

Smooth or singular solutions to the Navier–Stokes system ?

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Abstract

The existence of singular solutions of the incompressible Navier-Stokes system with singular external forces, the existence of regular solutions for more regular forces as well as the asymptotic stability of small solutions (including stationary ones), and a pointwise loss of smoothness for solutions are proved in the same function space of pseudomeasure type.

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1 Introduction

So far, only two ways for attacking the Cauchy problem for the Navier–Stokes equations are known: the first is due to J. Leray [27], and the second is due to T. Kato [18]. None of them can be considered the “golden rule” for solving the Navier–Stokes equations because they both leave open the following celebrated question. In three dimensions, does the velocity field of a fluid flow that starts smooth remain smooth and unique for all time ?

The concept of “weak” solutions introduced by J. Leray in 1933, permits the study of functions in much larger classes than the classical spaces used to describe the motion of a fluid. It is easier to prove the existence of a solution (regular or singular) in a larger class, but such a solution may not be unique. Based on *a priori* energy estimates, Leray’s theory gives the existence of global weak, possibly irregular and possibly non-unique solutions to the Navier–Stokes equations. On the other hand, a completely different theory introduced by T. Kato in 1984, based on semigroups techniques and the fixed point scheme, gives the existence of a global unique regular “mild” solution, under the restrictive assumption of small initial data. A second restriction is given by the fact that Kato’s algorithm does not provide a framework for studying *a priori* singular solutions. In fact, in order to overcome the difficulty (and sometimes the impossibility) of proving the continuity of the bilinear estimate in the, so-called, critical spaces, Kato’s algorithm makes clever use of a combination of two estimates in two different norms, the natural one and a regularizing norm. As such, Kato’s approach imposes *a priori* a regularization effect on solutions we look for. In other words, they are considered as fluctuations around the solution of the heat equation with same initial data. For people who believe in blow up and singularities, this *a priori* condition coming from the “two norms approach” is indeed very strong. However, there exist two exceptions, more exactly two critical spaces where Kato’s method applies with just one norm: the Lorentz space $L^{3,\infty}$ (considered independently by M. Yamazaki [33] and by Y. Meyer [29]) and the pseudomeasure space of Y. Le Jan and A.S. Sznitman [25], [7]. Here we will not go into the technical details arising from these critical spaces and we refer the reader to the recent surveys contained in [4] and in [26].

In this paper we will show how the approach with only one norm gives existence and uniqueness of a (small) solution in a larger space which, in our case, contains genuinely singular solutions that are not smoothed out by the action of the nonlinear semigroup associated. More exactly, in the case of the pseudomeasure space we can prove the following results. The existence of singular solutions associated to singular (e.g. the Dirac delta) external forces thus allowing to describe the solutions considered by L.D. Landau in [23] and by G. Tian and Z. Xin in [32]. The existence of regular solutions for more regular external forces. The asymptotic stability of small solutions including stationary ones. A pointwise loss of smoothness for solutions.

The study of the Navier–Stokes equations written in terms of the vorticity and with measures as initial data started in the 80s in a series of papers by G. Benfatto, R. Esposito, M. Pulvirenti [1], G.-H. Cottet and J. Soler [9, 10], and Y. Giga, T. Miyakawa

and H. Osada [12, 13]. We refer the reader as well to the more recent results obtained by T. Kato in [19] and Y. Giga in [11]. On the other hand, the case of external forces that can be singular atomic measures was studied by H. Kozono and M. Yamazaki [20]. Here we want to provide, among others, such kind of results.

2 One-point singular solutions

As observed by J. Heywood in [15], in principle “*it is easy to construct a singular solution of the NS equations that is driven by a singular force. One simply constructs a solenoidal vector field u that begins smoothly and evolves to develop a singularity, and then defines the force to be the residual.*” In this section we want to give an explicit example of this mathematical evidence. Our example arises from the physical experiment described by L.D. Landau in [23] (see also [24, Sec. 23]), where an axially symmetric jet discharging from a thin pipe into the unbounded space is studied. Passing to the limit with the diameter of the pipe, this “plunged” jet can be regarded as emerging from a point source (i.e. driven by the delta function). Landau provided a mathematical setting for explaining this phenomenon by using the classical incompressible Navier–Stokes system and deriving an explicit “solution” for it.

To be more precise, let us recall the famous Navier–Stokes equations, describing the evolution of the velocity field u and pressure field p of a three-dimensional incompressible viscous fluid at time t and the position $x \in \mathbb{R}^3$. These equations are given by

$$u_t - \Delta u + (u \cdot \nabla)u + \nabla p = F, \quad (2.1)$$

$$\nabla \cdot u = 0, \quad (2.2)$$

$$u(0) = u_0. \quad (2.3)$$

where the external force F and initial velocity u_0 are assigned.

Recently, G. Tian and Z. Xin [32] also found explicit formulas for a one-parameter family of stationary “solutions” of the three-dimensional Navier–Stokes system “with $F \equiv 0$ ” which are regular except at a given point. Due to the translation invariance of the Navier–Stokes system, one can assume that the singular point corresponds to the origin. These explicit “solutions” by Tian and Xin agree with those obtained by Landau for special values of the parameter. More exactly, the main theorem from [32] reads as follows. *All solutions to system (2.1)–(2.3) (with $F \equiv 0$) $u(x) = (u_1(x), u_2(x), u_3(x))$ and $p = p(x)$ which are steady, symmetric about x_1 -axis, homogeneous of degree -1 , regular except $(0, 0, 0)$ are given by the following explicit formulas:*

$$\begin{aligned} u_1(x) &= 2 \frac{c|x|^2 - 2x_1|x| + cx_1^2}{|x|(c|x| - x_1)^2}, & u_2(x) &= 2 \frac{x_2(cx_1 - |x|)}{|x|(c|x| - x_1)^2}, \\ u_3(x) &= 2 \frac{x_3(cx_1 - |x|)}{|x|(c|x| - x_1)^2}, & p(x) &= 4 \frac{cx_1 - |x|}{|x|(c|x| - x_1)^2} \end{aligned} \quad (2.4)$$

where $|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and c is an arbitrary constant such that $|c| > 1$.

Remark 2.1 Note that in the formula [32, (2.1)] the numerator of the fraction defining $u_1(x)$ should read $cr^2 - 2r(x_1 - x_1^0) + c(x_1 - x_1^0)^2$. The factor “2” was missing in that formula what can be inferred from [32, (2.40)] or [24, (23,16)–(23,19)]. On the other hand, the sign “–” in the formula [24, (23,20)] for the pressure is wrong. \square

Before commenting this result, we think it is necessary to clarify the meaning of “solution of the Navier–Stokes equations”, for, since the appearance of the pioneer papers of Leray, the word “solution” has been used in a more or less generalized sense giving origin to so many different definitions of “solutions”, distinguished only by the class of functions they are supposed to belong to: *classical, strong, mild, weak, very weak, uniform weak* and *local Leray* solutions of the Navier–Stokes equations ! We will not present all the possible (more or less well-known) definitions here and refer the reader to [4] and the references therein.

Let us first remark that there is no hope to describe the “solutions” given by equations (2.4) in Leray’s theory, because they are not globally of finite energy, in other words they do not belong to $L^2(\mathbb{R}^3)$. However, they do belong to $L^2_{loc}(\mathbb{R}^3)$ and this is at least enough to allow us to give a (distributional) meaning to the nonlinear term $(v \cdot \nabla)v = \nabla \cdot (v \otimes v)$. Moreover, the “solutions” discover by Tian and Xin cannot be analyzed by Kato’s two norms method either, because they are global but not smooth, more exactly they are singular at the origin with a singularity of the kind $\sim 1/|x|$ for all time.

We will provide in the following section an *ad hoc* framework for studying such singularity within the fixed point scheme and without using the two norms approach. As recalled in the introduction, this can be done in principle either in a Lorentz or in a pseudomeasure space and they both contain singularities of the type $\sim 1/|x|$. However, we will chose the latter space not only because the proofs will be very elementary, but also because this choice will allow us to treat singular (Delta type) external force, that precisely arise from Landau and Tian and Xin “solutions”.

More exactly, by straightforward calculations, one can check that, indeed, the functions $(u_1(x), u_2(x), u_3(x))$ and $p(x)$ given by (2.4) satisfy (2.1)–(2.3) with $F \equiv 0$ in the *pointwise sense* for every $x \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$. On the other hand, if one treats $(u(x), p(x))$ as a *distributional or generalized* solution to (2.1)–(2.3) in the whole \mathbb{R}^3 , they correspond to the very singular external force $F = (b\delta_0, 0, 0)$, where the parameter $b \neq 0$ depends on c and δ_0 stands for the Dirac delta. Let us state this fact more precisely.

Proposition 2.1 *Let $u = (u_1, u_2, u_3)$ and p be defined by (2.4). For every test function $\varphi \in C_c^\infty(\mathbb{R}^3)$ the following equalities hold true:*

$$\int_{\mathbb{R}^3} u \cdot \nabla \varphi \, dx = 0 \quad (2.5)$$

and

$$\int_{\mathbb{R}^3} \left(\nabla u_k \cdot \nabla \varphi - u_k u \cdot \nabla \varphi - p \frac{\partial}{\partial x_k} \varphi \right) dx = \begin{cases} -b(c)\varphi(0) & \text{if } k = 1 \\ 0 & \text{if } k = 2, 3, \end{cases} \quad (2.6)$$

where

$$b(c) = 4\pi \left(4c + 2c^2 \log \frac{c-1}{c+1} + \frac{16}{3} \frac{c}{c^2-1} \right). \quad (2.7)$$

In particular, the function $b = b(c)$ is decreasing on $(-\infty, -1)$ and $(1, +\infty)$. Moreover, $\lim_{c \searrow -1} b(c) = \infty$, $\lim_{c \nearrow -1} b(c) = -\infty$ and $\lim_{|c| \rightarrow \infty} b(c) = 0$.

