Approximation of the stochastic Rayleigh-Bénard problem near the onset of convection and related problems

DIRK BLÖMKER[1,2]

[1] Mathematisches Institut, RWTH Aachen, D- 52056 Aachen, Germany

BLOEMKER@INSTMATH.RWTH-AACHEN.DE

[2] MRC, University of Warwick, Coventry CV4 7AL - UK

June 5, 2003

Abstract

Complicated stochastic evolution systems on bounded domains are often dominated by slow modes near a change of stability. We approximate the evolution of the system by stochastic ordinary differential equations describing the amplitudes of these dominating modes changing on a slow time-scale.

We focus on equations with quadratic nonlinearity and give applications to the Rayleigh-Bénard problem and a model from surface growth. The unique existence of global solutions is not necessary.

Keywords: Amplitude equation, Rayleigh Bénard, multiscale analysis, approximate center manifold, surface growth, fractional Brownian motion, systems of SPDEs

Classification: 60H15, 35R60, 60H10, 76E06, 35Q72

1 Introduction

Bifurcation points play a central role in the theory of evolution equations given by deterministic partial differential equations (PDEs). At these points the dynamical behavior of solutions changes drastically, one standard example being a pitchfork-bifurcation, where the number of stable fixed points of the evolution changes from one to two. But for stochastic systems the question how a bifurcation occurs and what is an analog to a deterministic bifurcation is not completely settled even for stochastic ordinary equations (see e.g. [Ar98, Sec. 9] or [CF98]).

Our approach will describe the transient dynamics of some in general very complicated system of stochastic PDEs near a deterministic bifurcation. In that case a natural separation of time-scales is present, which allows to separate finitely many dominating modes from the infinite-dimensional problem. Moreover the dynamics is completely described by these finitely many degrees of freedom. The method presented could also be referred to as an approximate center manifold. Nevertheless the vector space given by the dominant modes is only a first order approximation of the true invariant manifold.

A related result is [FvE03], where the stochastic flow of a stochastically perturbed gradient system is investigated on a slow time-scale in some slow manifold, which is finite-dimensional. Moreover in [K01] envelope-equations for noise induced oscillations in delay-equations are derived by multi-scale analysis. But neither result contains a rigorous proof of the reduction.

The approximation of stochastic PDEs in a neighborhood of a change of stability by a set of simple stochastic ODEs or SPDEs is on a formal level well-understood (see e.g. [H83] or [W97]). Despite of that, the mathematical theory was rigorously only developed for cubic type nonlinearities (see [B03, BMS01]). Moreover, the theory for general odd nonlinearities, which was not investigated, is completely similar to the cubic case. The main advantage in both cases is a simple separation of dynamics, where the dominating modes decouple easily from the fast modes.

In contrast to that even nonlinearities like the quadratic one in the Navier-Stokes equation are much more involved, as they tend to mix modes much stronger. We will see that the main problem to overcome is the fact that the nonlinearity does not act directly on the dominant modes, but only influences them through non-dominant modes.

In our abstract setting we consider equations of the following type:

$$\partial_t u(t) = Lu(t) + \varepsilon^2 Au(t) + B(u(t), u(t)) + \varepsilon^{2H+1}\xi(t), \quad u(0) = u_0. \tag{1}$$

The equation is considered in some Banach space X, where L and A

are some (unbounded) operators, B is a bilinear mapping, and ξ is some Gaussian noise, both in space and time, where the correlation in time should be fractional noise with Hurst-parameter $H \in [\frac{1}{2}, 1)$. This means we have (for $H > \frac{1}{2}$) formally

$$\mathbb{E}\xi(t,x) = 0 \qquad \text{and} \qquad \mathbb{E}\xi(t,x)\xi(s,y) = c_H|t-s|^{2H-2}q(x,y)$$

where q is the kernel of the corresponding covariance operator, and c_H some constant depending only on $H > 0$. The case $H = \frac{1}{2}$ corresponds to white noise in time (see e.g [B02] for further references).

In [B03, BMS01] the scaling of the noise and the deterministic perturbation $\varepsilon^2 Au$ should be of the same order, to get an interesting stochastic approximation. But if we increase the time-regularity of the noise, the scaling allowing the noise to show up in the amplitude equations becomes different.

Our major examples will be the Rayleigh-Bénard problem. Therefore, we omit cubic or higher order terms in (1) for simplicity of presentation. Nevertheless we can simply deal with that. Cubic nonlinearities will give a contribution to the amplitude equation (like in [B03]), while higher order nonlinearities give no contribution at all, at least not on the time-scale of interest. It is even possible that (1) is unstable and exhibits blow-ups in finite time, but if the amplitude equation is stable, this will happen only after a very long time.

The main observation for our abstract system is that the projection onto the kernel $N(L)$ of L lives on a much slower time-scale, as it is not subject to an exponential decay on a time-scale of order $\mathcal{O}(1)$. We can now perform a simple formal calculation. Denote the projection onto $N(L)$ by P_c and define $P_s = I - P_c$.

Consider the ansatz $u(t) = \varepsilon\Phi_c(\varepsilon^2 t) + \varepsilon^2\psi_s(t)$, where $\Phi_c \in N(L)$ and $\psi_s \in P_s X$. Let us just remark, that the ansatz $u(t) = \varepsilon\Phi_c(\varepsilon^2 t)$ is not appropriate, as the nonlinearity usually maps $N(L)$ into $P_s X$. The approach leads just to a linear equation, which cannot be justified rigorously. The higher order corrections are necessary, although in the final result, they play no role, as the approximation error will be of order $\mathcal{O}(\varepsilon^2)$.

Using (1) we obtain in lowest order in ε the following system of formal approximations. First of order $\mathcal{O}(\varepsilon^2)$ on the fast time-scale in $P_s X$.

$$\partial_t\psi_s(t) = L\psi_s(t) + P_s B(\Phi_c(\varepsilon^2 t), \Phi_c(\varepsilon^2 t)) + \varepsilon^{2H-1} P_s\xi(t). \tag{2}$$

Secondly we get in $N(L)$ in order ε^2

$$P_c B(\Phi_c(T), \Phi_c(T)) = 0 \tag{3}$$

and in order ε^3

$$\Phi_c'(T) = P_c A \Phi_c(T) + 2P_c B(\Phi_c(T), \psi_s(\varepsilon^{-2}T)) + P_c\hat{\xi}(T), \tag{4}$$

where $T = \varepsilon^2 t$ is the slow time-scale and $\hat{\xi}(T) = \varepsilon^{2H-2}\xi(\varepsilon^{-2}T)$ is a rescaled version of our noise, i.e. both stochastic processes posess the same distributions.

Now (3) simply leads to an Assumption on B_c (cf. Assumption 2.5). The other two equations are on one hand a dominating equation (4) on a slow time-scale coupled to an equation (2) on the fast time-scale.

Equations with a similar structure are treated in [PS03a, PS03b] where tracers in a fast moving velocity field are considered. Moreover, the problem is similar to the averaging principles. See e.g. Khasminskii or Krylov [KK01] for a recent reference or Freidlin [Fr96] for Hamiltonian systems. Nevertheless the structure of our problem is slightly different, as we want to get an effective equation for the slow component completely independent of the fast modes.

In order to simplify (2) and (4), rescale (2) to the slow time-scale T by defining $\psi_s(t) = \Phi_s(\varepsilon^2 t)$. We obtain

$$\varepsilon^2 \Phi_s'(T) = L\Phi_s(T) + P_s B(\Phi_c(T), \Phi_c(T)) + \varepsilon P_s \hat{\xi}(T)$$

Noting that L is invertible on $P_s X$, we get in lowest order of ε that $\psi_s(\varepsilon^{-2}T) = -L^{-1}P_s B(\Phi_c(T), \Phi_c(T))$. Plugging this into (4) we end up with a single approximation equation for the amplitudes of the slow modes.

$$\Phi_c'(T) = P_c A \Phi_c(T) - 2P_c B\Big(\Phi_c(T), L^{-1}P_s B\Big(\Phi_c(T), \Phi_c(T)\Big)\Big) + P_c\hat{\xi}(T), \tag{5}$$

Surprisingly, this equation involves a cubic nonlinearity, although the nonlinearity in the original equation was quadratic.

Our main results will show that these formal calculations can be made rigorous. First the *attractivity* result (Theorem 2.13) justifies the ansatz for small initial conditions after a short time of order $\mathcal{O}(\ln(\varepsilon^{-1}))$. This involves basically only exponential decay on the fast modes.

The *approximation* result (Theorem 2.18) is more involved. It verifies the formal calculation for a solution u of (1). Provided $u(0) = \varepsilon a(0) \cdot e + \mathcal{O}(\varepsilon^2)$

then $u(t) = \varepsilon a(\varepsilon^2 t) \cdot e + \mathcal{O}(\varepsilon^2)$ on a time-scale of order $\mathcal{O}(\varepsilon^{-2})$, where $e = (e_1, \ldots, e_n)$ is a basis of $N(L)$ and the amplitudes $a \in \mathbb{R}^n$ fulfill a stochastic ordinary differential equation given by (5).

There are numerous examples of equations of type (1) in the physics literature. One example is the growth of rough amorphous surfaces, which (after rescaling to the right dimension free quantities) is described by

$$\partial_t h = -\Delta^2 h - \nu \Delta h - \Delta |\nabla h|^2 + \varepsilon^{2H+1} \xi \tag{6}$$

Here $h = h(t, x)$ is the height profile of a growing surface over $x \in [0, 2\pi]^d$, $d = 1, 2$. See for example [RLH01, RM+00] for physical derivation, and [BGR02] for the existence of global solutions. We suppose periodic boundary conditions for simplicity. The noise ξ is gaussian space-time noise, which has to be ideally space-time white, but we will have to specify regularity conditions later on. The model is known to exhibit an instability for large ν leading to the growth of hills (see e.g. [RLH00]).

This model was already proposed in [SP94] for the growth of crystalline surfaces. Another related model is the well-known Kuramoto-Shivashinski equation, where the term $\Delta |\nabla h|^2$ is replaced by $|\nabla h|^2$. This model originally describes the propagation of flames, but was recently also proposed to model surfaces obtained by ion-sputtering (see e.g. [CB95, FB+02]).

Our main example is the *Rayleigh-Bénard problem* (see Subsection 5.2) which is the paradigm of pattern formation in convection problems. It is described by the Navier-Stokes equation coupled to a heat equation (see e.g. [Ge98, W97]). For simplicity we consider the equation only in a strip. The full three dimensional problem is more technical, but the result is similar. For the set of equations see (45) - (47). One main problem arises due to the fact that the linearized operator L is not self-adjoint.

Let us finally comment on some limitations of our approach. The mean values of solutions or correlation functions can also be approximated by this method, but this will not be covered in this article. Nevertheless, the power of the presented approach is that it still works, if equation (1) exhibits blow-ups in finite time, and statistical quantities are no longer defined. This can also be the case in both our examples. For surface growth in dimension $d = 1, 2$ and Bénards problem for $d = 3$, we do not know uniqueness of global solutions. Therefore our result uses only the stability and global existence of unique solutions for the amplitude equation.

Our approach neglects all the other modes, apart from the dominating ones. Therefore it is limited to bounded domains. The main drawback is the following. Our proofs rely heavily on the existence of a spectral gap between the first non-zero eigenvalue and 0 (ω in Assumption 2.3). It is well-known in most examples that this gap shrinks to 0, if the domain size is large, hence leading to a blow up of constants. (See e.g. the proof of Theorem 2.13, where we use a constant $\int_0^\infty \tau^{-\alpha} e^{-\tau\omega} d\tau$.) Another example is the timescale of attractivity, which is $\frac{1}{\omega}\ln(\varepsilon^{-2})$.

Therefore the presented approach is limited to bounded domains only, and the validity of the theory is restricted to $\varepsilon \in (0, \varepsilon_0)$, where $\varepsilon_0 \to 0$ for domain-size to infinity. For unbounded (or just very large) domains, we would need modulating equations like in e.g. [KSM92, S93] or many other works by these authors. This will be the topic of further research.

The paper is organized as follows. Section 2 contains a rigorous formulation of our problem, the assumptions that are necessary, and the statement of the main results. Section 3 provides the proof of the attractivity result, and Section 4 the proof of the approximation. In Section 5 we present two applications. The final Section 6 establishes technical results used throughout the proofs of the main results.

