x,\omega,\varepsilon)|_E \ : \ 0 \leq t \leq L,\ |x|_E \leq \overline{R}_0,\ \omega \in \Omega\} \to 0 \tag{74}$$

as $\varepsilon \to 0$.

Proof. The proof below goes along the same lines as the proofs of the corresponding results from [15],[19] and [20]. We set $v(t) := \varphi(t,x,\omega,\varepsilon) - \bar{\varphi}(t,x,\omega,\varepsilon)$. Subtracting the equation (68) from the equation (69), we obtain

$$v(t) = \int_0^t e^{(t-s)A}(-B(v(s),\varphi(s,x,\omega,\varepsilon)) - B(\bar{\varphi}(s,x,\omega,\varepsilon),v(s)))ds + \int_0^t e^{(t-s)A} f_1(\omega s)ds \tag{75}$$

According to Theorem 2.8 $|\varphi(t,x,\omega,\varepsilon)|, |\bar{\varphi}(t,x,\omega,\varepsilon)| \leq r_0$ for all $t \geq 0$, where $r_0 := \max\{\frac{\|f\|}{\alpha}, \frac{\|f_1\|}{\alpha}\}$. In view of (75) $v(t)$ satisfies the inequality

$$|v(t)|_E \leq |\int_0^t e^{(t-s)A}(B(v(s),\varphi(s,x,\omega,\varepsilon)) + B(\bar{\varphi}(s,x,\omega,\varepsilon),v(s)))ds|_E + |\int_0^t e^{(t-s)A} f_1(\omega\frac{s}{\varepsilon})ds|_E\ ,\ t \in [0,L]. \tag{76}$$

By (26) and (27) we see that the first term on the right-hand side of (76) is less than

$$2r_0 \cdot K \cdot C_B \int_0^t e^{-a(t-s)}(t-s)^{\alpha_1}|v(s)|_E ds\ .$$

We now show that the second term in (76) tends to 0 as $\varepsilon \to 0$ uniformly w.r.t. $t \in [0, L], |x| \le R_0$ and $\omega \in \Omega$. Integrating by part in s and taking into account inequalities (10)-(11), (15)-(16) and (64) we find

$$|\int_0^t e^{(t-s)A} f_1(\omega \frac{s}{\varepsilon}) ds|_E = |e^{At} \int_0^t f_1(\omega \frac{s}{\varepsilon}) ds + \int_0^t A e^{(t-s)A} \int_t^s f_1(\omega \frac{\tau}{\varepsilon}) d\tau|_E ds \le$$
$$\|e^{A\tau}\|_{X\to E} | \int_0^t f_1(\omega \frac{s}{\varepsilon}) ds|_X + \int_0^t \|A e^{A(t-s)}\|_{X \to E} | \int_t^s f_1(\omega \frac{s}{\varepsilon}) ds|_X \le$$
$$K t^{1-\beta_1} e^{-at} k_1(\frac{t}{\varepsilon}) + \int_0^t K (t-s)^{1-\beta_2} e^{-a(t-s)} k_1(\frac{t-s}{\varepsilon}) ds.$$

Let $\alpha \in [0, 1), \nu \in (0, 1)$ and $\beta \in [0, 2)$. Since

$$t^\alpha k_1(\frac{t}{\varepsilon}) \le \sup_{0 \le t \le \varepsilon^\nu} t^\alpha k_1(\frac{t}{\varepsilon}) + \sup_{\varepsilon^\nu \le t \le L} t^\alpha k_1(t) \le$$
$$\varepsilon^{\alpha\nu} k_1(0) + L^\alpha k_1(\varepsilon^{\nu-1}) := c_1(\varepsilon) \to 0 \text{ as } \varepsilon \to 0$$

and

$$\int_0^t s^{1-\beta} k_1(\frac{s}{\varepsilon}) ds = \int_0^{\varepsilon^\nu} s^{1-\beta} k_1(\frac{s}{\varepsilon}) ds + \int_{\varepsilon^\nu}^t s^{1-\beta} k_1(\frac{s}{\varepsilon}) ds \le$$
$$k_1(0) \frac{\varepsilon^{\nu(2-\beta)}}{2-\beta} + k(\varepsilon^{\nu-1}) \frac{(t^{2-\beta} - \varepsilon^{\nu(2-\beta)})}{2-\beta} \le$$
$$k_1(0) \frac{\varepsilon^{\nu(2-\beta)}}{2-\beta} + k(\varepsilon^{\nu-1}) \frac{(L^{2-\beta} - \varepsilon^{\nu(2-\beta)})}{2-\beta} := c_2(\varepsilon) \text{ as } \varepsilon \to 0$$

uniformly w.r.t. $t \in [0, L]$ and $\omega \in \Omega$, then

$$\sup_{0 \le t \le L} | \int_0^t e^{(t-s)A} f_1(\omega \frac{s}{\varepsilon}) ds|_E \to 0 \text{ as } \varepsilon \to 0.$$