Proof. Equality (2.5) says that the velocity u is weakly divergence-free in \mathbb{R}^3 . This can be shown by a standard argument involving integration by parts, since each component of u is homogeneous of degree -1 and thus belongs to $W_{loc}^{1,p}(\mathbb{R}^3)$ with $1 \leq p < 3/2$ and $(\nabla \cdot u)(x) = 0$ for all $x \in \mathbb{R}^3 \setminus \{0\}$.

Next, due to singularities of u and p at the origin, we fix $\varepsilon > 0$ and we integrate in equations (2.6) for $|x| \geq \varepsilon$, only. Integrating by parts, we obtain

$$\begin{aligned} & \int_{|x| \geq \varepsilon} \left(\nabla u_k \cdot \nabla \varphi - u_k u \cdot \nabla \varphi - p \frac{\partial}{\partial x_k} \varphi \right) dx \\ &= \int_{|x| \geq \varepsilon} \left(-\Delta u_k + \nabla \cdot (u_k u) + \frac{\partial}{\partial x_k} p \right) \varphi dx \\ & \quad + \int_{|x|=\varepsilon} \left((\nabla u_k - u_k u) \cdot \frac{x}{\varepsilon} - p \frac{x_k}{\varepsilon} \right) \varphi d\sigma(x), \end{aligned} \quad (2.8)$$

because x/ε is the unit vector normal to the sphere $\{x \in \mathbb{R}^3 : |x| = \varepsilon\}$. Obviously, the first term on the right-hand side of (2.8) disappears, and our goal is to compute the limit as $\varepsilon \searrow 0$ of the second one.

For this reason, note first that each term ∇u_k , $u_k u$, and p is homogeneous of degree -2 . Hence, changing variables $x = \varepsilon y$ in the integral $\int_{|x|=\varepsilon} \dots d\sigma(x)$ in (2.8), and next passing to the limit with $\varepsilon \searrow 0$ we show by the Lebesgue Dominated Convergence Theorem that it converges toward

$$\varphi(0) \int_{|x|=1} ((\nabla u_k - u_k u) \cdot x - p x_k) d\sigma(x). \quad (2.9)$$

To complete this proof, it remains to compute the surface integral in (2.9). First, however, we simplify it a little by using the Euler theorem for homogeneous functions which in this case gives $x \cdot \nabla u_k = -u_k$. Moreover, it follows from the definition of u_k and p that

$$u_1 = \frac{1}{2} p x_1 + \frac{2}{c|x| - x_1}, \quad u_2 = \frac{1}{2} p x_2, \quad u_3 = \frac{1}{2} p x_3.$$

Consequently, for $k = 2, 3$, the integral in (2.9) equals

$$- \int_{|x|=1} (u_k + u_k(u \cdot x) + 2u_k) d\sigma(x) = 0,$$

because u_2 and u_3 are odd functions with respect to x_2 and x_3 , respectively, and $u \cdot x$ is even. In case of $k = 1$, we use the identities

$$u_1(x) = c + (c^2 - 1) \left(\frac{c}{(c - x_1)^2} - \frac{2}{c - x_1} \right) \quad \text{and} \quad p x_1 = 2u_1 - \frac{4}{c - x_1}$$

valid for $|x| = 1$, and the polar coordinates to show that

$$\begin{aligned} & \int_{|x|=1} \left(u_1 + u_1(u \cdot x) + 2u_1 - \frac{4}{c-x_1} \right) d\sigma(x) \\ &= 2\pi \int_{-1}^1 2 \left(\left(c + (c^2 - 1) \left(\frac{c}{(c-x_1)^2} - \frac{2}{c-x_1} \right) \right) \left(1 + 2\frac{c^2-1}{(c-x_1)^2} \right) - \frac{2}{c-x_1} \right) dx_1 \\ &= b(c). \end{aligned}$$

Here, we skip these long but rather elementary calculations. \square

Remark 2.2 As we have already emphasized, the stationary solutions defined in (2.4) are singular with singularity of the kind $\mathcal{O}(1/|x|)$ as $|x| \rightarrow 0$. This is the critical singularity in the context of Proposition 2.1, because as it was shown by H.J.Choe and H.Kim [8], every pointwise stationary solution to system (2.1)–(2.3) with $F \equiv 0$ in $B_R \setminus \{0\} = \{x \in \mathbb{R}^3 : 0 < |x| < R\}$ satisfying $u(x) = o(1/|x|)$ as $|x| \rightarrow 0$ is also a solution in the sense of distributions in the whole B_R . Moreover, it is shown in [8] that under the additional assumption $u \in L^q(B_R)$ for some $q > 3$, then the stationary solution $u(x)$ is smooth in the whole ball B_R . In other words, if $u(x) = o(1/|x|)$ as $|x| \rightarrow 0$ and $u \in L^q(B_R)$ for some $q > 3$, then the singularity at the origin is removable. \square

3 Definitions and spaces

We will study global-in-time solutions $u = u(x, t)$ to the Cauchy problem in \mathbb{R}^3 for the incompressible Navier–Stokes equations (2.1)–(2.2). As far as $u = u(x, t)$ is a sufficiently regular function, the equations (2.1)–(2.2) can be rewritten as

$$u_t - \Delta u + \nabla \cdot (u \otimes u) + \nabla p = F, \quad \nabla \cdot u = 0.$$

If we recall that the Leray projector on solenoidal vector fields is given by the formula

$$\mathbb{P}v = v - \nabla \Delta^{-1}(\nabla \cdot v) \tag{3.1}$$

for sufficiently smooth functions $v = (v_1(x), v_2(x), v_3(x))$, we formally transform the system (2.1)–(2.2) into

$$u_t - \Delta u + \mathbb{P}\nabla \cdot (u \otimes u) = \mathbb{P}F, \quad \nabla \cdot u = 0.$$

Finally, let us emphasize that we shall study the problem (2.1)–(2.3) via the following integral equation obtained from the Duhamel principle

$$\begin{aligned} u(t) &= S(t)u_0 - \int_0^t S(t-\tau)\mathbb{P}\nabla \cdot (u \otimes u)(\tau) d\tau \\ &\quad + \int_0^t S(t-\tau)\mathbb{P}F(\tau) d\tau, \end{aligned} \tag{3.2}$$

where $S(t)$ is the heat semigroup given as the convolution with the Gauss–Weierstrass kernel: $G(x, t) = (4\pi t)^{-3/2} \exp(-|x|^2/(4t))$. To give a meaning to the Leray projector

$\mathcal{I}\mathcal{P}$ (defined in (3.1)), let us first recall that the Riesz transforms R_j are the pseudodifferential operators defined in the Fourier variables as $\widehat{R_k f}(\xi) = \frac{i\xi_k}{|\xi|} \widehat{f}(\xi)$. Here and in what follows the Fourier transform of an integrable function v is given by $\widehat{v}(\xi) \equiv (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} v(x) dx$. Using these well-known operators we define

$$(\mathcal{I}\mathcal{P}v)_j = v_j + \sum_{k=1}^3 R_j R_k v_k;$$

moreover, in our considerations below, we shall often denote by $\widehat{\mathcal{I}\mathcal{P}}(\xi)$ the symbol of the pseudodifferential operator $\mathcal{I}\mathcal{P}$ which is the matrix with components

$$(\widehat{\mathcal{I}\mathcal{P}}(\xi))_{j,k} = \delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2}.$$

All these components are bounded on \mathbb{R}^3 and we put

$$\kappa = \max_{1 \leq j,k \leq 3} \sup_{\xi \in \mathbb{R}^3 \setminus \{0\}} |(\widehat{\mathcal{I}\mathcal{P}}(\xi))_{j,k}|. \quad (3.3)$$

We are now in a position to introduce the Banach functional spaces relevant to our study of solutions of the Cauchy problem for the system (2.1)–(2.3):

$$\mathcal{PM}^a \equiv \{v \in \mathcal{S}'(\mathbb{R}^d) : \widehat{v} \in L^1_{loc}(\mathbb{R}^d), \|v\|_{\mathcal{PM}^a} \equiv \operatorname{ess\,sup}_{\xi \in \mathbb{R}^3} |\xi|^a |\widehat{v}(\xi)| < \infty\},$$

where $a \geq 0$ is a given parameter. The notation \mathcal{PM} stands for *pseudomeasure*, and the classical space of pseudomeasures introduced in harmonic analysis (i.e. those distributions whose Fourier transforms are bounded) corresponds to $a = 0$.

Definition 3.1 *By a solution of (2.1)–(2.3) we mean in this paper a function $u = u(t) = (u_1(t), u_2(t), u_3(t))$ with each component u_i belonging to the space of vector-valued functions $\mathcal{X} = \mathcal{C}_w([0, T]; \mathcal{PM}^2)$, $0 < T \leq \infty$, and such that*

$$\begin{aligned} \widehat{u}(\xi, t) &= e^{-t|\xi|^2} \widehat{u}(\xi, 0) + \int_0^t e^{-(t-\tau)|\xi|^2} \widehat{\mathcal{I}\mathcal{P}}(\xi) i\xi \cdot (u \otimes u)(\xi, \tau) d\tau \\ &\quad + \int_0^t e^{-(t-\tau)|\xi|^2} \widehat{\mathcal{I}\mathcal{P}}(\xi) \widehat{F}(\xi, \tau) d\tau \end{aligned} \quad (3.4)$$

for all $0 \leq t \leq T$.

The space \mathcal{PM}^2 is chosen because it contains homogeneous functions of degree -1 which are sufficiently regular on the unit sphere. In particular, one can easily check that this is the case for the one-point singular solutions defined in (2.4).