2 Notation and Formulation of the Problem

Let X be some Banach space with norm $\|\cdot\|$. For the linear operator L in (1), we assume the following:

Assumption 2.1 (Linear Operator) *Suppose L is an unbounded linear operator on X. Denote the kernel (or nullspace) of L by $\mathcal{N}$, and suppose that $n := \dim(\mathcal{N}) < \infty$. Define by $e = (e_1, \ldots, e_n) \in X^n$ a basis of $\mathcal{N}$.*

Definition 2.2 (Projections) *We fix a projection onto $\mathcal{N}$ by P_c. One complementary projection to P_c is then given by $P_s := I - P_c$.*

As the dimension of $\mathcal{N}$ is finite, it is well-known that both P_c and P_s are bounded linear operators on X (cf. [W80]), i.e. $P_c, P_s \in \mathcal{L}(X)$. Later on we will restrict the choice of P_c by further assumptions.

The second assumption on L, which induces the separation of time-scales is the following:

Assumption 2.3 (Semigroup and the space Y) *The operator L from Assumption 2.1 generates an analytic semigroup $\{\mathrm{e}^{tL}\}_{t\geq 0}$ of linear operators on X such that there are constants $\omega > 0$ and $M \geq 1$ with*

$$\|\mathrm{e}^{tL} P_s x\| \leq M \mathrm{e}^{-t\omega} \|x\| \quad \text{for all } t \geq 0,\ x \in X. \tag{7}$$

Suppose there is a second Banach space Y, such that X is continuously imbedded into Y. Assume e^{tL} can be extended to a semigroup on Y, and for some $\alpha \in [0,1)$ we have

$$\|P_s \mathrm{e}^{tL} y\| \leq M(1 + t^{-\alpha})\mathrm{e}^{-t\omega} \|y\|_Y \quad \text{for all } t > 0,\ y \in Y. \tag{8}$$

Moreover suppose that P_c and hence P_s commutes with e^{tL}.

As e^{tL} is also a semigroup on Y we can also extend L_s^{-1} to a continuous operator from $P_s Y$ to $P_s X$, e.g. by writing $L_s^{-1} = \int_0^\infty \mathrm{e}^{\tau L} P_s d\tau$.

As $\mathcal{N} \subset X \subset Y$, we can extend P_c and hence P_s to projections in the space Y, which are both still continuous linear operators (i.e., $P_c,\ P_s \in \mathcal{L}(Y, Y)$).

As $L \equiv 0$ on $\mathcal{N}$ it is easy to verify that $\mathrm{e}^{tL} = Id$ on $\mathcal{N}$ for all $t \geq 0$. Therefore, we can assume without loss of generality that M is large enough such that

$$\|\mathrm{e}^{tL} x\| \leq M\|x\| \quad \text{for all } t \geq 0,\ x \in X.$$

Moreover, as $\mathcal{N}$ is finite dimensional the norms $\|\cdot\|$ and $\|\cdot\|_Y$ are equivalent on $\mathcal{N}$. Hence, we can also assume that M is sufficiently large such that

$$\|\mathrm{e}^{tL} P_c y\| \leq M\|P_c y\|_Y \quad \text{for all } t \geq 0, y \in Y.$$

For the stochastic perturbation let the following assumption be true. For a detailed discussion of Q-Wiener processes and stochastic convolutions see for white noise [dPZ92] and for fractional noise [DPM02].

Assumption 2.4 (Noise) *Suppose that the fractional noise process ξ is the generalized derivative of some fractional Q-Wiener process $\{W(t)\}_{t\geq 0}$ with Hurst parameter $H \in [\frac{1}{2}, 1)$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that the stochastic convolution*

$$W_L(t) = \int_0^t \mathrm{e}^{(t-\tau)L} dW(\tau) \tag{9}$$

is a well-defined stochastic process with continuous paths in X.

To be more precise, we assume that there is a basis of X consisting of eigenfunctions f_k of Q with $Qf_k = \alpha_k^2 f_k$, and $W(t) = \sum_{k=1}^\infty \alpha_k \beta_k^H(t) f_k$, where β_k^H are i.i.d. real (fractional) Brownian motions with Hurst parameter $H \geq \frac{1}{2}$.

We say that the noise (or W) is of trace-class, if $\operatorname{tr}(Q) < \infty$.

The case with $H = \frac{1}{2}$ corresponds to white noise, and we will obtain a lot of technical difficulties for that case. Therefore we will discuss this case usually separately. Some results will also apply to $H < \frac{1}{2}$, but we will not focus on that.

Note that due to scaling properties $\varepsilon^{2H} W(\varepsilon^{-2} t)$ is a version of the fractional Wiener process W for all $\varepsilon > 0$.

As the projections commute with the semigroup, it is straightforward to verify that

$$P_s[W_L(t)] = \int_0^t \mathrm{e}^{(t-\tau)L} dP_s W(\tau) \quad \text{and} \quad P_c[W_L(t)] = P_c W(t).$$

Assumption 2.5 (Bilinear Operator B**)** *Suppose $B : X \times X \to Y$ is a bilinear and continuous mapping which is symmetric, i.e. $B(u,v) = B(v,u)$ and there is a constant $C_B > 0$ such that $\|B(u,v)\|_Y \leq C_B \|u\| \|v\|$.*

Denote $B(u) = B(u,u)$, $P_s B(u,v) = B_s(u,v)$, and $P_c B(u,v) = B_c(u,v)$.

The key assumption on B is $B_c \equiv 0$ on $P_c X$, which means $B_c(P_c \cdot, P_c \cdot) = 0$. This was already indicated in (3) in the formal calculation.

Assumption 2.6 (Linear Operator A**)** *Suppose we have a continuous linear operator $A : X \to Y$, i.e. there is a constant $C_A > 0$ such that $\|Au\|_Y \leq C_A \|v\|$.*

Define $A_s = P_s A$ and $A_c = P_c A$, which are both still bounded linear operators from X to Y.

It is possible to consider a different Y here that also fulfills the conditions in Assumption 2.3. It does not make any difference for the proof.

To give a meaning to (1) we will always consider mild solutions. The following Assumption is mainly for convenience. Under the previous Assumptions it is usually easy to verify it.

Assumption 2.7 (Mild Solutions) *We assume that for all (stochastic) initial conditions $u_0 \in X$ equation (1) has a mild local solution u. This* *means we have a stopping time $t^* > 0$ and a stochastic process u such that $u : [0, t^*] \to X$ is $\mathbb{P}$-a.s. a solution of*

$$u(t) = \mathrm{e}^{tL}u_0 + \int_0^t \mathrm{e}^{(t-\tau)L}\Big[\varepsilon^2 Au(\tau) + B(u(\tau))\Big]d\tau + \varepsilon^{2H+1}W_L(t) \qquad \text{for } t \leq t^*. \tag{10}$$

Moreover, either $t^* = \infty$ or $\|u(t)\| \to \infty$ for $t \to t^*$.

The proof of existence of unique local solutions is standard under our assumptions. See e.g. [dPZ92] for a textbook. For L^p-theory with application to Navier-Stokes eq. see e.g. [BP99, BP00]. Moreover there are results on Kuramoto-Shivashinsky (see [DJ01]) and the surface growth equation from (6) (see e.g. [BG03]). Note that most of the above cited articles do not use fractional noise, but as everything is done pathwise, the results usually carry over to our case. Only regularity of the stochastic convolution W_L is needed.

We can split the variation of constants formula (10) into two parts:

$$P_s u(t) = \mathrm{e}^{tL}P_s u_0 + \int_0^t \mathrm{e}^{(t-\tau)L}\Big[\varepsilon^2 A_s u(\tau) + B_s(u(\tau))\Big]d\tau + \varepsilon^{2H+1}\int_0^t \mathrm{e}^{(t-\tau)L}dP_s W(t) \tag{11}$$

and

$$P_c u(t) = P_c u_0 + \int_0^t \Big[\varepsilon^2 A_c u(\tau) + B_c(u(\tau))\Big]d\tau + \varepsilon^{2H+1}P_c W(t). \tag{12}$$

Definition 2.8 *We call $u_s(t) = P_s u(t)$* fast modes*, as they are subject to a deterministic exponential decay on a time-scale of order $\mathcal{O}(1)$. Moreover $u_c(t) = P_c u(t)$ will be the* slow modes.

2.1 The amplitude equation

The amplitude equation is a SODE (or a system of SODEs) that describes the essential dynamics of mild solutions of (1) near 0. It was formally derived in (5). Here we state the rigorous formulation. First we introduce some notation.

Definition 2.9 *For $a \in \mathbb{R}^n$ denote $a \cdot e = \sum_{k=1}^n a_k e_k$. Define the canonical projection $\Pi : X \to \mathbb{R}^n$ by $\Pi(a \cdot e + z) = a$ for all $a \in \mathbb{R}^n$ and $z \in \mathrm{kernel}(\mathrm{P_c})$.*

As the spaces $\mathcal{N}$ and $\mathbb{R}^n$ are finite dimensional, we easily obtain that Π is continuous, i.e., there is a constant $C_\pi > 0$ such that $|\Pi(x)| \leq C_\pi \|x\|$ for all $x \in X$, where $|\cdot|$ denotes the standard euclidean norm on $\mathbb{R}^n$.

We discussed after Assumption 2.3 that $L_s := P_s L$ is invertible on $P_s X$, as $N(L_s) = \{0\}$. Hence, the mapping $L_s^{-1} B_s(\cdot) : P_c X \to P_s X$ is well-defined.

Definition 2.10 *Define the cubic nonlinearity* $\Gamma : \mathbb{R}^n \to \mathbb{R}^n$ *by*

$$\begin{aligned} \Gamma[a] &= -2\Pi\{B_c\left(a \cdot e, L_s^{-1} B_s(a \cdot e)\right)\} \\ &= -2 \sum_{i,j,k=1}^{n} a_i a_j a_k \Pi B_c\left(e_i, L_s^{-1} B_s(e_j, e_k)\right)\} \end{aligned} \tag{13}$$

and the linearity $\nu : \mathbb{R}^n \to \mathbb{R}^n$

$$\nu(a) = \Pi\{A_c(a \cdot e)\} = \sum_{i=1}^{n} a_i \Pi\{A_c(e_i)\}. \tag{14}$$

Definition 2.11 *Define the* amplitude equation *by*

$$a(T) = a(0) + \int_0^T \nu(a(s))ds + \int_0^T \Gamma[a(s)]ds + \beta(T) \tag{15}$$

where $\{\beta(T)\}_{T\geq 0}$ *is a fractional Wiener process in* $\mathbb{R}^n$ *given by* $\beta(T) = \varepsilon^{2H}\Pi(W(\varepsilon^{-2}T))$.

Remark 2.12 *The distribution of* β *is actually independent of* ε *due to the scaling properties of a fractional Wiener process (see e.g. [DÜ99]) Hence, the distribution of solutions of (15) is independent of* ε.

2.2 Main results

Our main results are the attractivity (see Theorem 2.13) and the approximation (see Theorem 2.18). The main ingredient for proving the attractivity is the existence of a spectral gap ω (see Assumption 2.3) leading to an exponential decay in $P_s X$.

Theorem 2.13 (Attractivity) *Suppose all assumptions of Section 2 are true. Fix the time* $t_\varepsilon = \frac{1}{\omega}\ln(\varepsilon^{-2})$ *with* ω *from (7). We can write the mild solution of (1) as*

$$u(t_\varepsilon) = \varepsilon a_\varepsilon \cdot e + \varepsilon^2 R_\varepsilon$$

with $a_\varepsilon \in \mathbb{R}^n$ *and* $R_\varepsilon \in P_sX$ *such that for sufficiently small* $\varepsilon \in (0,1]$

$$\Big\{ \|u_0\| \leq \delta\varepsilon, \quad \sup_{t\in[0,t_\varepsilon]} \|W_L(t)\| \leq \varepsilon^{-1}, \quad \|P_s W_L(t_\varepsilon)\| \leq C_w \Big\} \tag{16}$$

$$\Rightarrow \quad \Big\{ |a_\varepsilon|_{\mathbb{R}^n} \leq C_\pi(M\delta + 2), \quad \|R_\varepsilon\| \leq K+1 \Big\}, \tag{17}$$

where $K = M(1 + C_B D)D \int_0^\infty (1+\tau^{-\alpha})\mathrm{e}^{-\tau\omega} d\tau + C_w$.

Note that the constant K and the time t_ε defined in the previous theorem both blow up for $\omega \to 0$.

