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Incomplete Exponential and Hypergeometric Functions with Applications to Noncentral χ^2-Distribution

by

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Abstract

Corresponding to the incomplete gamma functions, found useful in many problems, we propose incomplete exponential functions. Like the generalized incomplete gamma functions, the proposed functions have an additional parameter. It is shown that these functions can be related to Bessel functions. This leads us naturally to an incomplete extension of the hypergeometric function. The usefulness of these functions in the closed form representation of the noncentral χ^2-distributions is demonstrated.

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1. Introduction

The purpose of this paper is to propose a "break-up" of the exponential function into two parts, in a manner similar to the break-up of the gamma function into the two incomplete gamma functions ([1], p. 260–262). It may be wondered why one should want to break up such an elegant and simple function as the exponential into parts that must lose (at least) some of its simplicity and elegance.

Of course, there are break-ups of the exponential in common use. For many purposes of actual computation, one truncates the Taylor series representation of the exponential function. Another common split is into the even and odd parts as

$$e^x = \cosh x + \sinh x, \tag{1.1}$$

which are of use in themselves. The distinguishing property of the exponential is that it is its own derivative and integral. This is not true for the truncated series. The hyperbolic functions inherit part of that property, in that they are each other's derivatives and integrals.

There are infinitely many ways to break up any transcendental function, most of which would be wholly un-interesting. Allowing the possibility that there may be a mathematically interesting split, one still needs a way to identify it. One possibility is that it arises directly in applications, like the incomplete gamma functions do. (For a historical aspect of the gamma function, we refer to [12], for recent monographs on the subject, we refer to [3], [34], and for a detailed discussion of the incomplete gamma functions and their generalizations, we refer to [6], [7], [8], [24].) Another criterion must be that it inherits some (or most) of the properties of the original function, in some elegantly modified way. An additional requirement would be that the new functions connect with other known and commonly used functions. We have found applications of these functions in the closed form representation of the cumulative distribution and the corresponding survival function of the noncentral χ^2-distributions.

We present a split of the exponential which satisfies the first two conditions. The properties do extend elegantly, and the new functions do relate to the previously known functions, as required. It is hoped that further applications will come in due course.

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