Remark 3.1 Given $f \in \mathcal{S}'(\mathbb{R}^3) \cap L^1_{loc}(\mathbb{R}^3)$ we denote the rescaling $f_\lambda(x) = f(\lambda x)$. In a standard way, we extend this definition to all tempered distributions. It follows from elementary calculations that $\widehat{f}_\lambda(\xi) = \lambda^{-3} \widehat{f}(\lambda^{-1}\xi)$. Hence, for every $\lambda > 0$, we obtain the scaling property of the norm in \mathcal{PM}^a

$$\|f(\lambda \cdot)\|_{\mathcal{PM}^a} = \lambda^{a-3} \|f\|_{\mathcal{PM}^a}. \quad (3.5)$$

In particular, the norm \mathcal{PM}^2 is invariant under rescaling $f \mapsto \lambda f(\lambda \cdot)$. Moreover, it follows from (3.5) that for $a = 3(1 - 1/p)$ the norms $\|\cdot\|_{\mathcal{PM}^a}$ and $\|\cdot\|_{L^p(\mathbb{R}^3)}$ have the same scaling property. \square

Remark 3.2 \mathcal{C}_w denotes, as usual (cf. [3]), the space of vector-valued functions which are weakly continuous as distributions in t . This is an additional difficulty caused by the fact that the heat semigroup $(S(t))_{t \geq 0}$ is not strongly continuous on the spaces of pseudomeasures but only weakly continuous (cf. Lemma 4.2, below). \square

Remark 3.3 Usually, a *mild solution* of an evolution equation like (2.1)–(2.3) is defined as a solution to the integral equation (3.2) and the integral is understood as the Bochner integral. However, such a meaning of a solution is not suitable for our construction of solutions of the Cauchy problem and, in particular, of self-similar solutions. Indeed, for stationary and homogeneous of degree -1 solutions u (given, e.g., by (2.4)), the nonlinear term corresponds to a tempered distribution which is homogeneous of degree -3 , hence, there exists a distribution H such that

$$S(t)IP\nabla \cdot (u \otimes u) = t^{-3/2} H \left(\frac{\cdot}{\sqrt{t}} \right).$$

Now, computing the \mathcal{PM}^2 norm and using the scaling relation (3.5), we obtain

$$\|S(t)IP\nabla \cdot (u \otimes u)\|_{\mathcal{PM}^2} = t^{-1} \|H\|_{\mathcal{PM}^2}.$$

So, $S(t)(IP\nabla \cdot (u \otimes u))$ is not Bochner integrable as a mapping on $[0, T)$ with values in \mathcal{PM}^2 . On the other hand, the Fourier transform of this quantity equals to $e^{-t|\xi|^2} \widehat{IP}(\xi)(u \widehat{\otimes} u)(\xi)$ and the singularity at $t = 0$ does not appear. Hence, the integral with respect to τ in equation (3.2) should be defined in a weak sense like, e.g., it was done in [33, Def. 2]. For more explanations, we refer the reader to [26], because our spaces \mathcal{PM}^a are the example of the shift-invariant Banach spaces of distributions systematically used in that book. \square

Nevertheless, a distributional solution of system (2.1)–(2.3) is a solution of the integral equation of (3.4), and vice versa. This equivalence can be proved by a standard reasoning, and we refer the interested reader to [33, Th. 5.2] for details of such computations.

To simplify the notation, the quadratic term in (3.2) will be denoted by

$$B(u, v)(t) = - \int_0^t S(t - \tau) IP\nabla \cdot (u \otimes v)(\tau) d\tau, \quad (3.6)$$

where $u = u(t)$ and $v = v(t)$ are functions defined on $[0, T)$ with values in a vector space (here most frequently \mathcal{PM}^2).

4 Global-in-time solutions

As in [3], the proof of our basic theorem on the existence, uniqueness and stability of solutions to the problem (2.1)–(2.3) is based on the following abstract lemma, whose slightly more general form is taken from [26].

Lemma 4.1 *Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a Banach space and $B : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ a bounded bilinear form satisfying $\|B(x_1, x_2)\|_{\mathcal{X}} \leq \eta \|x_1\|_{\mathcal{X}} \|x_2\|_{\mathcal{X}}$ for all $x_1, x_2 \in \mathcal{X}$ and a constant $\eta > 0$. Then, if $0 < \varepsilon < 1/(4\eta)$ and if $y \in \mathcal{X}$ such that $\|y\| < \varepsilon$, the equation $x = y + B(x, x)$ has a solution in \mathcal{X} such that $\|x\|_{\mathcal{X}} \leq 2\varepsilon$. This solution is the only one in the ball $\bar{B}(0, 2\varepsilon)$. Moreover, the solution depends continuously on y in the following sense: if $\|\tilde{y}\|_{\mathcal{X}} \leq \varepsilon$, $\tilde{x} = \tilde{y} + B(\tilde{x}, \tilde{x})$, and $\|\tilde{x}\|_{\mathcal{X}} \leq 2\varepsilon$, then*

$$\|x - \tilde{x}\|_{\mathcal{X}} \leq \frac{1}{1 - 4\eta\varepsilon} \|y - \tilde{y}\|_{\mathcal{X}}.$$

Proof. Here, the reasoning is based on the standard Picard iteration technique completed by the Banach fixed point theorem. For other details of the proof, we refer the reader to [26, Th. 13.2]. \square

Our goal is to apply Lemma 4.1 in the space

$$\mathcal{X} = \mathcal{C}_w([0, \infty), \mathcal{PM}^2) \quad (4.1)$$

to the integral equation (3.2) which has the form $u = y + B(u, u)$, where the bilinear form is defined in (3.6) and $y = S(t)u_0 + \int_0^t S(t - \tau) \mathbb{P}F(\tau) d\tau$. We need some preliminary estimates.

Lemma 4.2 *Given $u_0 \in \mathcal{PM}^2$, we have $S(\cdot)u_0 \in \mathcal{X}$.*

Proof. By the definition of the norm in \mathcal{PM}^2 , it follows that

$$\|S(t)u_0\|_{\mathcal{PM}^2} = \operatorname{ess\,sup}_{\xi \in \mathbb{R}^3} |\xi|^2 \left| e^{-t|\xi|^2} \hat{u}_0(\xi) \right| \leq \operatorname{ess\,sup}_{\xi \in \mathbb{R}^3} |\xi|^2 |\hat{u}_0(\xi)| = \|u_0\|_{\mathcal{PM}^2},$$

so, $S(\cdot)u_0 \in L^\infty([0, \infty), \mathcal{PM}^2)$.

Now, let us prove the weak continuity with respect to t , and, by the semigroup property of $S(t)$, it suffices to do this for $t = 0$ only. For every $\varphi \in \mathcal{S}(\mathbb{R}^3)$, by the Plancherel formula, we obtain

$$\begin{aligned} |\langle S(t)u_0 - u_0, \varphi \rangle| &= \left| \int \left(e^{-t|\xi|^2} - 1 \right) \hat{u}_0(\xi) \hat{\varphi}(\xi) d\xi \right| \\ &\leq t \operatorname{ess\,sup}_{\xi \in \mathbb{R}^3} \left| \frac{e^{-t|\xi|^2} - 1}{t|\xi|^2} \right| \|u_0\|_{\mathcal{PM}^2} \|\hat{\varphi}\|_{L^1 \mathbb{R}^3} \rightarrow 0 \quad \text{as } t \searrow 0. \end{aligned}$$

\square

Lemma 4.3 *Given $F \in \mathcal{C}_w([0, \infty), \mathcal{PM})$, it follows that*

$$w \equiv \int_0^t S(t - \tau) \mathbb{P}F(\tau) d\tau \in \mathcal{X}.$$

Moreover, $\|w\|_{\mathcal{X}} \leq \|F\|_{\mathcal{C}_w([0, \infty), \mathcal{PM})}$.

Proof. Similarly as in the proof of Lemma 4.2 we get

$$\begin{aligned} \|w(t)\|_{\mathcal{PM}^2} &= \operatorname{ess\,sup}_{\xi \in \mathbb{R}^3} |\xi|^2 \left| \int_0^t e^{-(t-\tau)|\xi|^2} \widehat{P}(\xi) \widehat{F}(\xi, \tau) \, d\tau \right| \\ &\leq \kappa |\xi|^2 \int_0^t e^{-(t-\tau)|\xi|^2} \, d\tau \|F\|_{\mathcal{C}_w([0, \infty), \mathcal{PM})} \\ &\leq \kappa \|F\|_{\mathcal{C}_w([0, \infty), \mathcal{PM})}. \end{aligned}$$

Let us skip the proof of the weak continuity of $w(t)$ because the reasoning is more or less standard. Similar arguments can be found e.g. either in [29, Ch. 18, Lemma 24] or in [33, Th. 3.1]. \square

The goal of the next proposition is to prove that the bilinear form $B(\cdot, \cdot)$ defined in (3.6) is continuous on the space $\mathcal{X} = \mathcal{C}_w([0, \infty), \mathcal{PM}^2)$. This fact is well-known and the proof appeared for the first time in [25] and [7]. Here, however, we repeat that reasoning because we want to control better all the constants which appear in the estimates below.

Proposition 4.1 *The bilinear operator $B(\cdot, \cdot)$ is continuous on the space \mathcal{X} defined in (4.1). Hence, there exists a constant $\eta > 0$ such that for every $u, v \in \mathcal{X}$, it follows*

$$\|B(u, v)\|_{\mathcal{X}} \leq \eta \|u\|_{\mathcal{X}} \|v\|_{\mathcal{X}}.$$

Proof. We do all the calculations in the Fourier variables. Recall that the constant κ is defined in (3.3). Using elementary properties of the Fourier transform we obtain

$$\begin{aligned} \left| \widehat{\mathcal{IP}}(\xi)(\widehat{u \otimes v})(\xi, \tau) \right| &\leq \kappa \int_{\mathbb{R}^3} \frac{dz}{|\xi - z|^2 |z|^2} \|u(\tau)\|_{\mathcal{PM}^2} \|v(\tau)\|_{\mathcal{PM}^2} \\ &= \frac{\eta}{|\xi|} \|u(\tau)\|_{\mathcal{PM}^2} \|v(\tau)\|_{\mathcal{PM}^2}. \end{aligned}$$

In the computations above, we use the equality $|\xi|^{-2} * |\xi|^{-2} = \pi^3 |\xi|^{-1}$. A detailed analysis concerning such convolutions can be found in [28, Th. 5.9] or [31, Ch. V, Sec.1, (8)], see also [2, Lem. 2.1]. Hence, $\eta = \kappa \pi^3$.