Remark 2.14 *For a result stating that (17) is true with high probability, we have to bound the probability of the event in (16) from below.*

The first condition depends only on the initial condition. We just have to suppose small initial conditions to get a nice bound. Next $\mathbb{P}(\|P_s W_L(t_\varepsilon)\| > C_w)$ *is arbitrarily small for large* C_w *independent of* ε*, as the law of* $\|P_s W_L(t_\varepsilon)\|$ *converges to a unique measure for* $t_\varepsilon \to \infty$ *(see [DPM02, Prop. 3.4]).*

There are no bounds on $\mathbb{P}(\sup_{t\in[0,t_\varepsilon]} \|W_L(t)\| > \varepsilon^{-1})$ *for* $H > \frac{1}{2}$ *available in the mathematics literature, but it is straightforward to use the factorization method similar to [DPM02, Prop. 3.2] to establish an analog to the well-known case* $H = \frac{1}{2}$*.*

Define now the approximation $\varepsilon\psi$ by

$$\varepsilon\psi(t) := \varepsilon \underbrace{a(\varepsilon^2 t) \cdot e}_{=:\psi_c(t)} + \varepsilon^2 \psi_s(t) \tag{18}$$

where a is a solution of the amplitude equation (15) representing the slow evolution on the slow modes, and ψ_s is the correction on the fast modes satisfying

$$\psi_s(t) = \mathrm{e}^{tL}\psi_s(0) + \varepsilon^{2H-1} P_s W_L(t) + \int_0^t \mathrm{e}^{(t-\tau)L} B(\psi_c(\tau)) d\tau. \tag{19}$$

Note that $B(\psi_c) \in P_sX$ by Assumption 2.5. Hence no projection is needed. As discussed during the derivation of the formal calculation in the introduction, the term ψ_s is a higher order correction, in order to deal with the coupling of modes in $\mathcal{N}$ to modes in P_sX due to the nonlinearity.

The residual of $\varepsilon\psi$ is given by

$$\mathrm{Res}(\varepsilon\psi)(t) = -\varepsilon\psi(t)+\mathrm{e}^{tL}\varepsilon\psi(0)+\int_0^t \mathrm{e}^{(t-\tau)L}[\varepsilon^2 A\varepsilon\psi(\tau)+B(\varepsilon\psi(\tau))]d\tau+\varepsilon^{2H+1}W_L(t). \tag{20}$$

In order to show that $\varepsilon\psi$ is a good approximation of a solution u of (10), we have to control the residual. This basically involves showing that the formal calculation in the introduction can be done rigorously. We do this in two steps. First we will discuss $P_c\mathrm{Res}(\varepsilon\psi)$, which involves the use of (15). The second step uses (19) and bounds $P_s\mathrm{Res}(\varepsilon\psi)$. It is crucial for our result, that the latter bound is much better than the first one.

Fix some $1 \gg \kappa \geq 0$. For $H = \frac{1}{2}$ we will need $\kappa > 0$, but for $H > \frac{1}{2}$, we can simply choose $\kappa = 0$. Given C_w, δ, $C_a > 0$, and a time $T_0 > 0$ define

$$\mathcal{B}_\varepsilon = \mathcal{B}_\varepsilon(C_w,\delta,C_a,T_0) := \Big\{ \sup_{t\in[0,T_0\varepsilon^{-2}]} \|P_sW_L(t)\| \leq C_w\varepsilon^{1-2H-\kappa},$$
$$\int_0^{T_0} \|P_sW_L(t\varepsilon^{-2})\|dt \leq C_w, \quad \|\psi_s(0)\| \leq \delta, \quad \sup_{T\in[0,T_0]} |a(T)| \leq C_a\Big\}. \tag{21}$$

Remark 2.15 *Let us now briefly comment, why we expect the probability of* $\mathcal{B}_\varepsilon$ *to be large. On* $\|\psi_s(0)\| \leq \delta$ *we will comment after Theorem 2.18.*

The term $\sup_{t\in[0,T_0\varepsilon^{-2}]} \|P_sW_L(t)\|$ *was already discussed in Remark 2.14. Note that for* $H = \frac{1}{2}$ *and* $\kappa = 0$ *we have* $\mathbb{P}(\sup_{t\in[0,T_0\varepsilon^{-2}]} \|P_sW_L(t)\| \leq C_w) \to 0$. *Hence,* $\kappa > 0$ *is essential for* $\mathbb{P}(\mathcal{B}_\varepsilon)$ *to be large.*

A very simple large deviation result for $\|a\|_{L^\infty([0,T_0])}$ *was already verified in [B03] for white noise. Nevertheless, that result did not use* $H = \frac{1}{2}$, *and carries immediately over to amplitude equations driven by fractional Brownian motion, using large deviation results for the latter.*

The probability of the integral being bounded is high in C_w/T_0. *This follows easily from the fact that the distribution of* $P_sW_L(t)$ *converges, as* $t \to \infty$.

For $0 < q < H$ fixed later define furthermore the following sets, which are additional bounds used only for $P_c\mathrm{Res}(\varepsilon\psi)$.

$$\mathcal{D}_\varepsilon := \Big\{\|a\|_{C^q([0,T_0])} \leq C_a\Big\} \tag{22}$$

$$\mathcal{D}_\varepsilon^{\mathrm{tr}} := \Big\{\|a\|_{C^q([0,T_0])} \leq C_a, \quad \|P_sW(\varepsilon^{-2}\cdot)\|_{C^q([0,T_0],X)} \leq C_w\varepsilon^{-2H}\Big\} \tag{23}$$

Here $C^q([0,T_0])$ denotes the space of Hölder-continuos functions from $[0,T_0]$ to either $\mathbb{R}^n$ or $\mathbb{R}$ with Hölder exponent q.

Remark 2.16 *Large deviation results for $\|a\|_{C^q([0,T_0])}$ are much more involved than the ones in $L^\infty([0,T_0])$, but it should be straightforward to carry them over from analogous results for the fractional Brownian motion. Nevertheless, it is well known that due to the regularity of a and β we can choose $q \in (\frac{1}{2}, H)$ for fractional noise with $H > \frac{1}{2}$, but for white noise we need $q < \frac{1}{2}$. Nevertheless, in this case we can choose q arbitrarily close to $\frac{1}{2}$*

Note that we have $\|W(\varepsilon^{-2}\cdot)\|_{C^q([0,T],X)} \overset{d)}{=} \varepsilon^{-2H}\|W\|_{C^q([0,T],X)}$ in law, due to the scaling invariance of the noise. Hence, the bound in $\mathcal{D}_\varepsilon^{\mathrm{tr}}$ is natural, and is fulfilled with high probability, if C_w is large.

Hence, $\mathbb{P}(\mathcal{D}_\varepsilon)$ and $\mathbb{P}(\mathcal{D}_\varepsilon^{\mathrm{tr}})$ are near one, provided, we choose the right q and make the constants C_a and C_w large.

Theorem 2.17 (Residual) *Suppose all assumptions of Section 2 are true. For constants $\delta, C_a, C_w > 0$, $q \in (0,H)$, some small $1 \gg \kappa \geq 0$, and some time $T_0 > 0$ consider the sets $\mathcal{B}_\varepsilon$ defined in (21) and $\tilde{\mathcal{D}}_\varepsilon = \mathcal{D}_\varepsilon$ or $\tilde{\mathcal{D}}_\varepsilon = \mathcal{D}_\varepsilon^{\mathrm{tr}}$ for trace-class noise, where the sets are defined in (22) and (23), respectively.*

Then there exist positive constants $C_{\mathrm{res}}^{(c)}$ and $C_{\mathrm{res}}^{(s)}$ such that for sufficiently small $\varepsilon > 0$ we obtain for all solutions a of (15)

$$\mathcal{B}_\varepsilon \cap \tilde{\mathcal{D}}_\varepsilon \;\Rightarrow\; \Big\{ \sup_{t\in[0,T_0\varepsilon^{-2}]} \|P_c \mathrm{Res}(\varepsilon\psi(t))\| \leq C_{\mathrm{res}}^{(c)}(\varepsilon^{\gamma^*} + \varepsilon^{2-2\kappa}), \quad \sup_{t\in[0,T_0\varepsilon^{-2}]} \|P_s \mathrm{Res}(\varepsilon\psi(t))\| \leq C_{\mathrm{res}}^{(s)}\varepsilon^{3-\kappa}\Big\}.$$

For $q > \frac{1}{2}$ in $\tilde{\mathcal{D}}_\varepsilon$, the exponent for the general case is $\gamma^* = 2H > 1$, and $\gamma^* = \max\{3-2H, 2H\} \geq \frac{3}{2}$ for trace-class noise. For $q < \frac{1}{2}$ and trace-class noise we have $\gamma^* = 2 - \eta(q)$, where $\eta(q) \to 0$ for $q \to \frac{1}{2}$. We have $\eta(q) \sim \frac{2}{3}\sqrt{\frac{1}{2} - q}$ for $q \to \frac{1}{2}$.

For $q < \frac{1}{2}$, we could also use the general case, but a bound of order $\mathcal{O}(\varepsilon)$ is not sufficient for proving the approximation result. Nevertheless, it is possible to improve γ^* for noise with $\mathrm{tr}(Q) = \infty$ when we have certain bounds on the eigenvalues of Q. This is much more involved, as we need more knowledge of the space, in which W is defined (e.g. some fractional power spaces of L), and we do not treat this here.

Theorem 2.18 (Approximation) *Suppose all assumptions of Section 2 are true. For constants $\delta > 0$, $C_a, C_w > 0$, $q \in (0, H)$, some small $1 \gg \kappa \geq 0$, and some time $T_0 > 0$ consider the sets $\mathcal{B}_\varepsilon$ defined in (21) and $\tilde{\mathcal{D}}_\varepsilon = \mathcal{D}_\varepsilon$ or $\tilde{\mathcal{D}}_\varepsilon = \mathcal{D}_\varepsilon^{\mathrm{tr}}$ for trace-class noise, where the sets are defined in (22) and (23), respectively.*

Then there is a constant $C_{\mathrm{att}} > 0$ such that for sufficiently small $\varepsilon > 0$ we obtain for all solutions u of (10) and all solutions a of (15)

$$\mathcal{B}_\varepsilon \cap \tilde{\mathcal{D}}_\varepsilon \cap \left\{ \|P_c(u_0 - \varepsilon\psi(0))\| \leq \delta\varepsilon^2, \quad \|P_s(u_0 - \varepsilon\psi(0))\| \leq \delta\varepsilon^3 \right\}$$

$$\Rightarrow \sup_{t \in [0, T_0\varepsilon^{-2}]} \|u(t) - \varepsilon\psi(t)\| \leq C_{\mathrm{att}}(\varepsilon^{\gamma^*} + \varepsilon^{2-2\kappa}),$$

where the constant $\gamma^*(H)$ was defined in Theorem 2.17.

Remark 2.19 *The conditions $\|P_c(u_0 - \varepsilon\psi(0))\| \leq \delta\varepsilon^2$, $\|P_s(u_0 - \varepsilon\psi(0))\| \leq \delta\varepsilon^3$, and $\|\psi_s(0))\| \leq \delta\varepsilon^2$ from $\mathcal{B}_\varepsilon$ are easily assured if we suppose initial conditions which are given by the attractivity result and define $\varepsilon^2\psi_s(0) = P_s u_0$ and $\varepsilon\psi_c(0) = \varepsilon a(0)\cdot e = P_c u_0$. Moreover, we have to choose $\delta > 0$ in Theorem 2.18 larger than the two bounds in (17)*

Note that we have to be careful, when doing a timeshift, as our processes are for $H > \frac{1}{2}$ not Markovian. Nevertheless, we can apply the attractivity on $[0, t_\varepsilon]$ and the approximation on $[t_\varepsilon, t_\varepsilon + T_0\varepsilon^{-2}]$ Combining both results a sketch of the typical dynamics is given in Figure 1.

Let us finally remark that we can give estimates for the stopping time t^* from Assumption 2.7, as $t^* \geq T_e\varepsilon^{-2}$ with high probability. This is especially important, as the solutions of our examples do not exhibit unique global solutions. Here we obtain, that the unique local solution will exist for a very long time with very high probability.

The $\mathcal{O}$-notation is used in the following way. A term $G_\varepsilon = \mathcal{O}(g_\varepsilon)$ if and only if there are positive constants ε_0 and C depending only on other constants such that $|G_\varepsilon| \leq Cg_\varepsilon$ for all $\varepsilon \in (0, \varepsilon_0]$.