Thus,

$$|v(t)|_E \le C(\varepsilon) + D \cdot \int_0^t (t-s)^{-\alpha_1} |v(s)|_E ds$$

for all $t \in [0, L]$, where $C(\varepsilon) := c_1(\varepsilon) + +c_2(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $D := 2R_0 C_B K$.

We now use the known inequality [18, Ch.7]. If

$$u(t) \le a + b \int_0^t (t-s)^{\beta-1} u(s) ds, \ 0 < \beta \le 1,$$

then

$$u(t) \le a G_\beta([b\Gamma(\beta)]^{1/\beta} t),$$

where $G_\alpha(x)$ is a monotone function, while $\Gamma(\beta)$ is a gamma function.

In our case we have

$$|v(t)|_E \le C(\varepsilon) G_\beta([b\Gamma(\beta)]^{1/\beta} t) \le$$
$$C(\varepsilon) G_\beta([b\Gamma(\beta)]^{1/\beta} L) := d(\varepsilon) \to 0 \ (\beta := 1 - \alpha_1 \in (0, 1])$$

as $\varepsilon \to 0$ uniformly w.r.t. $\omega \in \Omega$, $x \in B[0, R_0]$ and $t \in [0, L]$ for every $L > 0$. The theorem is proved. $\square$

6. Global averaging principle for the nonautonomous Navier-Stokes equations

Let Ω be a compact metric space, $(\Omega, \mathbb{R}, \sigma)$ be a dynamical system on Ω, E be a Banach space and $\langle E, \varphi, (\Omega, \mathbb{R}, \sigma)\rangle$ be a cocycle on $(\Omega, \mathbb{R}, \sigma)$ with fiber E.

A family of nonempty compact sets $\{ I_\omega \mid \omega \in \Omega\}$ $(I_\omega \subset E)$ is called a local compact attractor (local compact forward attractor) if the following conditions are fulfilled:

(i)
$$I = \bigcup \{I_\omega \ : \ \omega \in \Omega\}$$
is compact;

(ii)
$$\varphi_{\lambda_0}(t, I^{\lambda_0}_\omega, \omega) = I^{\lambda_0}_{\sigma(t,\omega)}$$
for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$;

(iii) there exists $R_0 > 0$ such that $I \subset B(0, R_0) := \{x \in E |\ |x| < R_0\}$ and
$$\lim_{t\to\infty} \sup_{\omega\in\Omega} \beta(\varphi(t, B[0, R_0], \omega), I) = 0$$
(respectively $\lim\limits_{t\to\infty} \sup\limits_{\omega\in\Omega} \beta(\varphi(t, B[0, R_0], \omega), I_{\omega t}) = 0$)

Theorem 6.1. *Let Λ be a compact metric space, E be a Banach space and φ_λ $(\lambda \in \Lambda)$ be a cocycle on $(\Omega, \mathbb{R}, \sigma)$ with fiber E. Suppose that the following conditions are fulfilled:*

(i) *the cocycle ϕ_{λ_0} admits a local compact forward attractor,*

(ii) *the following relation takes place*

$$m_L(\lambda) := \sup\{|\varphi_\lambda(t, x, \omega) - \varphi_{\lambda_0}(t, x, \omega)| \ : \ 0 \le t \le L, \omega \in \Omega, |x| \le R_0\} \to 0 \tag{80}$$

as $\lambda \to \lambda_0$ for every positive number L;

(iii) *every cocycle φ_λ is asymptotically compact.*

Then the following statements are valid:

a. *there exists a positive number μ such that for all $\lambda \in B[\lambda_0, \mu] := \{\lambda \in \Lambda \ : \ \rho(\lambda, \lambda_0) \le \mu\}$ the cocycle φ_λ admits a forward attractor $\{I_\omega \ : \ \omega \in \Omega\}$ in $B[0, R_0]$;*

b.
$$\sup_{\omega\in\Omega} \beta(I^\lambda_\omega, I^{\lambda_0}_\omega) \to 0$$
as $\lambda \to \lambda_0$.