Now, the boundedness of the bilinear form on \mathcal{X} results from the following estimates

$$\begin{aligned} |\xi|^2 \left| \int_0^t e^{-(t-\tau)|\xi|^2} \widehat{\mathcal{IP}}(\xi) i\xi \cdot (\widehat{u \otimes v})(\xi, \tau) \, d\tau \right| \\ \leq \eta |\xi|^2 \int_0^t e^{-(t-\tau)|\xi|^2} \, d\tau \|u\|_{\mathcal{X}} \|v\|_{\mathcal{X}} \\ \leq \eta \|u\|_{\mathcal{X}} \|v\|_{\mathcal{X}}. \end{aligned}$$

It remains to show the weak continuity of $B(u, v)(t)$ with respect to t , but this follows again from standard arguments, cf. the remark at the end of the proof of Lemma 4.3. \square

Now, the main theorem of this section results immediately from Lemma 4.1 combined with Lemmata 4.2–4.3 and Proposition 4.1.

Theorem 4.1 *Assume that $u_0 \in \mathcal{PM}^2$ and $F \in \mathcal{C}_w([0, \infty), \mathcal{PM})$ satisfy $\|u_0\|_{\mathcal{PM}^2} + \|F\|_{\mathcal{C}_w([0, \infty), \mathcal{PM})} < \varepsilon$ for some $0 < \varepsilon < 1/(4\eta)$ where η is defined in Proposition 4.1. There exists a global-in-time solution of (2.1)–(2.3) in the space $\mathcal{X} = \mathcal{C}_w([0, \infty), \mathcal{PM}^2)$. This is the unique solution satisfying the condition $\|u\|_{\mathcal{C}_w([0, \infty), \mathcal{PM}^2)} \leq 2\varepsilon$. Moreover, this solution depends continuously on initial data and external forces in the sense of Lemma 4.1. \square*

Assume, for a moment, that $F \equiv 0$. Homogeneity properties of the problem (2.1)–(2.2) imply that if u solves the Cauchy problem, then the rescaled function $u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t)$ is also a solution for each $\lambda > 0$. Thus, it is natural to consider solutions which satisfy the scaling invariance property $u_\lambda \equiv u$ for all $\lambda > 0$, i.e. *forward self-similar* solutions. By the very definition, they are global-in-time, and one may expect that they describe the large time behavior of general solutions of (2.1)–(2.3). Indeed, if $\lim_{\lambda \rightarrow \infty} \lambda u(\lambda x, \lambda^2 t) = U(x, t)$ in an appropriate sense, then $tu(xt^{1/2}, t) \rightarrow U(x, 1)$ as $t \rightarrow \infty$ (take $t = 1$, $\lambda = t^{1/2}$), and $U \equiv U_\lambda$ is scale invariant. Hence U is a self-similar solution, and

$$U(x, t) = t^{-1/2}U(x/t^{1/2}, 1) \quad (4.2)$$

is thus determined by a function of d variables $U(y) \equiv U(y, 1)$, $y = x/t^{1/2}$ being the Boltzmann substitution.

If $u_\lambda \equiv u$ for all $\lambda > 0$, then from the self-similar form (4.2), the initial condition (2.3) $\lim_{t \searrow 0} u(x, t)$ is a distribution homogeneous of degree -1 at the origin. Of course, one-point singular solutions defined in (2.4) are self-similar solutions which are time independent.

Self-similar solutions can be obtained directly from Theorem 4.1 by taking u_0 homogeneous of degree -1 of small \mathcal{PM}^2 norm. By the uniqueness property of solutions of the Cauchy problem constructed in Theorem 4.1, they have the form (4.2).

The same reasoning can be applied to the case when external forces are present. Indeed, if the initial datum u_0 is homogeneous of degree -1 and if the external force $F(x, t)$ satisfies

$$\lambda^3 F(\lambda x, \lambda^2 t) = F(x, t) \quad \text{for all } \lambda > 0 \quad (4.3)$$

(here, the scaling is understood in the distributional sense), the solution obtained in Theorem 4.1 is self-similar. Note that, in particular, we can take

$$F(x, t) = F(x) = (b_1 \delta_0, b_2 \delta_0, b_3 \delta_0)$$

(the multiples of the Dirac delta) for sufficiently small $|b|$. In other words, the existence of the solutions introduced by Tian and Xin and described in the previous section can be ensured by the fixed point method for large values of the parameter c (this is possible because of the particular expression of the function $b(c)$ in (2.7)). We will clarify this fact in Section 6.

Proceeding in this way we arrive at

Corollary 4.1 *Suppose that the initial condition $u_0 \in \mathcal{PM}^2$ is homogeneous of degree -1 and $F \in \mathcal{C}_w([0, \infty), \mathcal{PM})$ satisfies (4.3). Let u_0 and F satisfy, moreover, the assumptions of Theorem 4.1. The corresponding unique solution constructed in Theorem 4.1 is self-similar.* \square

Remark 4.1 The self-similar solutions constructed in such a way can have singularities for any time. This is the case, for instance, for the self-similar (stationary) solutions by Landau and Tian and Xin. On the other hand, when using the two norms approach of Kato as in [3, 4], the self-similar solutions that arise from this construction are *instantaneously* smoothed out for $t > 0$ and the only singularity (of the type $\sim 1/|x|$) can be found at $t = 0$. We will remark on this important point in Section 7. \square

Remark 4.2 An alternative way to prove the existence of self-similar solutions is to convert (2.1)–(2.3) into the integral formulation (3.2) and check that the form B reproduces the scale-invariant form (4.2) of u . Thus, the equation (3.2) can be solved in a subspace of \mathcal{X} formed by self-similar functions, as was done in [3], [29]. \square

Remark 4.3 The existence and the stability results from this section are closely related to those from the paper by Yamazaki [33] where he studied the Navier–Stokes system in the weak L^p -spaces in an exterior domain Ω . In those considerations, Yamazaki applied the Kato algorithm in the space $\mathcal{C}_w([0, \infty), L^{3,\infty}(\Omega))$ without *a priori* assumptions on the decay of solutions. Our approach involving the \mathcal{PM}^2 space is much more elementary than that from [33]. Moreover, we can treat more singular external forces, and we obtain a kind of asymptotic stability of solutions (see the next section). \square

Remark 4.4 Solutions to the Navier–Stokes system corresponding to singular external forces can also be obtained from very general results by Kozono and Yamazaki [20] where they use the Sobolev-type spaces based on homogeneous Morrey spaces. Their proof of existence of stationary solutions relies on the inverse function theorem and subtle estimates of the Stokes operator. Next, they investigate properties of a perturbation of the Stokes operator and they show resolvent estimates in the Morrey spaces needed in the proof of stability of stationary solutions. Here, our space \mathcal{PM}^2 is much smaller than those from [20]. Our approach, however, besides its simplicity, does not require separate reasoning for stationary solutions and unsteady ones. Moreover, we believe that such an elementary idea will allow to understand better properties of large solutions (see Section 8). \square

5 Asymptotic behavior of solutions

In our investigations concerning the large time behavior of solutions to problem (2.1)–(2.3) we need the following improvement of Lemma 4.3.

Lemma 5.1 *Assume that $F \in \mathcal{C}_w([0, \infty), \mathcal{PM})$ satisfies $\lim_{t \rightarrow \infty} \|F(t)\|_{\mathcal{PM}} = 0$. Then*

$$\lim_{t \rightarrow \infty} \left\| \int_0^t S(t - \tau) \mathcal{P}F(\tau) d\tau \right\|_{\mathcal{PM}^2} = 0.$$

Proof. It follows from the definition of the norm $\|\cdot\|_{\mathcal{PM}^2}$ that

$$\begin{aligned} \left\| \int_0^t S(t-\tau) \mathcal{P}F(\tau) d\tau \right\|_{\mathcal{PM}^2} &\leq \kappa \sup_{\xi \in \mathbb{R}^3} \int_0^t |\xi|^2 e^{-(t-\tau)|\xi|^2} \|F(\tau)\|_{\mathcal{PM}} d\tau \\ &\leq \kappa \left(\sup_{\xi \in \mathbb{R}^3} \int_0^{t/2} \dots d\tau + \sup_{\xi \in \mathbb{R}^3} \int_{t/2}^t \dots d\tau \right). \end{aligned}$$

Using the substitution $\xi = w\sqrt{t-\tau}$, we first obtain

$$\begin{aligned} \sup_{\xi \in \mathbb{R}^3} \int_0^{t/2} |\xi|^2 e^{-(t-\tau)|\xi|^2} \|F(\tau)\|_{\mathcal{PM}} d\tau &\leq \int_0^{t/2} (t-\tau)^{-1} \sup_{w \in \mathbb{R}^3} |w|^2 e^{-|w|^2} \|F(\tau)\|_{\mathcal{PM}} d\tau \\ &\leq C \int_0^{t/2} (t-\tau)^{-1} \|F(\tau)\|_{\mathcal{PM}} d\tau \\ &= C \int_0^{1/2} (1-s)^{-1} \|F(ts)\|_{\mathcal{PM}} d\tau. \end{aligned}$$

Now, the right-hand side of the above inequality tends to 0 as $t \rightarrow \infty$ by the Lebesgue Dominated Convergence Theorem.

We estimate the term containing the integral $\int_{t/2}^t \dots d\tau$ in the most direct way by

$$\left(\sup_{\xi \in \mathbb{R}^3} \int_{t/2}^t |\xi|^2 e^{-(t-\tau)|\xi|^2} d\tau \right) \sup_{t/2 \leq \tau \leq t} \|F(\tau)\|_{\mathcal{PM}} \leq C \sup_{t/2 \leq \tau \leq t} \|F(\tau)\|_{\mathcal{PM}} \rightarrow 0$$

as $t \rightarrow \infty$ by the assumption on F . □

Theorem 5.1 *Let the assumptions of Theorem 4.1 hold true. Assume that u and v are two solutions of (2.1)–(2.3) constructed in Theorem 4.1 corresponding to the initial conditions $u_0, v_0 \in \mathcal{PM}^2$ and external forces $F, G \in \mathcal{C}_w([0, \infty), \mathcal{PM})$, respectively. Suppose that*

$$\lim_{t \rightarrow \infty} \|S(t)(u_0 - v_0)\|_{\mathcal{PM}^2} = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \|F(t) - G(t)\|_{\mathcal{PM}} = 0. \quad (5.1)$$

Then

$$\lim_{t \rightarrow \infty} \|u(\cdot, t) - v(\cdot, t)\|_{\mathcal{PM}^2} = 0 \quad (5.2)$$

holds.