3 The Attractivity

This section follows basically the proof of analogous results in [B03]. First we establish a bound on mild solutions u of (10). Using straightforward

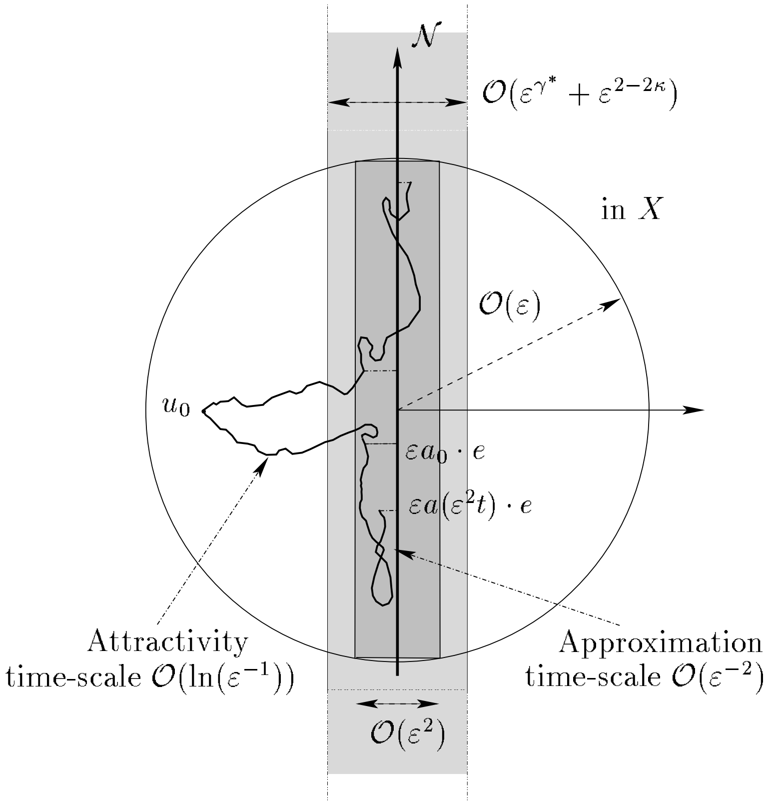

Figure 1: Two typical trajectories of mild solutions of (1)

estimates, we show in Lemma 3.1 that solutions with initial conditions of order $\mathcal{O}(\varepsilon)$ stay of order $\mathcal{O}(\varepsilon)$ on a large time-scale of order $\mathcal{O}(\varepsilon^{-1})$. It is not trivial to extend this result to time-scales of order $\mathcal{O}(\varepsilon^{-2})$. We need the approximation result for this.

Lemma 3.1 *Suppose all assumptions of Section 2 are true. For all times* $t_\varepsilon \leq \varepsilon^{-1}$ *and all constants* $\delta > 0$ *and* $C_w > 0$ *we obtain with* $D := M\delta + 2$ *and* $\varepsilon \in (0, 1]$ *sufficiently small that*

$$\Big\{ \sup_{t\in[0,t_\varepsilon]} \|W_L(t)\| \leq \varepsilon^{-1}, \quad \|u_0\| \leq \delta\varepsilon \Big\} \Rightarrow \sup_{t\in[0,t_\varepsilon]} \|u(t)\| \leq D\varepsilon. \tag{24}$$

Proof: By Assumptions 2.5 and 2.6 we easily show

$$\|B(v) + \varepsilon^2 A(v)\|_Y \leq \varepsilon^2 C_A \|v\| + C_B \|v\|^2. \tag{25}$$

Define the exit time $\tau_\varepsilon^* := \inf\{\tau > 0 \ : \ \|u(\tau)\| > D\varepsilon\}$. Hence, as long as $\tau < \tau_\varepsilon^*$ we obtain

$$\|B(u(\tau)) + \varepsilon^2 A(u(\tau))\|_Y \leq \varepsilon^2 (C_A \varepsilon + C_B D) D. \tag{26}$$

Now we derive from (10) for all $t \leq \min\{t_\varepsilon, \tau_\varepsilon^*\}$

$$\begin{aligned}\|u(t)\| &\leq M\|u_0\| + M\int_0^t (1+(t-\tau)^{-\alpha})\|B(v)+\varepsilon^2 A(v)\|_Y d\tau + \varepsilon^{2H+1}\|W_L(t)\| \\ &\leq [M\delta + \varepsilon^{2H}]\varepsilon + M\varepsilon^2(C_A\varepsilon + C_B D)D\int_0^{t_\varepsilon}(1+\tau^{-\alpha})d\tau \\ &\leq [M\delta+1]\varepsilon + M(C_A\varepsilon + C_B D)D\frac{2-\alpha}{1-\alpha}\cdot\varepsilon^2 \\ &< D\varepsilon\end{aligned}$$

for $\varepsilon > 0$ sufficiently small. This yields immediately $\tau_\varepsilon^* \geq t_\varepsilon$ on the set on interest, which finishes the proof. □

Proof: (Theorem 2.13) Define $\varepsilon a_\varepsilon = \Pi(u(t_\varepsilon))$ and $\varepsilon^2 R_\varepsilon = P_s u(t_\varepsilon)$. By Lemma 3.1 all we need to show is a bound on $P_s u$, as $|\varepsilon a_\varepsilon| = |\Pi(u(t_\varepsilon))| \leq C_\pi D\varepsilon$ with C_π from Definition 2.9.

Using (11) and then (7) we obtain

$$\begin{aligned}\|P_s u(t_\varepsilon)\| &\leq M\mathrm{e}^{-\omega t_\varepsilon}\|u_0\| + \varepsilon^{2H+1}\|P_s W_L(t_\varepsilon)\| \\ &\quad + M\int_0^{t_\varepsilon}(1+(t_\varepsilon-\tau)^{-\alpha})\mathrm{e}^{-(t_\varepsilon-\tau)\omega}\|\varepsilon^2 A(u(\tau)) + B(u(\tau))\|_Y d\tau.\end{aligned}$$

As $\tau \leq t_\varepsilon \leq \varepsilon^{-1}$ and $\|u(\tau)\| \leq D\varepsilon$ (by Lemma 3.1) we use (26) to finally end up with

$$\|P_s u(t_\varepsilon)\| \leq M\delta\varepsilon^3 + M(C_A\varepsilon + C_B D)\varepsilon^2 D\int_0^\infty (1+\tau^{-\alpha})\mathrm{e}^{-\tau\omega}d\tau + C_w\varepsilon^{2H+1}.$$

This implies the result. □

4 Residual and Approximation

On the set $\mathcal{B}_\varepsilon$ from (21), we readily obtain bounds for ψ_s and ψ_c from (18) The proof of the following lemma is straightforward.

Lemma 4.1 *Suppose all assertions of Section 2 are true. Then for all constants C_w, δ, $C_a > 0$, $1 \gg \kappa \geq 0$, and all times T_0, there are constants C_c and C_s such that for sufficiently small $\varepsilon > 0$*

$$\mathcal{B}_\varepsilon \Rightarrow \Big\{ \sup_{t\in[0,\frac{T_0}{\varepsilon^2}]} \|\psi_s(t)\| \leq C_s\varepsilon^{-\kappa}, \quad \sup_{t\in[0,\frac{T_0}{\varepsilon^2}]} \|\psi_c(t)\| \leq C_c, \quad \int_0^{\frac{T_0}{\varepsilon^2}} \|\psi_s(t)\| \leq C_s \Big\}.$$

Let us now turn to the residual from (20) of our approximation $\varepsilon\psi$. Split $\mathrm{Res}(\varepsilon\psi)(t) = P_c\mathrm{Res}(\varepsilon\psi)(t) + P_s\mathrm{Res}(\varepsilon\psi)(t)$. Projecting (20) with P_s we obtain

$$\begin{aligned}
&P_s\mathrm{Res}(\varepsilon\psi)(t)\\
&= -\varepsilon^2\psi_s(t) + \mathrm{e}^{tL}\varepsilon^2\psi_s(0) + \varepsilon^{2H+1}P_sW_L(t)\\
&\quad + \int_0^t \mathrm{e}^{(t-\tau)L}\Big[\varepsilon^2 A_s\Big(\varepsilon\psi_c(\tau) + \varepsilon^2\psi_s(\tau)\Big) + B_s\Big(\varepsilon\psi_c(\tau) + \varepsilon^2\psi_s(\tau)\Big)\Big]d\tau\\
&= \varepsilon^2\Big[-\psi_s(t) + \mathrm{e}^{tL}\psi_s(0) + \varepsilon^{2H-1}P_sW_L(t) + \int_0^t \mathrm{e}^{(t-\tau)L}B_s(\psi_c(\tau))]d\tau\Big]\\
&\quad +2\varepsilon^3\int_0^t \mathrm{e}^{(t-\tau)L}B_s(\psi_c(\tau),\psi_s(\tau))d\tau + \varepsilon^3\int_0^t \mathrm{e}^{(t-\tau)L}A_s\psi_c(\tau)d\tau\\
&\quad +\varepsilon^4\int_0^t \mathrm{e}^{(t-\tau)L}\Big[A_s\psi_s(\tau) + B_s(\psi_s(\tau))\Big]d\tau.
\end{aligned}$$

Define

$$K_{\alpha,\omega} = \int_0^\infty (1+\tau^{-\alpha})\mathrm{e}^{-\omega\tau}d\tau. \tag{27}$$

Using (8), Assumptions 2.5 and 2.6, Lemma 4.1 and (19) to cancel terms of order ε^2, we readily obtain:

Lemma 4.2 *Under the assumptions of Lemma 4.1*

$$\mathcal{B}_\varepsilon \Rightarrow \sup_{t\in[0,T_0\varepsilon^{-2}]} \|P_s\mathrm{Res}(\varepsilon\psi(t))\| \le C_{\mathrm{res}}^{(s)}\varepsilon^{3-\kappa}$$

with $C_{\mathrm{res}}^{(s)} = K_{\alpha,\omega}M\|P_s\|_{\mathcal{L}(X)}[2C_sC_cC_B + 1 + C_AC_c]$ *provided* $\varepsilon(C_AC_s + C_BC_s^2\varepsilon^{-\kappa}) \le 1$.

For $P_c\mathrm{Res}(\varepsilon\psi(t))$ the bound is a little bit more involved. First we obtain from (20)

$$\begin{aligned}
P_c\mathrm{Res}(\varepsilon\psi)(t) &= -\varepsilon\psi_c(t) + \varepsilon\psi_c(0) + \varepsilon^{2H+1}P_cW(t)\\
&\quad + \int_0^t \Big[\varepsilon^2 A_c\Big(\varepsilon\psi_c(\tau) + \varepsilon^2\psi_s(\tau)\Big) + B_c\Big(\varepsilon\psi_c(\tau) + \varepsilon^2\psi_s(\tau)\Big)\Big]d\tau.
\end{aligned}$$

Using $B_c(\psi_c) = 0$ from Assumption 2.5 we derive

$$\begin{aligned}
&P_c\mathrm{Res}(\varepsilon\psi)(t)\\
&= \varepsilon\cdot\Big(\psi_c(0) - \psi_c(t) + \varepsilon^{2H}P_cW(t) + \varepsilon^2\int_0^t \Big[A_c(\psi_c(\tau) + 2B_c(\psi_c(\tau),\psi_s(\tau))\Big]d\tau\Big)\\
&\quad +\varepsilon^4\cdot\int_0^t \Big[B_c(\psi_s(\tau)) + A_c\psi_s(\tau)\Big]d\tau.
\end{aligned} \tag{28}$$

For $t \leq T_e\varepsilon^{-2}$ the last term in (28) is bounded by $\mathcal{O}(\varepsilon^{2-2\kappa})$ due to Lemma 4.1. Using the substitution $T = \varepsilon^2 t$ to the slow time-scale and (15) to cancel out most $\mathcal{O}(\varepsilon)$-terms, we derive

$$P_c\mathrm{Res}(\varepsilon\psi)(t) = 2\varepsilon\int_0^T \Big[B_c\Big(a(s)\cdot e, \psi_s(s\varepsilon^{-2})\Big) + B_c\Big(a(s)\cdot e, L^{-1}B_s(a(s)\cdot e)\Big)\Big]ds + \mathcal{O}(\varepsilon^{2-2\kappa})$$

Inserting the definition of ψ_s from (19) we obtain

$$\int_0^T B_c\Big(a(s)\cdot e, \Big[\psi_s(s\varepsilon^{-2}) + L^{-1}B_s(a(s)\cdot e)\Big]\Big)ds \tag{29}$$

$$= \int_0^T B_c\Big(a(s)\cdot e, \mathrm{e}^{s\varepsilon^{-2}L}\psi_s(0))\Big)ds \tag{30}$$

$$+ \int_0^T B_c\Big(a(s)\cdot e, \varepsilon^{2H-1}P_sW_L(s\varepsilon^{-2}))\Big)ds \tag{31}$$

$$+ \int_0^T B_c\Big(a(s)\cdot e, \int_0^{\frac{s}{\varepsilon^2}} \mathrm{e}^{(\frac{s}{\varepsilon^2}-\tau)L}B_s(\psi_c(\tau))d\tau + L^{-1}B_s(\psi_c(s))\Big)ds. \tag{32}$$

Now we have to bound the terms in (30) - (32) separately. The main formal idea is $\varepsilon^{-2}L\mathrm{e}^{t\varepsilon^{-2}L} \to \delta_0 I$ for $\varepsilon \to 0$, where δ_0 is the Delta-distribution in time. Unfortunately, we do not have enough regularity, to preform this limit in a simple way. For the current proof we postpone details to forthcoming Lemmas 4.4, 4.5, and 4.6.