Proof. Let $\rho > 0$ be an arbitrary small number such that $B[I^{\lambda_0}, \rho] \subset B[0, R_0]$. We choose $L = L(\frac{\rho}{3})$ according to the condition

$$\varphi_{\lambda_0}(t, B[0, R_0], \omega) \subset B[I^{\lambda_0}_\omega, \frac{\rho}{3}]$$

for all $\omega \in \Omega$ and $t \ge L(\frac{\rho}{3})$. Now we choose $\varepsilon_0 = \varepsilon_0(L)$ so that $m(\lambda) < \frac{\rho}{3}$ for all $\lambda \in B[\lambda_0, \varepsilon_0]$.

Let $t_1 := L$, then we have $\varphi_{\lambda_0}(t_1, x, \omega) \in B[I^{\lambda_0}_{\omega t_1}, \frac{\rho}{3}]$ and $\varphi_{\lambda}(t_1, x, \omega) \in B[I^{\lambda_0}_{\omega t_1}$. We take the point $x_1 := \varphi_{\lambda}(t_1, x, \omega)$ as initial and consider $\varphi_{\lambda}(t, x_1, \omega t_1)$ on the segment $[0, L]$,

$$\varphi_{\lambda_0}(t, x_1, \omega t_1);\ \varphi_{\lambda}(t, x_1, \omega t_1) = \varphi_{\lambda}(t, \varphi_{\lambda}(t_1, x, \omega), \omega t_1) = \varphi_{\lambda}(t + t_1, x, \omega).$$

On this segment $\varphi_{\lambda}(t, x_1, \omega t_1)$ and $\varphi_{\lambda_0}(t, x_1, \omega t_1)$ will diverge by the value less than $\frac{\rho}{3}$. Since $\varphi_{\lambda_0}(t, x_1, \omega t_1) \in B[I_{\omega t_1}, \frac{\rho}{3}]$, we get $\varphi_{\lambda}(2t_1, x, \omega) \in B[I_{\omega 2t_1}, \frac{2\rho}{3}]$.

If we take the point $x_2 := \varphi_{\lambda}(2t_1, x, \omega)$ as initial one, then we see that the situation is similar to that which occurred at the previous step.

Repeating this process, we arrive at the conclusion that $\varphi_{\lambda}(t, x, \omega) \in B[I^{\lambda_0}_{\omega t}, \rho] \subset B(0, R_0)$ for all $t \geq L(\frac{\rho}{3})$ and $\omega \in \Omega$. Since the cocycle φ_{λ} is asymptotical compact then according to Theorem 3.4 and Corollary 3.5 it admits a forward attractor $\{I_{\lambda}\ :\ \omega \in \Omega\}$ such that $I^{\lambda} := \bigcup \{I^{\lambda}_{\omega}\ :\ \omega \in \Omega\} \subseteq B[I^{\lambda_0}, \rho]$ and, consequently, $\beta(I^{\lambda}, I^{\lambda_0}) \to 0$ as $\lambda \to \lambda_0$.

Below we proved the inclusion $\varphi_{\lambda}(t, B[0, R_0], \omega) \subseteq B[I^{\lambda_0}_{\omega t}, \frac{\rho}{3}]$ for all $t \geq L$ and $\omega \in \Omega$ and, consequently, we obtain

$$\varphi_{\lambda}(t, B[0, R_0], \omega_t) \subseteq B[I^{\lambda_0}_{\omega}, \frac{\rho}{3}] \tag{81}$$

for all $t \geq L$ and $\omega \in \Omega$. Taking into consideration that

$$I^{\lambda}_{\omega} = \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} \varphi_{\lambda}(\tau, B[0, R_0], \sigma(-\tau, \omega))} \tag{82}$$

we see that from (81) and (82) follows that $I^{\lambda}_{\omega} \subseteq B[I^{\lambda_0}_{\omega}, \rho]$ for all $\omega \in \Omega$ and $\lambda \in B[\lambda_0, \varepsilon_0]$ and, consequently, $\sup\limits_{\omega \in \Omega} \beta(I^{\lambda}_{\omega}, I^{\lambda_0}_{\omega}) \to 0$ as $\lambda \to \lambda_0$. The theorem is proved. □