This result means that if the difference of the solutions of the heat equation issued from u_0, v_0 becomes negligible as $t \rightarrow \infty$ (e.g., if the difference of the initial data $u_0 - v_0$ is not too singular) and if $F(t)$ and $G(t)$ have the same large time asymptotics, the solutions of the nonlinear problem $u(t), v(t)$ behave similarly for large times. It can be interpreted as a kind of asymptotic stability result if the choice of v_0 is restricted to the initial data in a neighborhood of u_0 satisfying additionally (5.1). It is easy to verify that the first condition in (5.1) is satisfied if, e.g., $|\xi|^2(\hat{u}_0(\xi) - \hat{v}_0(\xi)) \rightarrow 0$ as $\xi \rightarrow 0$.

Proof of Theorem 5.1. First, let us recall that, by Theorem 4.1, we have

$$\sup_{t \geq 0} \|u(t)\|_{\mathcal{PM}^2} \leq 2\varepsilon < \frac{1}{2\eta} \quad \text{and} \quad \sup_{t \geq 0} \|v(t)\|_{\mathcal{PM}^2} \leq 2\varepsilon < \frac{1}{2\eta}. \quad (5.3)$$

We subtract the integral equation (3.2) for v from the analogous expression for u . Next, computing the norm $\|\cdot\|_{\mathcal{PM}^2}$ of the resulting equation and repeating the calculations from the proof of Proposition 4.1 we obtain the following inequality

$$\begin{aligned} & \|u(t) - v(t)\|_{\mathcal{PM}^2} \\ & \leq \|S(t)(u_0 - v_0)\|_{\mathcal{PM}^2} + \left\| \int_0^t S(t - \tau) \mathcal{IP}(F(t) - G(t)) d\tau \right\|_{\mathcal{PM}^2} \\ & \quad + \eta \sup_{\xi \in \mathbb{R}^3} \int_0^{\delta t} |\xi|^2 e^{-(t-\tau)|\xi|^2} (\|u(\tau)\|_{\mathcal{PM}^2} + \|v(\tau)\|_{\mathcal{PM}^2}) \|u(\tau) - v(\tau)\|_{\mathcal{PM}^2} d\tau \\ & \quad + \eta \sup_{\xi \in \mathbb{R}^3} \int_{\delta t}^t |\xi|^2 e^{-(t-\tau)|\xi|^2} (\|u(\tau)\|_{\mathcal{PM}^2} + \|v(\tau)\|_{\mathcal{PM}^2}) \|u(\tau) - v(\tau)\|_{\mathcal{PM}^2} d\tau. \end{aligned} \quad (5.4)$$

where small constant $\delta > 0$ will be chosen later.

In the term on the right-hand side of (5.4) containing the integral $\int_0^{\delta t} \dots d\tau$, we change the variables $\tau = ts$ and we use the identity

$$\sup_{\xi \in \mathbb{R}^3} |\xi|^2 e^{-(1-s)t|\xi|^2} = ((1-s)t)^{-1} \sup_{w \in \mathbb{R}^3} |w|^2 e^{-|w|^2} = ((1-s)t)^{-1} e^{-1}$$

in order to estimate it by

$$\begin{aligned} & \eta \sup_{\xi \in \mathbb{R}^3} \int_0^{\delta} t |\xi|^2 e^{-(1-s)t|\xi|^2} \|u(ts) - v(ts)\|_{\mathcal{PM}^2} ds \\ & \quad \times \left(\sup_{\tau > 0} \|u(\tau)\|_{\mathcal{PM}^2} + \sup_{\tau > 0} \|v(\tau)\|_{\mathcal{PM}^2} \right) \\ & \leq 4\epsilon\eta e^{-1} \int_0^{\delta} (1-s)^{-1} \|u(ts) - v(ts)\|_{\mathcal{PM}^2} ds. \end{aligned} \quad (5.5)$$

We deal with the term in (5.4) containing $\int_{\delta t}^t \dots d\tau$ estimating it directly by

$$\begin{aligned} & \eta \left(\sup_{\xi \in \mathbb{R}^3} \int_{\delta t}^t |\xi|^2 e^{-(t-\tau)|\xi|^2} d\tau \right) \left(\sup_{\delta t \leq \tau \leq t} \|u(\tau) - v(\tau)\|_{\mathcal{PM}^2} \right) 4\epsilon \\ & = 4\epsilon\eta \sup_{\delta t \leq \tau \leq t} \|u(\tau) - v(\tau)\|_{\mathcal{PM}^2}, \end{aligned} \quad (5.6)$$

since $\sup_{\xi \in \mathbb{R}^3} \int_{\delta t}^t |\xi|^2 e^{-(t-\tau)|\xi|^2} d\tau = \sup_{\xi \in \mathbb{R}^3} (1 - e^{-(\delta t)|\xi|^2}) = 1$.

Now, we denote

$$g(t) = \|S(t)(u_0 - v_0)\|_{\mathcal{PM}^2} + \left\| \int_0^t S(t - \tau) \mathcal{IP}(F(t) - G(t)) d\tau \right\|_{\mathcal{PM}^2},$$

and it follows from the assumptions, (5.1) and Lemma 5.1 that

$$g \in L^\infty(0, \infty) \quad \text{and} \quad \lim_{t \rightarrow \infty} g(t) = 0. \quad (5.7)$$

Hence, applying (5.5) and (5.6) to (5.4) we obtain

$$\begin{aligned} \|u(t) - v(t)\|_{\mathcal{PM}^2} & \leq g(t) + 4\epsilon\eta e^{-1} \int_0^{\delta} (1-s)^{-1} \|u(ts) - v(ts)\|_{\mathcal{PM}^2} ds \\ & \quad + 4\epsilon\eta \sup_{\delta t \leq \tau \leq t} \|u(\tau) - v(\tau)\|_{\mathcal{PM}^2} \end{aligned} \quad (5.8)$$

for all $t > 0$.

Next, we put

$$A = \limsup_{t \rightarrow \infty} \|u(t) - v(t)\|_{\mathcal{PM}^2} \equiv \lim_{k \in \mathbb{N}, k \rightarrow \infty} \sup_{t \geq k} \|u(t) - v(t)\|_{\mathcal{PM}^2}.$$

The number A is nonnegative and finite because both $u, v \in L^\infty([0, \infty), \mathcal{PM}^2)$, and our claim is to show that $A = 0$. Here, we apply the Lebesgue Dominated Convergence Theorem to the obvious inequality

$$\sup_{t \geq k} \int_0^\delta (1-s)^{-1} \|u(ts) - v(ts)\|_{\mathcal{PM}^2} ds \leq \int_0^\delta (1-s)^{-1} \sup_{t \geq k} \|u(ts) - v(ts)\|_{\mathcal{PM}^2} ds,$$

and we obtain

$$\limsup_{t \rightarrow \infty} \int_0^\delta (1-s)^{-1} \|u(ts) - v(ts)\|_{\mathcal{PM}^2} ds \leq A \int_0^\delta (1-s)^{-1} ds = A \log \left(\frac{1}{1-\delta} \right). \quad (5.9)$$

Moreover, since

$$\sup_{t \geq k} \sup_{\delta t \leq \tau \leq t} \|u(\tau) - v(\tau)\|_{\mathcal{PM}^2} \leq \sup_{\delta k \leq \tau < \infty} \|u(\tau) - v(\tau)\|_{\mathcal{PM}^2},$$

we have

$$\limsup_{t \rightarrow \infty} \sup_{\delta t \leq \tau \leq t} \|u(\tau) - v(\tau)\|_{\mathcal{PM}^2} \leq A. \quad (5.10)$$

Finally, computing $\limsup_{t \rightarrow \infty}$ of the both sides of inequality (5.8), and using (5.7), (5.9), and (5.10) we get

$$A \leq \left(4\varepsilon\eta e^{-1} \log \left(\frac{1}{1-\delta} \right) + 4\varepsilon\eta \right) A.$$

Consequently, it follows that $A = \limsup_{t \rightarrow \infty} \|u(t) - v(t)\|_{\mathcal{PM}^2} = 0$ because

$$4\varepsilon\eta \left(e^{-1} \log \left(\frac{1}{1-\delta} \right) + 1 \right) < 1,$$

for $\delta > 0$ sufficiently small, by the assumption of Theorem 4.1 saying that $0 < \varepsilon < 1/(4\eta)$. This completes the proof of Theorem 5.1. \square

As a direct consequence the proof of Theorem 5.1, we have also necessary conditions for (5.2) to hold. We formulate this fact in the following corollary.