First by Lemma 4.4 we obtain for $T \leq T_0$

$$\|(30)\| \leq C\varepsilon^2\|a\|_{L^\infty([0,T])}\|\psi_s(0)\| \leq C\varepsilon^2 C_a\delta = \mathcal{O}(\varepsilon^2),$$

where we used $\mathcal{B}_\varepsilon$.

For the third term we derive for $q > \frac{1}{2}$ by Lemma 4.6 using $\mathcal{D}_\varepsilon$ and $T \leq T_0$

$$\|(32)\| \leq C\varepsilon^2\|a\|_{C^q([0,T])} = \mathcal{O}(\varepsilon^2).$$

For $q < \frac{1}{2}$ there is, again by Lemma 4.6 some $\eta(q)$ with $\eta(q) \to 0$ for $q \to \frac{1}{2}$, such that

$$\|(32)\| \leq C\varepsilon^{2-\eta(q)}\|a\|_{C^q([0,T])} = \mathcal{O}(\varepsilon^{2-\eta(q)}),$$

where we can easily establish the asymptotics for η. For (31) we establish two bounds. The direct estimate gives

$$\|(31)\| \leq C\varepsilon^{2H-1}\|a\|_{L^\infty([0,T])}\int_0^T \|P_sW_L(s\varepsilon^{-2})\|ds = \mathcal{O}(\varepsilon^{2H-1}).$$

But for small H and trace-class noise, we get a better bound by Lemma 4.5. As long as $q > \frac{1}{2}$ and $T \le T_0$

$$\|(31)\| \le C\varepsilon^2 \|a\|_{C^q([0,T])} \|P_s W(\varepsilon^{-2}\cdot)\|_{C^q([0,T],X)} = \mathcal{O}(\varepsilon^{2-2H}).$$

Moreover, for $q < \frac{1}{2}$ and $T \le T_0$ we derive again by Lemma 4.5

$$\|(31)\| \le C\varepsilon^{2-\eta(q)} \|a\|_{C^q([0,T])} \|P_s W(\varepsilon^{-2}\cdot)\|_{C^q([0,T],X)} = \mathcal{O}(\varepsilon^{2-2H-\eta(q)}),$$

where $\eta(q)$ as above.

Hence, we can bound (29) for $q > \frac{1}{2}$ by $\mathcal{O}(\varepsilon^{2H-1})$. If additionally $\mathrm{tr}(Q) < \infty$, then we derive the bound $\mathcal{O}(\varepsilon^{2-2H})$. For $q < \frac{1}{2}$ we get roughly the same result, only our exponent is slightly smaller by some $\eta(q)$.

We finally proved the following Lemma 4.3.

Lemma 4.3 *Under the assumptions of Theorem 2.17*

$$\mathcal{B}_\varepsilon \cap \tilde{\mathcal{D}}_\varepsilon \Rightarrow \sup_{t\in[0,T_0\varepsilon^{-2}]} \|P_c \mathrm{Res}(\varepsilon\psi(t))\| \le C_{\mathrm{res}}^{(c)}(\varepsilon^{\gamma^*} + \varepsilon^{2-2\kappa}) \tag{33}$$

where $C_{\mathrm{res}}^{(c)}$ is some sufficiently large constant. For $H \in (\frac{1}{2}, 1)$ the exponent is either $\gamma^* = 2H > 1$ for the general case or $\gamma^* = \max\{3-2H, 2H\} \ge \frac{3}{2}$ for trace-class noise. For $H = \frac{1}{2}$ and $\mathrm{tr}(Q) < \infty$ we have $\gamma^* = 2 - \eta(q)$, where $\eta(q) \to 0$ for $q \to \frac{1}{2}$.

Proof: (Theorem 2.17) Combining the previous Lemma 4.3 with Lemma 4.2 immediately yields the result. □

Before proving Theorem 2.18, we establish some lemmas used in the proof of Lemma 4.3.

Lemma 4.4 *Let $a : [0,T] \to \mathbb{R}^n$ be a measurable function such that $\|a\|_{L^\infty} := \sup_{t\in[0,T]} |a(t)| < \infty$. Let Assumptions 2.3 and 2.5 be true. Furthermore fix $e = (e_1, \dots, e_n) \in X^n$ and define $\|e\|^2 := \sum_{i=1}^n \|e_i\|^2$. Then for all $w \in X$*

$$\| \int_0^T B_c\Big(a(\tau)\cdot e, \mathrm{e}^{\tau\varepsilon^{-2}L} w)\Big) d\tau \| \le \varepsilon^2 M C_B \int_0^\infty \mathrm{e}^{-s\omega} ds \|a\|_{L^\infty} \|e\| \|w\|.$$

Proof: This is a straightforward estimate. Note that the left hand side is obviously bounded by $MC_B \int_0^T \|a(\tau)\cdot e\| \mathrm{e}^{-\tau\varepsilon^{-2}\omega} d\tau \|w\|$. Now Cauchy-Schwarz inequality and the substitution $\tau' = \tau\varepsilon^{-2}$ gives the claim. □

Lemma 4.5 *Suppose that L as in Assumption 2.1 generates an analytic semigroup, as in Assumption 2.3, and W is a fractional Wiener-process as in Assumption 2.4 with $H \geq \frac{1}{2}$ and $\mathrm{tr}(Q) < \infty$. Then for $q > \frac{1}{2}$ and all Hölder continuous $a : [0,T] \to \mathbb{R}^n$*

$$\Big\| \underbrace{\int_0^T B_c\Big(a(\tau)\cdot e, P_s W_L(\tau\varepsilon^{-2})\Big) d\tau}_{=:I_w} \Big\| \leq C\varepsilon^2 \|a\|_{C^q([0,T],\mathbb{R}^n)} \cdot \|P_s W(\varepsilon^{-2}\cdot)\|_{C^q([0,T],X)}$$

Moreover, for all $r > 1$ we obtain for some constant $C_r > 0$

$$\|I_w\| \leq C_r \varepsilon^{2-\frac{3r+1}{2r(r+1)}} \|a\|_{C^{\frac{1}{2}-\frac{1}{4r(r+2)}}([0,T],\mathbb{R}^n)} \|P_s W(\varepsilon^{-2}\cdot)\|_{C^{\frac{1}{2}-\frac{r+1}{4r(r+2)}}([0,T],X)}.$$

Proof: First we establish a straightforward estimate using Assumption 2.5 and the fractional integration by parts from Lemma 6.4.

$$\begin{aligned}
&\Big\| \int_0^T B_c\Big(a(\tau)\cdot e, P_s W_L(\tau\varepsilon^{-2})\Big) d\tau \Big\| \\
&= \Big\| \sum_{i=1}^n B_c\Big(e_i, \int_0^T a_i(\tau) P_s W_L(\tau\varepsilon^{-2}) d\tau\Big) \Big\| \\
&\leq C_B \|P_c\|_{\mathcal{L}(X)} \sum_{i=1}^n \|e_i\| \cdot \Big\| \int_0^T a_i(\tau) \frac{\partial}{\partial \tau} \int_0^\tau P_s W_L(\sigma\varepsilon^{-2}) d\sigma d\tau \Big\| \\
&\leq C \|a\|_{C^q([0,T])} \cdot \Big\| t \mapsto \int_0^t P_s W_L(\tau\varepsilon^{-2}) d\tau \Big\|_{C^q([0,T],X)} \qquad (34)
\end{aligned}$$

for $q > \frac{1}{2}$.

For trace-class noise, it is well-known that $W(t)$ exhibits Hölder continuous paths in X. Hence, we obtain using integration by parts

$$\begin{aligned}
&\int_0^t P_s W_L(\tau\varepsilon^{-2}) d\tau \\
&= \int_0^t \int_0^{\tau\varepsilon^{-2}} \mathrm{e}^{(\tau\varepsilon^{-2}-\sigma)L} dP_s W(\sigma) d\tau \\
&= \int_0^t P_s W(\tau\varepsilon^{-2}) d\tau + L_s \int_0^t \int_0^{\tau\varepsilon^{-2}} \mathrm{e}^{(\tau\varepsilon^{-2}-\sigma)L} P_s W(\sigma) d\sigma d\tau \\
&= \int_0^t P_s W(\tau\varepsilon^{-2}) d\tau + \varepsilon^{-2} L_s \int_0^t \int_0^\tau \mathrm{e}^{(\tau-\sigma)L\varepsilon^{-2}} P_s W(\sigma\varepsilon^{-2}) d\sigma d\tau \\
&= \int_0^t \mathrm{e}^{(t-\sigma)L\varepsilon^{-2}} P_s W(\sigma\varepsilon^{-2}) d\sigma, \qquad (35)
\end{aligned}$$

where we used Fubini in the last step. The previous equation (35) can also be seen by integrating the equation $d(P_sW_L) = L_sP_sW_Ldt + d(P_sW)$.

Now (53) of Lemma 6.2 with $h = P_sW(\varepsilon^{-2}\cdot)$ together with (34) and (35) gives the first result.

For the second result choose some small $\alpha > 0$ fixed later and fix some large $r > 0$. Then by applying Lemma 6.4 with $p_1 = \frac{1}{2} - \alpha/(r+1)$ and $p_2 = \frac{1}{2} + 2\alpha/(r+1)$ we change (34) to

$$\|I_w\| \le C\|a\|_{C^{\frac{1}{2}-\frac{\alpha}{r+1}}([0,T])} \cdot \Big\|t \mapsto \int_0^t P_sW_L(\tau\varepsilon^{-2})d\tau\Big\|_{C^{\frac{1}{2}+\frac{2\alpha}{r+1}}([0,T],X)}$$

Now (35) together with Corollary 6.3 implies

$$\|I_w\| \le C\varepsilon^{2-\frac{1}{r+1}-2\alpha\frac{r+2}{r+1}} \cdot \|a\|_{C^{\frac{1}{2}-\frac{\alpha}{r+1}}([0,T])} \cdot \|P_sW(\varepsilon^{-2}\cdot)\|_{C^{\frac{1}{2}-\alpha}([0,T],X)}$$

Choosing $\alpha = \frac{r+1}{4r(r+2)}$ gives the assertion. An easy calculation shows that the condition $r > 1$ is obviously necessary to apply Corollary 6.3. $\square$

Lemma 4.6 *Let Assumptions 2.1, 2.3 and 2.5 be true. For* $a \in C^q([0,T],\mathbb{R}^n)$ *define*

$$J := \int_0^T B_c\Big(a(t)\cdot e, \int_0^{t\varepsilon^{-2}} \mathrm{e}^{(t\varepsilon^{-2}-\tau)L}B_s(a(\varepsilon^2\tau)\cdot e)d\tau + L_s^{-1}B_s(a(t)\cdot e)\Big)dt.$$

Then for $q > \frac{1}{2}$ *we obtain*

$$\|J\| \le C\varepsilon^2\|a\|^3_{C^q([0,T])}.$$

Moreover, for $r > 1$ *and* $q = \frac{1}{2} - 1/(4r(r+2))$ *we derive*

$$\|J\| \le C\varepsilon^{2-\frac{3r+1}{2r(r+1)}}\|a\|^3_{C^{\frac{1}{2}-\frac{1}{4r(r+2)}}([0,T])}.$$

Proof: Inserting $a\cdot e = \sum_{j=1}^n a_je_j$ we obtain

$$\begin{aligned} J &= \int_0^T B_c\Big(a(t)\cdot e, \varepsilon^{-2}\int_0^t \mathrm{e}^{(t-\tau)L\varepsilon^{-2}}B(a(\tau)\cdot e)d\tau + L_s^{-1}B_s(a(t)\cdot e)\Big)dt \\ &= \sum_{i,j,k=1}^n B_c\Big(e_i, L_s^{-1}\int_0^T a_i(t)\Big[\varepsilon^{-2}L_s\int_0^t \mathrm{e}^{(t-\tau)L\varepsilon^{-2}}B_s(e_j,e_k)a_j(\tau)a_k(\tau)d\tau \\ &\qquad\qquad + B_s(e_j,e_k)a_j(t)a_k(t)\Big]dt\Big) \end{aligned}$$