Remark 6.2. *1. The second condition of Theorem 6.1 is fulfilled, for example, if the space E is finite-dimensional and the mapping $\varphi : \mathbb{R}_+ \times E \times \Omega \times \Lambda \to E$, defined by the equality $\varphi(t, x, \omega, \lambda) := \varphi_{\lambda}(t, x, \omega)$, is continuous.*

In fact, if we suppose that it is not true, then there exist $L_0 > 0,\ \lambda_k \to \lambda_0,\ x_k \in B[0, R_0], t_l \in [0, L_0]$ and $\omega_k \in \Omega$ such that

$$|\varphi_{\lambda_k}(t_k, x_k, \omega_k) - \varphi_{\lambda_0}(t_k, x_k, \omega_k)| \geq \varepsilon_0 > 0. \tag{83}$$

Since the sets $B[0, R_0], \Omega$ and $[0, L_0]$ are compact, we can suppose that the sequences $\{x_k\},\ \{t_k\}$ and $\{\omega_k\}$ are convergent. Denote by $t_0 := \lim\limits_{k\to\infty} t_k,\ x_0 := \lim\limits_{k\to\infty} x_k$ and $\omega_0 := \lim\limits_{k\to\infty} \omega_k$. Passing to the limit in equality (83) and taking into account the continuity of the mapping φ we obtain $0 \geq \varepsilon_0$. The obtained contradiction proves our statement.

2. Under the conditions of Theorem 6.1 if we suppose that the cocyle φ_{λ_0} admits a compact global forward attractor $\{I^{\lambda_0}_{\omega}\ :\ \omega \in \Omega\}$, i.e.

$$\lim_{t\to\infty} \sup_{\omega \in \Omega} \beta(\varphi_{\lambda_0}(t, B[0, R], \omega), I_{\omega t}) = 0$$

for every $R > 0$, then there should be natural to hope that for λ sufficiently close to λ_0 the cocycle φ_{λ} also will admit a compact global forward attractor $\{I^{\lambda}_{\omega}\ :\ \omega \in \Omega\}$

in small neighborhood of I^{λ_0}. Unfortunately, generally speaking, this assertion is not true.

In fact, let φ_0 be a cocycle (dynamical system) generated by the equation $x' = -x$ and φ_λ be a cocycle generated by the equation $x' = -x + \lambda x^3$ ($\lambda > 0$). It is clear that the cocycle φ_0 (φ_λ) admits a compact global attractor $I^0 = \{0\}$ ($I^\lambda = [-\lambda^{-1/2}, \lambda^{-1/2}]$). In the small neighborhood of the attractor $I^0 = \{0\}$ the cocycle φ_λ (for small λ) admits a local (but not global) attractor $I^\lambda = \{0\}$.

Theorem 6.3. *Let Λ be a compact metric space, $(\Omega, \mathbb{R}, \sigma)$ be a dynamical system on the compact metric space Ω, E be a Banach space and φ_λ ($\lambda \in \Lambda$) be a cocycle on $(\Omega, \mathbb{R}, \sigma)$ with fiber E. Suppose that the following conditions are fulfilled:*

(i) *the cocycle φ_{λ_0} admits a compact global forward attractor;*
(ii) *the following relation takes place*

$$m_L(\lambda) := \sup\{|\varphi_\lambda(t,x,\omega) - \varphi_{\lambda_0}(t,x,\omega)| \ : \ 0 \le t \le L, \ \omega \in \Omega, \ |x| \le R_0\} \to 0$$

as $\lambda \to \lambda_0$ for every positive number L;
(iii) *every cocycle φ_λ admits a compact global attractor $\{I^\lambda_\omega \ : \ \omega \in \Omega\}$;*
(iv) *the set $I := \bigcup\{I^\lambda \ : \ \lambda \in \Lambda\}$ is bounded in E.*