Corollary 5.1 *Assume that $u, v \in \mathcal{C}_w([0, \infty), \mathcal{PM}^2)$ are solutions to system (2.1)–(2.3) corresponding to initial conditions $u_0, v_0 \in \mathcal{PM}^2$ and external forces $F, G \in \mathcal{C}_w([0, \infty), \mathcal{PM})$, respectively. Suppose that*

$$\lim_{t \rightarrow \infty} \|u(t) - v(t)\|_{\mathcal{PM}^2} = 0. \quad (5.11)$$

Then

$$\lim_{t \rightarrow \infty} \left\| S(t)(u_0 - v_0) + \int_0^t S(t-\tau) P(F(\tau) - G(\tau)) d\tau \right\|_{\mathcal{PM}^2} = 0.$$

Proof. As in the beginning of the proof of Theorem 5.1, we subtract the integral equation (3.2) for v from the same expression for u , and we compute the \mathcal{PM}^2 -norm

$$\begin{aligned} & \left\| S(t)(u_0 - v_0) + \int_0^t S(t - \tau) \mathcal{P}(F(\tau) - G(\tau)) d\tau \right\|_{\mathcal{PM}^2} \\ & \leq \|u(t) - v(t)\|_{\mathcal{PM}^2} \\ & \quad + \eta \sup_{\xi \in \mathbb{R}^3} \int_0^t |\xi|^2 e^{-(t-\tau)|\xi|^2} (\|u(\tau)\|_{\mathcal{PM}^2} + \|v(\tau)\|_{\mathcal{PM}^2}) \|u(\tau) - v(\tau)\|_{\mathcal{PM}^2} d\tau. \end{aligned} \quad (5.12)$$

The first term on the right-hand side of (5.12) tends to zero as $t \rightarrow \infty$ by (5.11). To show the decay of the second one, it suffices to repeat calculations from (5.4), (5.5), (5.6), and (5.9). Here, however, one should remember that now it is assumed that $A = 0$ and $\sup_{t>0} \|u(t)\|_{\mathcal{PM}^2} < \infty$ and $\sup_{t>0} \|v(t)\|_{\mathcal{PM}^2} < \infty$. \square

Remark 5.1 The Lyapunov stability of solutions (not necessarily stationary ones) follows immediately from the construction via the Banach fixed point theorem (cf. Lemma 4.1). This phenomenon was already observed and used several times, see e.g. the papers by H. Kozono and M. Yamazaki [20, Th. 2], [22, Th. 1], and by M. Yamazaki [33, Th. 1.3]. Theorem 5.1 extends those results by giving sufficient conditions on the asymptotic stability of solutions. In particular, Yamazaki [33, Remark 4.1] emphasized that the trivial solution 0 is stable but not asymptotically stable in the space $L^{3,\infty}(\mathbb{R}^3)$ (in contrast to the Lebesgue space $L^3(\mathbb{R}^3)$), because there exist self-similar solutions with constant $L^{3,\infty}$ -norm. Theorem 5.1 and Corollary 5.1 explain this phenomenon in the case of the space \mathcal{PM}^2 . Indeed, given $u_0 \in \mathcal{PM}^2$ such that $\|u_0\|_{\mathcal{PM}^2} < \varepsilon$ and $F \equiv 0$, the corresponding solution converges in \mathcal{PM}^2 to zero as $t \rightarrow \infty$ if and only if $\lim_{t \rightarrow \infty} \|S(t)u_0\|_{\mathcal{PM}^2} = 0$.

Note here, that if $U(x, t) = t^{-1/2}U(x/t^{1/2})$ is a self-similar solution to system (2.1)–(2.3), its \mathcal{PM}^2 -norm is constant in time by the scaling relation (3.5). As it is well-known, $U(x, t)$ corresponds to the initial condition $U_0(x)$ which is homogeneous of degree -1 , so $S(t)U_0(x) = t^{-1/2}S(1)U_0(xt^{-1/2})$. Consequently, by the scaling property of the norm, we have $\|S(t)U_0\|_{\mathcal{PM}^2} = \|S(1)U_0\|_{\mathcal{PM}^2}$, cf. Corollary 5.1 with $F = G \equiv 0$. \square

Remark 5.2 In the setting of the L^p -spaces and the homogeneous Besov spaces, the study of the asymptotic stability of self-similar solutions to the Navier–Stokes system began with the paper [30] of F. Planchon (see also the presentation of Planchon’s results in [26, Ch. 23.3]). As illustrated in the book by Y. Giga and M.-H. Giga [14] those ideas are quite universal and were used for other partial differential equations (e.g. the porous medium, the nonlinear Schrödinger and the KdV equations); they were applied for instance to study asymptotic properties of solutions to a large class of nonlinear parabolic equations [16] as well as of solutions with zero mass to viscous conservation laws [17]. In this section, we extend them on solutions which not necessarily decay to 0 as $t \rightarrow \infty$. \square

6 Stationary solutions

Our approach, described in previous sections, to study global-in-time solutions to the problem (2.1)–(2.3), as well as their large time behavior, can be also applied to stationary solutions. Below, we briefly describe some consequences of Theorems 4.1 and 5.1. The following proposition contains two equivalent integral equations satisfied by stationary solutions.

Proposition 6.1 *Assume that $u = u(x) \in \mathcal{PM}^2$ and $F \in \mathcal{PM}$. The following two facts are equivalent*

- 1) $u = u(x)$ is a stationary mild solution of system (2.1)–(2.2) in the sense of Definition 3.1. Hence, u is the solution of the integral equation

$$u = S(t)u - \int_0^t S(t-\tau) \mathbb{P} \nabla \cdot (u \otimes u) \, d\tau + \int_0^t S(\tau) \mathbb{P} F \, d\tau \quad (6.1)$$

for every $t > 0$;

- 2) u satisfies the integral equation

$$u = - \int_0^\infty S(\tau) \mathbb{P} \nabla \cdot (u \otimes u) \, d\tau + \int_0^\infty S(\tau) \mathbb{P} F \, d\tau, \quad (6.2)$$

where the integrals above should be understood in the Fourier variables for almost every ξ .

Proof. By Definition 3.1, the integral equation (6.1) can be rewritten as

$$\begin{aligned} \widehat{u}(\xi) &= e^{-t|\xi|^2} \widehat{u}(\xi) - \int_0^t e^{-(t-\tau)|\xi|^2} \, d\tau \widehat{\mathbb{P}}(\xi) i\xi \cdot (\widehat{u \otimes u})(\xi) \\ &\quad + \int_0^t e^{-(t-\tau)|\xi|^2} \, d\tau \widehat{\mathbb{P}}(\xi) \widehat{F}(\xi) \\ &= e^{-t|\xi|^2} \widehat{u}(\xi) - \frac{1 - e^{-t|\xi|^2}}{|\xi|^2} \widehat{\mathbb{P}}(\xi) i\xi \cdot (\widehat{u \otimes u})(\xi) + \frac{1 - e^{-t|\xi|^2}}{|\xi|^2} \widehat{\mathbb{P}}(\xi) \widehat{F}(\xi). \end{aligned} \quad (6.3)$$

for every $t > 0$. Passing to the limit as $t \rightarrow \infty$ in (6.3) and using the identity

$$\frac{1}{|\xi|^2} = \int_0^\infty e^{-\tau|\xi|^2} \, d\tau \quad \text{for } \xi \neq 0,$$

we obtain equation (6.2) in the Fourier variables.

Now, assume that u solves (6.2). Repeating the arguments above in the reverse order, we obtain that u is the solution of the equation

$$\widehat{u}(\xi) = - \frac{1}{|\xi|^2} \widehat{\mathbb{P}}(\xi) i\xi \cdot (\widehat{u \otimes u})(\xi) + \frac{1}{|\xi|^2} \widehat{\mathbb{P}}(\xi) \widehat{F}(\xi). \quad (6.4)$$

If we subtract from this equality the same expression multiplied by $e^{-t|\xi|^2}$ we get (6.3) which obviously is equivalent to (6.1). \square

Theorem 6.1 *Assume that $F \in \mathcal{PM}$ satisfies $\|F\|_{\mathcal{PM}} < \varepsilon < 1/(4\eta)$. There exists a stationary solution u_∞ to the Navier–Stokes system in the space \mathcal{PM}^2 with F as the external force. This is the unique solution satisfying the condition $\|u\|_{\mathcal{PM}^2} \leq 2\varepsilon$.*

Proof. This theorem results immediately from Lemma 4.1 applied to the integral equation (6.2) (or its equivalent version (6.4)). The bilinear form

$$B(u, v) = \int_0^\infty S(\tau) \mathbb{P} \nabla \cdot u \otimes v \, d\tau$$

is bounded on the space \mathcal{PM}^2 and the proof of this property of $B(\cdot, \cdot)$ is completely analogous to the one of Proposition 4.1. Let us also skip an easy proof that $y = \int_0^\infty S(\tau) \mathbb{P} F \, d\tau$ satisfies $\|y\|_{\mathcal{PM}^2} = \|F\|_{\mathcal{PM}}$. \square

Now, the application of Theorem 5.1 gives the following result on the asymptotic stability of stationary solutions.

Corollary 6.1 *Assume that u_∞ is the stationary solution constructed in Theorem 6.1 corresponding to the external force F . Suppose that $v_0 \in \mathcal{PM}^2$ and $G \in \mathcal{C}_w([0, \infty), \mathcal{PM})$ satisfy $\|v_0\|_{\mathcal{PM}^2} + \|G\|_{\mathcal{C}_w([0, \infty), \mathcal{PM})} \leq \varepsilon < 1/(4\eta)$ and, moreover,*

$$\lim_{t \rightarrow \infty} \|S(t)(v_0 - u_\infty)\|_{\mathcal{PM}^2} = 0, \quad \lim_{t \rightarrow \infty} \|G(t) - F\|_{\mathcal{PM}} = 0.$$

Then, the solution $v = v(x, t)$ of system (2.1)–(2.3) corresponding to v_0 and G converges toward the stationary solution u_∞ in the following sense

$$\lim_{t \rightarrow \infty} \|v(t) - u_\infty\|_{\mathcal{PM}^2} = 0.$$

Proof. Here, it suffices only to note that stationary solutions belong to the space $\mathcal{C}_w([0, \infty), \mathcal{PM}^2)$ (treated as constant functions on $[0, \infty)$ with values in \mathcal{PM}^2) and satisfy the integral equation (3.2) (see Proposition 6.1). So, Theorem 5.1 is applicable in this case. \square

Remark 6.1 Results from this section can be extended to solutions which exist for all $t \in \mathbb{R}$ (and not only for $t \geq 0$) as was done by M. Yamazaki [33]. In this case the corresponding integral equation (the counterpart of (6.1) and (6.2)) has the form

$$u(t) = - \int_0^\infty S(\tau) \mathbb{P} \nabla (u \otimes u)(t - \tau) \, d\tau + \int_0^\infty S(\tau) \mathbb{P} F(t - \tau) \, d\tau,$$

and, like in [33], by the application of Theorem 4.1, one obtains solutions which are, for example, time periodic or almost periodic with respect to $t \in \mathbb{R}$. In the same manner, Theorem 5.1 allows us to describe solutions which converge in \mathcal{PM}^2 as $t \rightarrow \infty$ toward given time periodic (or almost periodic) solution. \square

7 Smooth solutions

Solutions of problem (2.1)–(2.3) constructed in the space $\mathcal{X} = \mathcal{C}_w([0, \infty), \mathcal{PM}^2)$ are, in fact, smooth (for sufficiently regular external forces), and they agree with mild solutions obtained by T. Kato [18] and in [3] for $F \equiv 0$, and, more generally, with solutions obtained in [6] when $F \neq 0$.