Hence,

$$
\begin{aligned}
\|J\| &\leq C \sum_{i,j,k=1}^{n} \Big\| \int_0^T a_i(s) L_s^{-1} \Big[\varepsilon^{-2} L \int_0^s \mathrm{e}^{(s-\tau)L\varepsilon^{-2}} B_s(e_j, e_k) a_j(\tau) a_k(\tau) d\tau \\
&\qquad\qquad + B_s(e_j, e_k) a_j(s) a_k(s) \Big] ds \Big\| \\
&\leq C\varepsilon^2 \sum_{i,j,k=1}^{n} \|a_i\|_{C^q([0,T])} \|a_j\|_{C^p([0,T])} \|a_k\|_{C^p([0,T])}
\end{aligned}
$$

by Lemma 6.5, where $p \in (1-q, 1)$ arbitrary. The main idea of Lemma 6.5 is to rewrite the sum of the two terms as a derivative of a convolution integral, and use fractional integration by parts. Note that constants depend on p and q. Choosing $p \in (1-q, q]$, we derive

$$\|J\| \leq C\varepsilon^2 \|a\|^3_{C^q([0,T])}.$$

Moreover the second result of Lemma 6.5, immediately gives the second assertion of this lemma. □

Proof: (Theorem 2.18)
Define

$$u(t) - \varepsilon\psi(t) = \varepsilon^2 R = \varepsilon^2 R_c + \varepsilon^3 R_s, \tag{36}$$

where as usual $R_c \in \mathcal{N}$ and $R_s \in P_s X$. Our aim is now to prove that R_c, R_s are of order $\mathcal{O}(1)$. Our result will be slightly worse, but nevertheless sufficient for the proof. Taking the difference of (10) and (20) yields immediately

$$
\begin{aligned}
R(t) &= \mathrm{e}^{tL} R(0) + \varepsilon^2 \int_0^t \mathrm{e}^{(t-\tau)L} A(R(\tau)) d\tau - \varepsilon^{-2} \mathrm{Res}(\varepsilon\psi)(t) \\
&\quad + \varepsilon^{-2} \int_0^t \mathrm{e}^{(t-\tau)L} [B(u(\tau)) - B(\varepsilon\psi(\tau))] d\tau. \qquad (37)
\end{aligned}
$$

As $u = \varepsilon\psi + \varepsilon^2 R$,

$$
\begin{aligned}
B(u) - B(\varepsilon\psi) &= 2B(\varepsilon\psi_c + \varepsilon^2\psi_s, \varepsilon^2 R_c + \varepsilon^3 R_s) + B(\varepsilon^2 R_c + \varepsilon^3 R_s) \\
&= 2\varepsilon^3 B(\psi_c + \varepsilon\psi_s, R_c + \varepsilon R_s) + \varepsilon^4 B(R_c + \varepsilon R_s).
\end{aligned}
$$

Projecting (37) to $P_c X$ and expanding the quadratic terms we obtain

$$R_c(t) = R_c(0) + \varepsilon^2 \int_0^t A_c(R_c + \varepsilon R_s) d\tau - \varepsilon^{-2} P_c \mathrm{Res}(\varepsilon\psi)(t)$$

$$+2\varepsilon^2 \int_0^t [B_c(\psi_c, R_s) + B_c(\psi_s, R_c)]d\tau + \varepsilon^4 \int_0^t B_c(R_s)d\tau$$
$$+\varepsilon^3 \int_0^t [B_c(\psi_s, R_s) + 2B_c(R_c, R_s)]d\tau, \qquad (38)$$

where we used that $B_c = 0$ on $P_c X$ by Assumption 2.5. Analogously,

$$\begin{aligned} R_s(t) &= \mathrm{e}^{tL} R_s(0) + \varepsilon \int_0^t \mathrm{e}^{(t-\tau)L} A_s(R_c + \varepsilon R_s)d\tau \\ &+2 \int_0^t \mathrm{e}^{(t-\tau)L} B_s(\psi_c + \varepsilon\psi_s, R_c + \varepsilon R_s)d\tau \\ &+\varepsilon \int_0^t \mathrm{e}^{(t-\tau)L} B_s(R_c + \varepsilon R_s)d\tau - \varepsilon^{-3} P_s \mathrm{Res}(\varepsilon\psi)(t). \end{aligned} \qquad (39)$$

Define the stopping time

$$\tau_R^* := \inf\{t > 0 \ : \ \|R_s(t)\| > \varepsilon^{-1} \quad \text{or} \quad \|R_c(t)\| > \varepsilon^{-1}\}. \qquad (40)$$

Now we use Lemma 4.1 and Theorem 2.17 to obtain for $t \le \min\{\tau_R^*,\ T_0\varepsilon^{-2}\}$ first

$$\begin{aligned} \|R_c(t)\| &= \|R_c(0)\| + \mathcal{O}(\varepsilon^{-2\kappa} + \varepsilon^{\gamma^*-2}) \\ &+C\varepsilon^2 \int_0^t \Big[\|R_c(\tau)\|(1 + \|\psi_s(\tau)\|) + \|R_s(\tau)\|\Big] d\tau, \end{aligned} \qquad (41)$$

where we used Assumptions 2.5 and 2.6. Note that for $\kappa > 0$ we do not have a uniform $\mathcal{O}(1)$ bound on ψ_s. Therefore, we have to leave this terms in (41), in order to get the right estimate by Gronwall inequality.

Analogously we obtain,

$$\begin{aligned} \|R_s(t)\| &= \mathrm{e}^{-t\omega}\|R_s(0)\| + \mathcal{O}(\varepsilon^{-\kappa}) \\ &+C \int_0^t (1 + (t-\tau)^{-\alpha})\mathrm{e}^{-(t-\tau)\omega} \Big[\|R_c(\tau)\| + \varepsilon\|R_s(\tau)\|\Big] d\tau, \end{aligned} \qquad (42)$$

where we additionally used (7).

Now the claim follows from the generalized Gronwall inequality of Lemma 6.1 with $\gamma(t) = 1 + \|\psi_s(\tau)\|$ and Lemma 4.1 to bound $\varepsilon^2 \int_0^{T_0\varepsilon^{-2}} \gamma(\tau)d\tau = \mathcal{O}(1)$ appearing in the exponent. $\square$

5 Applications

5.1 Surface growth model

Consider the surface growth model from (6) with fractional noise of trace-class with Hurst parameter $H \in [\frac{1}{2}, 1)$.

$$\partial_t h = -\Delta_{\text{per}}^2 h - \nu \Delta_{\text{per}} h - \Delta_{\text{per}} |\nabla h|^2 + \varepsilon^{2H+1} \xi \tag{43}$$

Here Δ_{per} is the Laplacian w.r.t. periodic boundary conditions. Suppose initial condition $h(0) = 0$ corresponding to an initially flat surface, and assume for simplicity of presentation that h is a 2π-periodic height profile of the surface over $[0, 2\pi]^2$, which nevertheless corresponds to the full three-dimensional problem.

Let $H_{\text{per}}^s := H_{\text{per}}^s([0, 2\pi]^2) = D((1 - \Delta_{\text{per}})^{s/2})$ for $s \geq 0$ be the standard fractional Sobolev space, and consider H_{per}^{-s} to be the dual of H_{per}^s. Define $\mathcal{L}_\nu = -\Delta_{\text{per}}^2 + \nu \Delta_{\text{per}}$. We consider $\mathcal{L}_\nu$ as an operator on $X = H_{\text{per}}^{3/2}$ with domain $D(\mathcal{L}_\nu) = H_{\text{per}}^{11/2}$. We could also choose the Sobolev space $W_{\text{per}}^{1,4}([0, 2\pi]^2)$ as X. Nevertheless, we want to work in an Hilbert space setting, where the assumptions are easier to verify.

Obviously, there is an orthogonal basis $\{e_k\}_{k \in \mathbb{Z}^2}$ of eigenfunctions of both $\mathcal{L}_\nu$ and Δ_{per} given by $e_k(x) := \mathrm{e}^{ikx}$ for $k \in \mathbb{Z}^2$. We use complex functions for simplicity of notation. To obtain a real basis, we could simply consider products of $\sin(k_j x_j)$ and $\cos(k_j x_j)$.

The corresponding eigenvalues are given by

$$\lambda_k(\nu) = |k|^2 (\nu - |k|^2) \quad \text{for} \quad k \in \mathbb{Z}^2.$$

Obviously, $\lambda_0(\nu) = 0$, and $\lambda_k(\nu) = 0$ for $|k| = 1$ iff $\nu = \nu_c := 1$. Moreover, these λ_k all change stability at ν_c. Hence, we have a bifurcation.

We can apply the general theory to two different regimes. First for $\nu < \nu_c$. Here $\mathcal{N} = \text{span}\{1\}$, and we obtain a SODE for the mean value of the solution, but we will not focus on this example. The result is just, that the mean behaves like a rescaled fractional Brownian motion, as in this case it is easy to see that $B_s(1, \cdot) \equiv 0$ and $A_c(1) = 0$ leading to $\Gamma = 0$ and $\nu = 0$ in the amplitude equation. It could be seen also directly from the equation, that the constant mode decouples from the rest of the equation.

More interesting is the case when $\nu \approx \nu_c$ and the distance from bifurcation and the noise strength are of a comparable order, for example $\nu = \nu_c + \varepsilon^2 \nu_0$

for some $\nu_0 \in [-1,1]$. Now

$$L := \mathcal{L}_{\nu_c} = -\Delta_{\text{per}}^2 - \Delta_{\text{per}}, \qquad A := -\nu_0 \Delta_{\text{per}}$$

It is standard to verify that Assumptions 2.3 and 2.6 are true, if we choose for instance $Y = H_{\text{per}}^{-2}$. Moreover, we obtain $\mathcal{N} = \text{sp}\{e_k \ : \ |k| \in \{0,1\}\}$ and Assumption 2.1 is fulfilled with $n = 5$. With a slight abuse of notation, we fix a real basis of $\mathcal{N}$

$$e = (\cos(x_1),\ \sin(x_1),\ 1,\ \cos(x_2),\ \sin(x_2)).$$

The bilinear operator B is given by

$$B(u,v) = -\Delta_{\text{per}}(\nabla u \cdot \nabla v).$$

It is easy to check, that B fulfills Assumption 2.5. First $B_c = 0$ on $P_c X$ is straightforward, and the other assumptions follow from the continuity of the following mappings

$$X \longrightarrow W_{\text{per}}^{1,4}([0,2\pi]^2) \stackrel{|\nabla \cdot|^2}{\longrightarrow} L^2([0,2\pi]^2) \stackrel{-\Delta_{\text{per}}}{\longrightarrow} H_{\text{per}}^{-2}([0,2\pi]^2).$$

Assumption 2.7 is for instance verified in [BG03] for noise that is white in time, which nevertheless carries immediately over to our case, as we need only enough regularity of the stochastic convolution. Furthermore Assumption 2.4 is true for trace-class noise with $H \geq \frac{1}{2}$. (see e.g. [DPM02]). We already have $W_L(t) \in H_{\text{per}}^2$.

Using the definitions from (13) and (14), it is an elementary but long computation to derive the amplitude equation (15) explicitly. First

$$\Gamma[a] = -\frac{1}{3}\Big(a_1(a_1^2 + a_2^2),\ a_2(a_1^2 + a_2^2),\ 0,\ a_4(a_4^2 + a_5^2),\ a_5(a_4^2 + a_5^2)\Big)$$

and $\nu(a) = \nu_0(a_1, a_2, 0, a_4, a_5)$. It is an interesting observation, that in the deterministic part the equations decouples into three independent equations. One describing functions constant in x-direction, another constant in y-direction, an the last part describing the constant itself.

The amplitude equation is now given by

$$da(T) = \nu_0 a(T)dT + \Gamma[a(T)]dT + d\beta^H(T), \tag{44}$$

where β^H is a fractional Brownian motion in $\mathbb{R}^5$ with Hurst parameter H and covariance matrix $\Sigma \in \mathbb{R}^{n \times n}$ determined by the covariance operator Q of the original noise process. Here $\Sigma = \Pi Q \Pi^*$. We remark without proof that in case of spatially homogeneous noise this covariance matrix is diagonal (cf. e.g. [B02] for $H = \frac{1}{2}$).