Then the following equality

$$\lim_{\lambda \to \lambda_0} \sup_{\omega \in \Omega} \beta(I^\lambda_\omega, I^{\lambda_0}_\omega) = 0$$

is fulfilled and, in particular,

$$\lim_{\lambda \to \lambda_0} \beta(I^\lambda, I^{\lambda_0}) = 0.$$

Proof. Suppose that the conditions of the theorem are fulfilled. According to the condition (iv) there exists a positive number R_0 such that $I \subset B(0, R_0)$. Reasoning as in Theorem 6.1 for all $\rho > 0$ we will find $L = L(\frac{\rho}{3}) > 0$ and $\delta_0 = \delta_0(\rho) > 0$ such that

$$\varphi_\lambda(t, I^\lambda_\omega, \omega) \subseteq B[I^{\lambda_0}_{\omega t}, \rho]$$

for all $t \ge L$ and $\omega \in \Omega$ and, consequently,

$$I^\lambda_\omega = \varphi_\lambda(t, I_{\sigma(-t,\omega)}, \sigma(-t,\omega)) \subseteq B[I^{\lambda_0}_\omega, \rho]$$

for all $\omega \in \Omega$ and $\rho(\lambda, \lambda_0) < \delta_0$. The theorem is proved. □

Lemma 6.4. *([8]) Let Λ be a compact metric space and let $\varphi : \mathbb{T}_+ \times W \times \Lambda \times \Omega \mapsto W$ satisfy the following conditions :*

(i) *φ is continuous ;*
(ii) *for every $\lambda \in \Lambda$ the function $\varphi_\lambda = \varphi(\cdot, \cdot, \lambda, \cdot) : \mathbb{T}_+ \times W \times \Omega \mapsto W$ is a continuous cocycle on Ω with the fiber W;*
(iii) *the cocycle φ_λ admits a pullback attractor $\{I^\lambda_\omega \mid \omega \in \Omega\}$ for every $\lambda \in \Lambda$;*
(iv) *the set $\bigcup\{I^\lambda \mid \lambda \in \Lambda\}$ is precompact, where $I^\lambda = \bigcup\{I^\lambda_\omega \mid \omega \in \Omega\}$.*

Then the following equality

$$\lim_{\lambda \to \lambda_0, \omega \to \omega_0} \beta(I^\lambda_\omega, I^{\lambda_0}_{\omega_0}) = 0 \tag{84}$$

takes place for every $\lambda_0 \in \Lambda$ *and* $\omega_0 \in \Omega$ *and*

$$\lim_{\lambda \to \lambda_0} \beta(I_\lambda, I_{\lambda_0}) = 0 \tag{85}$$

for every $\lambda_0 \in \Lambda$.

Lemma 6.5. *(*[8]*)*

Let the conditions of Lemma 6.4 and additionally the following condition be fulfilled:

5. *for certain* $\lambda_0 \in \Lambda$ *the application* $F : \Omega \mapsto C(W)$*, defined by equality* $F(\omega) = I_\omega^{\lambda_0}$ *is continuous, i.e.* $\alpha(F(\omega), F(\omega_0)) \to 0$ *if* $\omega \to \omega_0$ *for every* $\omega_0 \in \Omega$*, where* α *is the full metric of Hausdorff, i.e.* $\alpha(A, B) = \max\{\beta(A, B), \beta(B, A)\}$.

Then the equality

$$\lim_{\lambda \to \lambda_0} \sup_{\omega \in \Omega} \beta(I_\omega^\lambda, I_\omega^{\lambda_0}) = 0 \tag{86}$$

takes place.

Theorem 6.6. *(*[8]*)*

Let W *possess the property* (S) *and let the cocycle* φ *admit a compact pullback attractor* $\{I_\omega \mid \omega \in \Omega\}$*, then :*

(i) *the set* I_ω *is connected for every* $\omega \in \Omega$*;*
(ii) *if the space* Ω *is connected, then the set* $I = \bigcup\{I_\omega | \omega \in \Omega\}$ *also is connected.*

Theorem 6.7. *Let* $\varepsilon \in (0, \varepsilon_0)$*,* Ω *be compact and connected and* φ_ε *(* $\bar{\varphi}_\varepsilon$*) be a cocycle generated by the equation (59) (respectively by the equation (65)).*

Suppose that the following conditions are fulfilled:

(i) $f(\omega) := f_0(\omega) + f_1(\omega)$ $(\omega \in \Omega)$, $f_0, f_1 \in C(\Omega, X)$*;*
(ii) *the average of* $f_1(\omega)$ *is equal to* 0*, i.e.* $\lim_{t \to \infty} \frac{1}{t} \int_0^t f_1(\omega s) ds = 0$ *uniformly w.r.t.* $\omega \in \Omega$*;*
(iii) *the cocycles* φ_ε *and* $\bar{\varphi}_\varepsilon$ *(*$\varepsilon \in (0, \varepsilon_0]$*)) are asymptotically compact .*

Then the following statements are true:

a. *for every* $\varepsilon \in (0, \varepsilon_0]$ *and* $\omega \in \Omega$ *the set* $I_\omega^\varepsilon := \{x \in E \ :$ *the solution* $\varphi_\varepsilon(t, x, \omega)$ *of equation (63) is defined and bounded on* $\mathbb{R}\}$ *(respectively* $\bar{I}_\omega^\varepsilon := \{x \in E \ :$ *the solution* $\bar{\varphi}_\varepsilon(t, x, \omega)$ *of equation (65) is defined and bounded on* $\mathbb{R}\}$*) is nonempty, compact and connected;*
b. *the cocycle* φ_ε *(*$\bar{\varphi}_\varepsilon$*) admits a compact global attractor* $\{I_\omega^\varepsilon \ : \ \omega \in \Omega\}$ *(respectively* $\{\bar{I}_\omega^\varepsilon \ : \ \omega \in \Omega\}$*);*
c. *the set* I^ε *(respectively* $\bar{I}^\varepsilon$*) is compact and connected;*
d. *the set* $I := \bigcup\{I^\varepsilon \ : \ \varepsilon \in [0, \varepsilon_0]\}$ *(respectively* $\bar{I} := \bigcup\{\bar{I}^\varepsilon \ : \ \varepsilon \in [0, \varepsilon_0]\}$*, where* $\bar{I}^\varepsilon := \bigcup\{\bar{I}_\omega^\varepsilon \ : \ \omega \in \Omega\}$*) is compact, where* $I^\varepsilon := \bigcup\{I_\omega^\varepsilon \ : \ \omega \in \Omega\}$*,* $I^0 = \bar{I}^0 := \bigcup\{\bar{I}_\omega^0 \ : \ \omega \in \Omega\}$ *and* $\{\bar{I}_\omega^0 \ : \ \omega \in \Omega\}$ *is a compact global attractor of the equation (65), when* $\varepsilon = 1$*;*
e. $\lim_{\varepsilon \to 0} \beta(I^\varepsilon, \bar{I}^0) = 0$*, where* β *is a semi-distance of Hausdorff;*

f. *If a dynamical system* $(\Omega, \mathbb{R}, \sigma)$ *is periodic, i.e. there exists* $\omega_0 \in \Omega$ *such that* $\omega_0 \tau = \omega_0$ *and* $\Omega = \{\omega_0 t \ : \ t \in [0, \tau)\}$*, then*

$$\lim_{\varepsilon \to 0} \sup\{\beta(I^\varepsilon_\omega, I^0_\omega) \ : \ \omega \in \Omega\} = 0.$$

Proof. Let $\varepsilon \in (0, \varepsilon_0)$, Ω be compact and connected and φ_ε ($\bar{\varphi}_\varepsilon$) be a cocycle generated by the equation (59) (respectively by the equation (65)), then we have

$$\varphi_\varepsilon(t, x, \omega) = \varphi(\varepsilon t, x, \omega, \varepsilon)(\text{respectively } \bar{\varphi}_\varepsilon(t, x, \omega) = \bar{\varphi}(\varepsilon t, x, \omega, \varepsilon)\ , \tag{87}$$

for all $t \in \mathbb{R}_+, x \in E$ and $\omega \in \Omega$, where $\varphi(\cdot, \cdot, \cdot, \varepsilon)$ (respectively $\bar{\varphi}(\cdot, \cdot, \cdot, \varepsilon)$) is a cocycle generated by the equation (66) (respectively (67)). From the equality (87) follows that $\{I^\varepsilon_\omega \ : \ \omega \in \Omega\}$ (respectively $\{\bar{I}^\varepsilon_\omega \ : \ \omega \in \Omega\}$,) is a compact global attractor of the equation (66) (respectively (67)). Now to finish the proof of theorem it is sufficient to apply Theorems 5.2, 6.3, 6.6 and Lemmas 6.4, 6.5. The theorem is proved. □

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