The goal of this section is to clarify this remark. First, let us recall that, in [3], solutions of (2.1)–(2.3) were constructed for sufficiently small initial conditions from the homogeneous Besov space $\dot{B}_p^{-1+3/p, \infty}(\mathbb{R}^3)$ with $3 < p < \infty$. The usual way of defining a norm in this space is based on the dyadic decomposition of tempered distributions. Here, however as in [3, 16] we prefer the equivalent norm whose definition involves the heat semigroup

$$\|v\|_{\dot{B}_p^{-\alpha, \infty}(\mathbb{R}^3)} \equiv \sup_{t>0} t^{\alpha/2} \|S(t)v\|_{L^p(\mathbb{R}^3)}.$$

Connections between \mathcal{PM}^2 and homogeneous Besov spaces are described in the following lemma.

Lemma 7.1 *For every $p \in (3, \infty]$ the following imbeddings $\mathcal{PM}^2 \subset \dot{B}_p^{-1+3/p, \infty}(\mathbb{R}^3)$ hold true and are continuous. Hence, there exists a constant $C = C(p)$ such that*

$$\sup_{t>0} t^{(1-3/p)/2} \|S(t)u_0\|_{L^p(\mathbb{R}^3)} \leq C \|u_0\|_{\mathcal{PM}^2}$$

for all $t > 0$ and $u_0 \in \mathcal{PM}^2$.

Proof. Here, our tool is the Hausdorff–Young inequality. For $1/p + 1/q = 1$ we obtain

$$\begin{aligned} \|S(t)u_0\|_{L^p(\mathbb{R}^3)}^q &\leq C \int_{\mathbb{R}^3} \left| e^{-t|\xi|^2} \widehat{u}_0(\xi) \right|^q d\xi \\ &\leq C \sup_{\xi \in \mathbb{R}^3} |\xi|^2 |\widehat{u}_0(\xi)| \int_{\mathbb{R}^3} \left| \frac{e^{-qt|\xi|^2}}{|\xi|^{2q}} \right| d\xi \\ &= C \|u_0\|_{\mathcal{PM}^2} t^{-3/2+q} \int_{\mathbb{R}^3} \left| \frac{e^{-q|w|^2}}{|w|^{2q}} \right| dw. \end{aligned}$$

In the calculations above, we assume that $2q < 3$ which is equivalent to $p > 3$. Since, $1/q = 1 - 1/p$ and $(3/2)(1 - 1/p) - 1 = (1/2)(1 - 3/p)$, we obtain

$$t^{(1/2)(1-3/p)} \|S(t)u_0\|_{L^p(\mathbb{R}^3)} \leq C \|u_0\|_{\mathcal{PM}^2}.$$

Note that this proof requires an obvious modification for $p = \infty$ and $q = 1$. One can also recall here the embedding of any “critical space” into the Besov space $\dot{B}_\infty^{-1, \infty}(\mathbb{R}^3)$, see [29, 4]. \square

Now, given $u_0 \in \mathcal{PM}^2$ with sufficiently small \mathcal{PM}^2 -norm, we may apply the theory described in [3] to get the solution $\tilde{u} = \tilde{u}(x, t)$ which is unique in the space

$$\mathcal{C}_w([0, \infty), \dot{B}_p^{-1+3/p, \infty}(\mathbb{R}^3)) \cap \{v : t^{(3/p-1)/2} \|v(t)\|_{L^p(\mathbb{R}^3)} < \infty\}$$

corresponding to u_0 as the initial condition and the zero external force. Moreover, this solution is smooth for all $t > 0$. On the other hand, our Theorem 4.1 gives a solution $u = u(x, t)$ in $\mathcal{C}_w([0, \infty), \mathcal{PM}^2)$.

Both constructions lead, in fact, to the same solution, and we show this by analyzing the parabolic regularization effect in problem (2.1)–(2.3) in the scale of spaces \mathcal{PM}^a . We begin by a definition.

Definition 7.1 *Let $2 \leq a < 3$. We define the Banach space*

$$\begin{aligned} \mathcal{Y}^a &\equiv \mathcal{C}_w([0, \infty), \mathcal{PM}^2) \\ &\cap \{v : (0, \infty) \rightarrow \mathcal{PM}^a : \|v\|_a \equiv \sup_{t>0} t^{a/2-1} \|v(t)\|_{\mathcal{PM}^a} < \infty\}. \end{aligned} \quad (7.1)$$

The space \mathcal{Y}^a is normed by the quantity $\|v\|_{\mathcal{Y}^a} = \|v\|_2 + \|v\|_a$. Of course, $\mathcal{Y}^2 \equiv \mathcal{X}$ with this definition.

Remark 7.1 The norm $\|\cdot\|_a$ is invariant under the rescaling $u_\lambda(x, t) = \lambda u(\lambda x, \lambda^2 t)$ for every $\lambda > 0$. This can be easily checked using the scaling property of the norm $\|\cdot\|_{\mathcal{PM}^a}$, see (3.5). \square

First we show an improvement of Proposition 4.1.

Proposition 7.1 *Let $2 \leq a < 3$. There exists a constant $\eta_a > 0$ such that for every $u \in \mathcal{C}_w([0, \infty), \mathcal{PM}^2)$ and $v \in \{v(t) \in \mathcal{PM}^a : \|v\|_a < \infty\}$ we have*

$$\|B(u, v)\|_a \leq \eta_a \|u\|_2 \|v\|_a.$$

Proof. First note that as in the proof of Proposition 4.1 we have

$$\begin{aligned} |(u \widehat{\otimes} v)(\xi, t)| &\leq \int_{\mathbb{R}^3} \frac{1}{|\xi - z|^2 |z|^a} dz \|u(t)\|_{\mathcal{PM}^2} \|v(t)\|_{\mathcal{PM}^a} \\ &= C |\xi|^{1-a} \|u(t)\|_{\mathcal{PM}^2} \|v(t)\|_{\mathcal{PM}^a}. \end{aligned}$$

Thus, for every $\xi \neq 0$ we obtain

$$|\xi|^a \left| \int_0^t e^{-(t-\tau)|\xi|^2} \widehat{P}(\xi) i\xi \cdot (u \widehat{\otimes} v)(\xi, \tau) d\tau \right| \leq C \int_0^t |\xi|^2 e^{-(t-\tau)|\xi|^2} \tau^{1-a/2} d\tau \|u\|_2 \|v\|_a.$$

The proof will be completed by showing that for every $a \geq 2$ the quantity

$$t^{a/2-1} \int_0^t |\xi|^2 e^{-(t-\tau)|\xi|^2} \tau^{1-a/2} d\tau \quad (7.2)$$

is bounded by a constant independent of ξ and t . Here, we decompose the integral with respect to τ into two parts $\int_0^t \dots d\tau = \int_0^{t/2} \dots d\tau + \int_{t/2}^t \dots d\tau$, and we deal with the each term separately.

In case of the integral over $[0, t/2]$, we estimate the above quantity by

$$t^{a/2-1} |\xi|^2 e^{-(t/2)|\xi|^2} \int_0^{t/2} \tau^{1-a/2} ds = C(t/2) |\xi|^2 e^{-(t/2)|\xi|^2} \leq C$$

where C is independent of ξ and t . For the interval $[t/2, t]$, the quantity is bounded by

$$t^{a/2-1}(t/2)^{1-a/2} \int_{t/2}^t |\xi|^2 e^{-(t-\tau)|\xi|^2} d\tau = (1/2)^{1-a/2} (1 - e^{-(t/2)|\xi|^2}) \leq (1/2)^{1-a/2}.$$

□

Next, we show that the heat semigroup regularizes distributions from \mathcal{PM}^2 .

Lemma 7.2 *For every $u_0 \in \mathcal{PM}^2$ and $t > 0$, it follows that $S(t)u_0 \in \mathcal{PM}^a$ with $a \geq 2$. Moreover, there exists C depending on the exponent a only such that*

$$\sup_{t>0} \left(t^{a/2-1} \|S(t)u_0\|_{\mathcal{PM}^a} \right) \leq C \|u_0\|_{\mathcal{PM}^2}.$$

Proof. Simple estimates (cf. Lemma 4.2) give

$$\begin{aligned} \sup_{t>0} \left(t^{a/2-1} \|S(t)u_0\|_{\mathcal{PM}^a} \right) &\leq \|u_0\|_{\mathcal{PM}^2} \sup_{\xi \in \mathbb{R}^3} \left(t^{a/2-1} |\xi|^{a-2} e^{-t|\xi|^2} \right) \\ &= C \|u_0\|_{\mathcal{PM}^2} \end{aligned}$$

where $C = \sup_{w \in \mathbb{R}^3} (|w|^{a-2} e^{-|w|^2})$. □

Let us also explain how to handle more regular external forces in the scale of the spaces \mathcal{PM}^a .