Note that we obtain a stable equation for a, where large deviation estimates are no problem. The *main results* are now the following. The attractivity is trivial with $t_\varepsilon = 0$, as $h(0) = 0$, and the approximation states that we have with high probability $h(t) \approx \varepsilon a(\varepsilon^2 t) \cdot e$ on a time-interval of order $\mathcal{O}(\varepsilon^{-2})$, where u is a solution of (44) with $a(0) = 0$. We refrain from reformulating the precise abstract result for this case.

5.2 Rayleigh-Bénards problem

We consider the two dimensional Bénard problem in a strip, where a fluid is heated from below. The three dimensional problem in a box is treated similarly, but the notation is much more involved.

In the following denote by (v, w) the velocity field of the fluid in $(y, z) \in D := [0, 2\pi] \times [0, \pi]$, where z is the vertical direction. Hence, the fluid is heated at $z \equiv 0$. Let p be the pressure and θ the normalized temperature, which means that $\theta \equiv 0$ and $(v, w) \equiv 0$ is heat transport without motion.

In dimension free quantities the governing Navier-Stokes and heat equation are given by (see e.g. [Ge98] or [W97])

$$\partial_t(v, w) + ((v, w) \cdot \nabla)(v, w)) = -\nabla p + (0, 1)\frac{R}{P}\theta + \Delta(v, w) \qquad (45)$$

$$\partial_t \theta - v + ((v, w) \cdot \nabla)\theta = \frac{1}{P}\Delta\theta + \varepsilon^{2H+1}\xi \qquad (46)$$

$$\operatorname{div}(v, w) = 0 \qquad (47)$$

We suppose periodic boundary conditions in y both for θ and (v, w). Moreover $\partial_z v = w = \theta = 0$ for $z = 0, \pi$. The noise ξ is trace-class fractional noise, corresponding to fluctuations in the temperature. We could also incorporate fluctuations in the velocity field, but we neglect this for simplicity.

In order to rule out motion of the whole fluid in the y-direction, we suppose vanishing mean flux $\int_0^\pi v dz$. We use the following constants: R is the Rayleigh number, P the Prantl number, and the quotient $\rho = R/P$ is the Reynolds number. The Rayleigh number is a dimension free measure of the heat difference between top and bottom of the strip, while the Prantl number depends only on the properties of the fluid.

Consider the following operator

$$\mathcal{L}u := \begin{pmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \frac{1}{P}\Delta \end{pmatrix} u + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \rho \\ 0 & 1 & 0 \end{pmatrix} u = \begin{pmatrix} \Delta v \\ \Delta w + \rho\theta \\ \frac{1}{P}\Delta\theta + v \end{pmatrix}$$

with domain

$$\mathcal{D} = \Big\{ (v,w,\theta) \in H^2(D,\mathbb{R}^3) : \ 2\pi - \text{periodic in } y,$$

$$\partial_z v = w = \theta \text{ for } z = 0,\pi, \quad \operatorname{div}(v,w) = 0, \ \int_0^\pi v dz = 0 \Big\}.$$

Let $\mathcal{Q}$ be the L^2-projection onto the functions that are divergence free in the first two components, and fulfill the boundary conditions of the problem. It is easy to check, that $\mathcal{Q}$ projects to the space $\{(v,w) \in H^2(D,\mathbb{R}^2) : \ (v,w,0) \in \mathcal{D}\}$ which is spanned by the following orthogonal basis of eigenfunctions of the Laplacian:

$$\begin{pmatrix} \cos(mz) \\ 0 \end{pmatrix}, \quad \begin{pmatrix} m\cos(mz)\cos(ky) \\ k\sin(mz)\sin(ky) \end{pmatrix}, \quad \begin{pmatrix} m\cos(mz)\sin(ky) \\ -k\sin(mz)\cos(ky) \end{pmatrix}$$

for $k,m \in \mathbb{N}$. Hence, $\mathcal{Q}$ and Δ commute, and

$$\mathcal{QL} := \begin{pmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \frac{1}{P}\Delta \end{pmatrix} + \mathcal{Q} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \rho \\ 0 & 1 & 0 \end{pmatrix}$$

Define furthermore for $u = (v,w,\theta)$

$$B_0(u,\tilde{u}) == -\mathcal{Q}[(v\partial_y + w\partial_z)\tilde{u}] \quad \text{and} \quad B(u,\tilde{u}) = \tfrac{1}{2}B_0(u,\tilde{u}) + \tfrac{1}{2}B_0(\tilde{u},u). \tag{48}$$

Now the equations (45) - (47) are given as an equation in $\mathcal{D}$ by

$$\partial_t u = \mathcal{QL}u + B(u,u) + \varepsilon^2(0,0,\xi)$$

Determine now the eigenfunctions of $\mathcal{QL}$ and the projection P_c. The major difficulties arise due to the fact that $\mathcal{QL}$ is in general not self-adjoint. Hence, the eigenfunctions are not orthogonal. Consider for $k \in \mathbb{Z}$ and $m \in \mathbb{N}$

$$\varphi^\pm(k,m)\mathrm{e}^{iky} = \Big(-ik\cos(mz), \ -\frac{k^2}{m}\sin(mz), \ -\frac{s(s+\lambda^\pm)}{\rho m}\sin(mz)\Big)\mathrm{e}^{iky}$$

where $s = k^2 + m^2$ and

$$\lambda^{\pm} = -\frac{1+P}{2P}s \pm \sqrt{\frac{k^2\rho}{s} + \frac{1}{4}\Big(\frac{1-P}{P}\Big)^2 s^2}.$$

In the beginning we use complex valued functions for simplicity of notation. It is easy to check, that $\varphi^{\pm}(k,m)\mathrm{e}^{iky}$ is divergence-free and fulfills the boundary conditions. Moreover, it is an eigenfunction of $\mathcal{QL}$ to the eigenvalue $\lambda^{\pm}$. We obtain a basis of eigenfunctions, if we include the functions $(\cos(mz), 0, 0)$ corresponding to eigenvalues $-m^2$ for $m > 1$.

For the other eigenvalues it is now easy to see that always $\lambda^- < 0$, and $\lambda^+ < 0$ as long as $\rho \leq s^3/Pk^2$. It is well know that the critical Reynolds number for the unbounded domain is $\rho_c = 27/4P$ with unstable wavenumber $s_c = \pm 1/\sqrt{2} \notin \mathbb{Z}$ (se e.g. [Ge98]). As we have a bounded domain with $s \in \mathbb{Z}$ our result is slightly different. We obtain:

$$\rho_c = \frac{8}{P}, \quad R_c = 8, \quad \text{and} \quad (m_c, k_c) = (1, \pm 1).$$

Now define $\rho = \rho_c + \varepsilon^2\rho_0$ with $\rho_0 \in [-1,1]$ and

$$L := \begin{pmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \frac{1}{P}\Delta \end{pmatrix} + \mathcal{Q}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \rho_c \\ 0 & 1 & 0 \end{pmatrix}, \quad A := \mathcal{Q}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \rho_0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, $\mathcal{N} = \operatorname{span}\{\varphi^+(1,1)\mathrm{e}^{iy},\ \varphi^+(1,-1)\mathrm{e}^{-iy}\} = \operatorname{span}\{e_1, e_2\}$ fulfilling Assumption 2.1 with

$$e_1 := \begin{pmatrix} \cos(z)\sin(y) \\ -\sin(z)\cos(y) \\ -P\sin(z)\cos(y)/2 \end{pmatrix} \quad \text{and} \quad e_2 := \begin{pmatrix} -\cos(z)\cos(y) \\ -\sin(z)\sin(y) \\ -P\sin(z)\sin(y)/2 \end{pmatrix}.$$

We now define the projection P_c to $\mathcal{N}$ as a projection along the other eigenspaces. This ensures, that P_c and L commute. It involves the dual basis of eigenfunctions of the adjoint L^*. The main advantage we obtain is that Assumption 2.3 will be very easy to verify, as all operators P_c, L, and e^{tL} are all given as explicit series expansions with respect to the same eigenfunctions.

For simplicity, we use the space $X = \mathcal{D}$ with high spatial regularity. It should be easy to relax this constraint significantly. Now $D(L) = H^4(D, \mathbb{R}^3) \cap$

X and $Y = \mathbb{L}^2(D)$ is the closure of X in $L^2(D, \mathbb{R}^3)$. Now Assumption 2.3 is easy to check due to the explicit basis of eigenfunctions.

Next it is easy to verify Assumption 2.6 is true. Moreover, as e_j is divergence-free and $\mathcal{Q}$ and the Laplacian commute (i.e., they exhibit joint invariant eigenspaces), it is an easy calculation, to verify that

$$Ae_j = \frac{\rho_0}{4}\frac{P}{1+P}e_j \quad \text{for} \quad j = 1, 2. \quad \text{Hence,} \quad \nu(a) = \frac{\rho_0}{4}\frac{P}{1+P}a.$$

Obviously, B from (48) fulfills Assumption 2.5, where it is elementary, but rather lengthy, to verify $P_c B(e_k, e_j) = 0$. Note that the nonlinearity without $\mathcal{Q}$ does not map into the right space.

It is an interesting observation, that $P_s B(e_k, e_k) = -\frac{P}{4}(0, 0, \sin(2z))$ and $P_s B(e_k, e_l) = (0, 0, 0)$ for $k \neq l$. Hence, all the coupling of the dominant modes is done via the heat equation.

Using $L_s(0, 0, \sin(2z)) = -\frac{4}{P}(0, 0, \sin(2z))$ it is now straightforward to compute $-2P_c B(e_j, L_s^{-1} P_s B(e_k, e_l))$, and we end up with (cf. (13))

$$\Gamma(a_1, a_2) = -\frac{\pi}{32}\frac{\sqrt{2}P^2\sqrt{8+P^2}}{1+P}(a_1^2 + a_2^2)\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Hence, the amplitude equation is given by

$$\partial_T \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{\rho_0}{8}\frac{P}{P+1}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} - \frac{\pi}{32}\frac{\sqrt{2}P^2\sqrt{8+P^2}}{1+P}(a_1^2 + a_2^2)\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \hat{\xi}$$

where $\hat{\xi}$ is some fractional noise in $\mathbb{R}^2$, which is derived from ξ by projection and rescaling of time.

Let us now comment on the remaining assumptions. First Assumption 2.7 is standard, as soon, as we have enough regularity of the stochastic convolution, which is assured by Assumption 2.4. The latter is also easy to verify, as the stochastic convolution with respect to L is just $(0, 0, W_{\Delta_\theta})$. Where W is the Wiener process in $H^2(D, \mathbb{R})$ corresponding to the noise ξ, and Δ_θ is the ordinary Laplacian in $L^2(D, \mathbb{R})$ subject to boundary condition of the temperature equation. Hence, we do not have a vector valued stochastic convolution, and especially the probability estimates are well known for that case.

6 Technical Lemmas

This section provides technical lemmas needed in the proof of the main theorems. The first lemma is a generalized Gronwall-type estimate.