Lemma 7.3 *Let $2 \leq a < 3$. Assume that $F(t) \in \mathcal{PM}^{a-2}$ for all $t > 0$ and*

$$\sup_{t>0} t^{a/2-1} \|F(t)\|_{\mathcal{PM}^{a-2}} < \infty. \quad (7.3)$$

There exists a constant C such that for $w(t) = \int_0^t S(t-\tau) \mathbb{P}F(\tau) d\tau$ it follows that

$$\|w\|_a \leq C \sup_{t>0} t^{a/2-1} \|F(t)\|_{\mathcal{PM}^{a-2}}.$$

Proof. As in the proof of Lemma 7.2, we obtain

$$\begin{aligned} \|w(t)\|_{\mathcal{PM}^a} &\leq \operatorname{ess\,sup}_{\xi \in \mathbb{R}^3} |\xi|^a \int_0^t \left| e^{-(t-\tau)|\xi|^2} \widehat{\mathbb{P}}(\xi) \widehat{F}(\xi, \tau) \right| d\tau \\ &\leq \kappa \int_0^t |\xi|^2 e^{-(t-\tau)|\xi|^2} \tau^{1-a/2} d\tau \sup_{t>0} t^{a/2-1} \|F(t)\|_{\mathcal{PM}^{a-2}}. \end{aligned}$$

From now on, it suffices to repeat the reasoning which leads to the estimates of the quantity in (7.2). □

Theorem 7.1 *Let $a \in [2, 3)$. There exists $\varepsilon > 0$ such that for every $u_0 \in \mathcal{PM}^2$ and $F \in \mathcal{C}_w([0, \infty), \mathcal{PM})$ satisfying (7.3) with*

$$\|u_0\|_{\mathcal{PM}^2} + \|F\|_{\mathcal{C}_w([0, \infty), \mathcal{PM})} + \sup_{t>0} t^{a/2-1} \|F(t)\|_{\mathcal{PM}^{a-2}} < \varepsilon,$$

the solution constructed in Theorem 4.1 satisfies $\|u\|_a \leq 2\varepsilon$.

Proof. It suffices to repeat the reasoning leading to Theorem 4.1 in the space

$$\mathcal{Y}^a = \mathcal{C}_w([0, \infty), \mathcal{PM}^2) \cap \{u : \sup_{t>0} t^{a/2-1} \|u(t)\|_{\mathcal{PM}^a} < \infty\}$$

involving Lemma 4.1. Here, the required estimate of the bilinear form $B(\cdot, \cdot)$ is proved in Propositions 4.1 and 7.1. Moreover, Lemmata 7.2 and 7.3 guarantee that $y = S(t)u_0 + \int_0^t S(t-\tau) \mathbb{P}F(\tau) d\tau$ belongs to \mathcal{Y}^a . \square

Let us formulate an interpolation inequality involving L^q and \mathcal{PM}^a norms.

Lemma 7.4 *Fix $a \in (2, 3)$. For every $q \in (3, \frac{3}{3-a})$ there exists a constant $C = C(a, q)$ such that*

$$\|v\|_{L^q(\mathbb{R}^3)} \leq C \|v\|_{\mathcal{PM}^2}^\beta \|v\|_{\mathcal{PM}^a}^{1-\beta} \quad (7.4)$$

for all $v \in \mathcal{PM}^2 \cap \mathcal{PM}^a$, where $\beta = \frac{1}{a-2} (1 - \frac{3}{q})$.

Proof. Assume that v is smooth and rapidly decreasing. Using the Hausdorff–Young inequality (with $1/p + 1/q = 1$ and $p \in [1, 2)$) and the definition of the \mathcal{PM}^a -norm we obtain

$$\begin{aligned} \|v\|_q^p &\leq C \|\widehat{v}\|_p^p \leq C \|v\|_2^p \int_{|\xi| \leq R} \frac{1}{|\xi|^{2p}} d\xi + C \|v\|_a^p \int_{|\xi| > R} \frac{1}{|\xi|^{ap}} d\xi \\ &\leq C \|v\|_2^p R^{3-2p} + C \|v\|_a^p R^{3-ap} \end{aligned} \quad (7.5)$$

for all $R > 0$ and C independent of v and R . In these calculations, we require $2p < 3$ which is equivalent to $q > 3$. Moreover, we have to assume that $ap > 3$ which leads to the inequality $q < 3/(3-a)$. Now, we optimize inequality (7.5) with respect to R to get (7.4). \square

Corollary 7.1 *Under the assumptions of Theorem 7.1 the constructed solution satisfies*

$$\|u(\cdot, t)\|_{L^q(\mathbb{R}^3)} \leq Ct^{-(1-3/q)/2}$$

for each $3 < q, 3/(3-a)$, all $t > 0$, and C independent of t .

Proof. It follows from Theorem 7.1 that the solution u satisfies $\|u(\cdot, t)\|_{\mathcal{PM}^a} \leq Ct^{1-a/2}$ for every $a \in [2, 3)$. Hence, to complete the proof of this corollary, it suffices to apply Lemma 7.4. \square

Let us finally prove that the difference of two (singular) solutions corresponding to the same external force is more regular than each term separately. This fact is in a perfect agreement with the regularity result for the bilinear term obtained in [7].

Theorem 7.2 *Assume that $u, v \in \mathcal{X}$ are solutions to (2.1)–(2.3) constructed in Theorem 4.1 corresponding to initial conditions $u_0, v_0 \in \mathcal{PM}^2$ and the same external force $F \in \mathcal{C}_w([0, \infty), \mathcal{PM})$. For every $2 \leq a < 3$ there exists $\varepsilon > 0$ such that for $\|u_0 - v_0\|_{\mathcal{PM}^2} < \varepsilon$ we have*

$$\|u - v\|_a \equiv \sup_{t>0} t^{a/2-1} \|u(t) - v(t)\|_{\mathcal{PM}^a} < \infty.$$

Moreover, $\sup_{t>0} t^{(1-3/q)/2} \|u(t) - v(t)\|_{L^q(\mathbb{R}^3)} < \infty$ for every $3 < q < 3/(3-a)$.

Proof. Here, the reasoning is similar to that presented above, hence we shall be brief in details. First, we subtract integral equations (3.2) for u and v to obtain

$$u(t) - v(t) = S(t)(u_0 - v_0) + B(u, u - v)(t) + B(u - v, v)(t).$$

We denote $z(t) = u(t) - v(t)$ and $z_0 = u_0 - v_0$, and we find the solution of the equation $z = S(\cdot)z_0 + B(u, z) + B(z, v)$ via the Banach fixed point theorem in the space \mathcal{Y}^a defined in (7.1). Here, Lemma 7.2 guarantees that $S(\cdot)z_0 \in \mathcal{Y}^a$ for every $2 \leq a < 3$. Moreover, Propositions 4.1 and 7.1 allow us to show the contractivity of the mapping $z \mapsto S(\cdot)z_0 + B(u, z) + B(z, v)$ for sufficiently small $\varepsilon > 0$ because, by Theorem 4.1, u and v satisfy (5.3). The second part of this theorem is deduced immediately from Lemma 7.4. \square

Remark 7.2 Given $u_0 \in \mathcal{PM}^2$ with sufficiently small norm and $F \equiv 0$, Theorem 4.1 guarantees the existence of a unique small solution $u \in \mathcal{C}_w([0, \infty), \mathcal{PM}^2)$. Next, our analysis in Corollary 7.1 allows us to show that $u(t) \in L^q(\mathbb{R}^3)$ for $q > 3$ and all $t > 0$. Hence, standard regularity theorems imply that $u(x, t)$ is a smooth function and satisfies the Navier–Stokes system in the classical sense.

Even if it is not written explicitly, the same conclusion can be deduced from Yamazaki’s results [33, Th. 1.3], where he showed that his solution belonging initially to $\mathcal{C}_w([0, \infty), L^{3, \infty}(\mathbb{R}^3))$ falls, in fact, into $L^{p, \infty}(\mathbb{R}^3)$ for every $3 < p < \infty$. Now, applying the Marcinkiewicz interpolation theorem for the identity mapping, one obtains immediately that $\bigcap_{3 < p < \infty} L^{p, \infty}(\mathbb{R}^3) \subset L^q(\mathbb{R}^3)$ for every $3 < q < \infty$. \square

We conclude this section by stressing again that the two norms approach by Kato imposes *a priori* a regularization effect on solutions we look for. In other words, they are considered as fluctuations around the solution of the heat equation $S(t)u_0$. The solutions appear to be unique locally in the space of more regular functions. The approach with the only one norm in Theorem 4.1 gives the local uniqueness in the larger space which, in our case, may contain genuinely singular solutions (like those in (2.4)) which are not smoothed out by the action of the nonlinear semigroup associated with (2.1)–(2.3).

8 Loss of smoothness for large solutions

As far as blow-up for Navier–Stokes several possibilities can be conjectured. One may imagine that blow-up of initially regular solutions never happens, or it becomes more likely as the initial norm increases, or that there is blow-up, but only on a very thin set, of measure zero.

As we have seen in the previous sections, when using a fixed point approach, existence and uniqueness of global solutions are guaranteed only under restrictive assumptions on the initial data and external forces, that are required to be small in some sense, i.e. in some functional space. In [3] we pointed out that fast oscillations are sufficient to make the fixed point scheme works, even if the norm in the corresponding function

space of the initial data is arbitrarily large (in fact, a different auxiliary norm turns out to be small). Here we want to suggest how some particular data, arbitrarily large (not oscillating) could give rise to irregular solutions. It is extremely unpleasant that we do not know in general whether for arbitrary large data the corresponding solution is regular or singular. More precisely:

Remark 8.1 Let us consider the Navier–Stokes equations (2.3) with external force $F \equiv 0$. Then, if one defines the functions $u_\varepsilon(x, 0) = \varepsilon u(x)$, where $u(x)$ is the (divergence free, homogeneous of degree -1) function given by (2.4) as the initial data, then for small ε the system has a global regular (self-similar) solution which is even more regular than *a priori* expected (Section 7) and for $\varepsilon = 1$ (and possibly for other large values of ε) the system has a singular “solution” for any time. \square

Unfortunately, this loss of smoothness for large data does not hold in the “distributional” sense, but as explained in Section 2, only “pointwise” for every $x \in \mathbb{R}^3 \setminus \{(0, 0, 0)\}$. However, for a model equation of gravitating particles this loss of smoothness for large data holds in the distributional sense and will be dealt with in a forthcoming paper [2].

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