Lemma 6.1 *Suppose we have two functions* $b, d : \mathbb{R} \to \mathbb{R}$ *and a positive function* $\gamma : \mathbb{R} \to \mathbb{R}_0^+$ *with* $\varepsilon^2 \int_0^{T_0\varepsilon^{-2}} \gamma(t)dt \leq K_\gamma$*. Suppose moreover that for some positive constants* ω, C_i, C_b *and* C_d *and* $\alpha \in [0,1)$ *we have for some* $\varepsilon > 0$ *and all* $t \in [0, T_0\varepsilon^{-2}]$

$$0 \leq d(t) \leq C_1\varepsilon^2 \int_0^t [d(\tau)(1+\gamma(\tau)) + b(\tau)]d\tau + C_d \tag{49}$$

$$0 \leq b(t) \leq C_2 \int_0^t (1+(t-\tau)^{-\alpha})\mathrm{e}^{-(t-\tau)\omega}[d(\tau)+\varepsilon b(\tau)]d\tau + C_b. \tag{50}$$

Define $K_{\alpha,\omega} = \int_0^\infty (1+\tau^{-\alpha})\mathrm{e}^{\tau\omega} d\tau$*. If* $\varepsilon \leq 1/(2C_2K_{\alpha,\omega})$ *then*

$$\sup_{t\in[0,T_0\varepsilon^{-2}]} |d(t)| \leq [C_d + 2C_bC_1T_0]\mathrm{e}^{C_1T_0(K_\gamma+2C_2K_{\alpha,\omega})}$$

and

$$\sup_{t\in[0,T_0\varepsilon^{-2}]} |b(t)| \leq 2C_b + 2C_2K_{\alpha,\omega}[C_d + 2C_bC_1T_0]\mathrm{e}^{C_1T_0(K_\gamma+2C_2K_{\alpha,\omega})}.$$

Proof: Define $S_b(t) := \sup_{s\in[0,t]} b(s)$ and similarly S_d. From (50) we obtain

$$b(t) \leq C_b + C_2K_{\alpha,\omega}[S_d(t') + \varepsilon S_b(t')]$$

for all $0 \leq t \leq t' \leq T_e\varepsilon^{-2}$. Hence

$$(1 - C_2K_{\alpha,\omega}\varepsilon)S_b(t) \leq C_b + C_2K_{\alpha,\omega}S_d(t),$$

and for $\varepsilon \leq 1/(2C_2K_{\alpha,\omega})$

$$S_b(t) \leq 2C_b + 2C_2K_{\alpha,\omega}S_d(t). \tag{51}$$

Now for $t \leq T_0\varepsilon^{-2}$

$$\begin{aligned} d(t) &\overset{(49)}{\leq} C_d + C_1\varepsilon^2 \cdot \int_0^t (S_d(\tau)\gamma(\tau) + S_b(\tau))d\tau \\ &\overset{(51)}{\leq} C_d + 2C_bC_1T_0 + C_1\varepsilon^2] \cdot \int_0^t (\gamma(\tau) + 2C_2K_{\alpha,\omega})S_d(\tau)d\tau. \end{aligned}$$

As the right hand side is monotone in t, we obtain

$$S_d(t) \leq C_d + 2C_bC_1T_0 + C_1\varepsilon^2 \cdot \int_0^t (\gamma(\tau) + 2C_2K_{\alpha,\omega})S_d(\tau)d\tau.$$

Now Gronwall implies for $t \leq T_0\varepsilon^{-2}$

$$S_d(t) \leq [C_d + 2C_bC_1T_0]\mathrm{e}^{C_1(K_\gamma + 2C_2K_{\alpha,\omega}T_0)}.$$

Using (51) the claim follows. □

The following lemma gives Hölder estimates for convolutions with the semigroup of L. Similar results are well-known (see e.g. [L94, Prop. 4.2.1]). Nevertheless, we give a simple proof, as we need explicitly the dependence of the constants on ε.

Lemma 6.2 *Consider L as in Assumption 2.3. Then for all $h \in L^\infty([0,T], P_sX)$ and $p \in (0,1)$ we obtain*

$$\Big\| \mapsto \int_0^t \mathrm{e}^{(t-s)L\varepsilon^{-2}} h(s)ds \Big\|_{C^p([0,T],X)} \leq C\varepsilon^{2-2p} \|h\|_{L^\infty([0,T],X)}, \qquad (52)$$

and for all $h \in C^p([0,T\varepsilon^{-2}], P_sX)$ with $h(0) = 0$

$$\Big\| \mapsto \int_0^t \mathrm{e}^{(t-s)L\varepsilon^{-2}} h(s)ds \Big\|_{C^p([0,T],X)} \leq C\varepsilon^2 \|h\|_{C^p([0,T],X))}, \qquad (53)$$

where the constants depend only on p, ω, and M.

We can also prove an intermediate result, which is useful for Hurst parameter $H = \frac{1}{2}$.

Corollary 6.3 *Under the assumptions of Lemma 6.2. For all $\alpha \in (0, \frac{1}{4})$ and $p \in (0, (1-4\alpha)/2\alpha)$ we obtain for all $h \in C^p([0,T\varepsilon^{-2}], P_sX)$ with $h(0) = 0$*

$$\Big\| t \mapsto \int_0^t \mathrm{e}^{(t-s)L\varepsilon^{-2}} h(s)ds \Big\|_{C^{\frac{1}{2}+\frac{2\alpha}{p+1}}([0,T],X)} \leq C\varepsilon^{2-\frac{1}{p+1}-2\alpha\frac{p+2}{p+1}} \cdot \|h\|_{C^{\frac{1}{2}-\alpha}([0,T],X)} \qquad (54)$$

where the constant depends only on p, α, ω, and M.

Proof: Define $\delta = \frac{1}{2} - \alpha$ and $\gamma = \frac{1}{2} + \alpha(p+2)$, which are both in $(0,1)$. Then we can use the following obvious interpolation formula for Hölder-norms:

$$\|u\|_{C^{\frac{p\delta+\gamma}{p+1}}([0,T],X)} \leq 2\|u\|^{\frac{p}{p+1}}_{C^\delta([0,T],X)} \cdot \|u\|^{\frac{1}{p+1}}_{C^\gamma([0,T],X)}$$

Now the claim follows immediately by using (52) and (53). □

Proof: (Lemma 6.2) First by (7) it is straightforward to verify

$$\Big\| \int_0^t \mathrm{e}^{(t-s)L\varepsilon^{-2}} h(s)ds \Big\| \leq M\varepsilon^2 \int_0^\infty \mathrm{e}^{-\omega t} dt \|h\|_{L^\infty([0,T],X)}.$$

This is the L^∞-bound. Let us now turn to the Hölder semi-norm. Consider for $\eta > 0$ and $t \in [0,T]$ such that $t + \eta \leq T$

$$\begin{aligned}
&\Big\| \int_0^{t+\eta} \mathrm{e}^{(t+\eta-s)L\varepsilon^{-2}} h(s)ds - \int_0^t \mathrm{e}^{(t-s)L\varepsilon^{-2}} h(s)ds \Big\| \\
\leq\ & \Big\| \int_t^{t+\eta} \mathrm{e}^{(t+\eta-s)L\varepsilon^{-2}} h(s)ds \Big\| + \Big\| (\mathrm{e}^{\eta L\varepsilon^{-2}} - 1) \int_0^t \mathrm{e}^{(t-s)L\varepsilon^{-2}} h(s)ds \Big\| \\
\leq\ & M\|h\|_{L^\infty([0,T],X)} \Big[\int_t^{t+\eta} \mathrm{e}^{-(t+\eta-s)\omega\varepsilon^{-2}} ds + M \int_0^\eta s^{-1+p} \mathrm{e}^{-s\omega\varepsilon^{-2}} ds \int_0^t s^{-p} \mathrm{e}^{-s\omega\varepsilon^{-2}} ds \Big] \\
\leq\ & \varepsilon^2 M\|h\|_{L^\infty([0,T],X)} \Big[\int_0^{\eta\varepsilon^{-2}} \mathrm{e}^{-s\omega} ds + M \int_0^{\eta\varepsilon^{-2}} s^{p-1} ds \int_0^\infty s^{-p} \mathrm{e}^{-s\omega} ds \Big] \\
\leq\ & C\varepsilon^{2(1-p)} \eta^p \|h\|_{L^\infty([0,T],X)}.
\end{aligned}$$

This easily implies the first assertion.

The second result is proven similar. First the L^∞-bound in $C^0([0,T],X)$ is completely analogous. Moreover, as $h(0) = 0$

$$\begin{aligned}
&\int_0^{t+\eta} \mathrm{e}^{(t+\eta-s)L\varepsilon^{-2}} h(s)ds - \int_0^t \mathrm{e}^{(t-s)L\varepsilon^{-2}} h(s)ds \\
=\ & \int_0^t \mathrm{e}^{(t-s)L\varepsilon^{-2}} [h(s+\eta) - h(s)]ds + \int_0^\eta \mathrm{e}^{(t+\eta-s)L\varepsilon^{-2}} [h(s) - h(0)]ds.
\end{aligned}$$

Now it is straightforward to verify the second assertion. □

The following lemma relies on fractional integration by parts and it is very useful to bound integrals:

Lemma 6.4 *For $p_1 \in (0,1)$ let $g \in C^{p_1}([0,T],\mathbb{R})$ and $\Phi \in C^1([0,T],X)$. Then*

$$\Big\| \int_0^T g(s)\partial_s\Phi(s)ds\Big\| \le C\|g\|_{C^{p_1}([0,T])}\|\Phi\|_{C^{p_2}([0,T],X)}$$

with $p_1+p_2>1$, where the constant $C=C(T,p_1,p_2)$ grows monotone in T.

We do not give a detailed proof of this lemma, as it is straightforward. We can for example first use the fractional integration by parts formula (see e.g. [Zä98]). This is to our knowledge only developed for real-valued functions, but it is easy to generalize this to our case by first applying an arbitrary element of the dual of X to the integral. Estimating the integral like for instance in [MN02] we obtain first fractional Sobolev norms, which then, by crude estimates on the fractional derivatives, give the Hölder norms.

Lemma 6.5 *Let Assumption 2.3 be true. Given $f \in C^q([0,T],\mathbb{R})$ and $g \in C^p([0,T],P_sY)$ with $q+p>1$, we obtain*

$$\begin{aligned}\|J_2\| &:= \|\int_0^T f(t)\Big(\varepsilon^{-2}L\int_0^t \mathrm{e}^{(t-s)L\varepsilon^{-2}}L^{-1}g(s)ds - L^{-1}g(t)\Big)dt\| \\ &\le C\varepsilon^2\|f\|_{C^q([0,T])}\|g\|_{C^p([0,T],Y)}.\end{aligned}$$

For $r>1$ define $p=\frac{1}{2}-1/(4r(r+2))$ and $q=\frac{1}{2}-(r+1)/(4r(r+2))$. Now

$$\|J_2\| \le C\varepsilon^{2-\frac{3r+1}{2r(r+1)}}\|f\|_{C^{\frac{1}{2}-\frac{1}{4r(r+2)}}([0,T],\mathbb{R})}\cdot\|g\|_{C^{\frac{1}{2}-\frac{r+1}{4r(r+2)}}([0,T],Y)}.$$

Proof: As $g \in C^q([0,T],Y)$ and hence $L^{-1}g \in C^q([0,T],X)$ we know (e.g. [L94, Thm. 4.3.1]) that $\int_0^t \mathrm{e}^{(t-s)L\varepsilon^{-2}}L^{-1}g(s)ds$ is in $C^q([0,T],D(L))$ and in $C^{1+q}([0,T],X)$. We obtain

$$\begin{aligned}\|J_2\| &= \|\int_0^T f(t)\partial_t\int_0^t \mathrm{e}^{(t-s)L\varepsilon^{-2}}L^{-1}g(s)dsdt\| \\ &\le \|\int_0^T f(t)\partial_t\int_0^t \mathrm{e}^{(t-s)L\varepsilon^{-2}}L^{-1}[g(s)-g(0)]dsdt\| \qquad (55)\\ &\quad +\|\int_0^T f(t)\partial_t\int_0^t \mathrm{e}^{(t-s)L\varepsilon^{-2}}dsdtL^{-1}g(0)\|, \qquad (56)\end{aligned}$$

where we induced $g(0)$ to apply Lemma 6.2. As

$$\partial_t\int_0^t \mathrm{e}^{(t-s)L\varepsilon^{-2}}ds = \partial_t\int_0^t \mathrm{e}^{sL\varepsilon^{-2}}ds = \mathrm{e}^{tL\varepsilon^{-2}},$$

the second term is easily bounded by

$$(56) \leq C\|f\|_{L^\infty(0,T)} \cdot \|g\|_{L^\infty([0,T],Y)} \cdot \varepsilon^2 \cdot \int_0^\infty \mathrm{e}^{-t\omega} dt.$$

For (55) we use fractional integration by parts (see Lemma 6.4) to obtain

$$\begin{aligned}(55) \quad &\leq \quad C\|f\|_{C^q([0,T])} \cdot \left\| t \mapsto \int_0^t \mathrm{e}^{-(t-s)L\varepsilon^{-2}} L^{-1}[g(s) - g(0)]ds \right\|_{C^p([0,T],X)} \\ &\leq \quad C\varepsilon^2 \|f\|_{C^q([0,T])} \cdot \|g\|_{C^p([0,T],Y)}\end{aligned}$$

by Lemma 6.2, and the first claim follows.

The second assertion is analogous. In the application of Lemma 6.4 we use $q = {}^1\frac{}{2} - \frac{\alpha}{r+1}$ and $p = \frac{1}{2} + \frac{2\alpha}{r+1}$ and then we Corollary 6.3 to obtain

$$\|J_2\| \leq C\varepsilon^{2-\frac{1}{r+1}-2\alpha\frac{r+2}{r+1}} \cdot \|f\|_{C^{\frac{1}{2}-\frac{\alpha}{r+1}}([0,T])} \cdot \|g\|_{C^{\frac{1}{2}-\alpha}([0,T],Y)}.$$

The claim follows by fixing $\alpha = (r+1)/4r(r+2)$. The condition $r > 1$ ensures that all conditions of Corollary 6.3 are fulfilled. □

7 Acknowledgements

This work was supported by the DFG Forschungsstipendium BL535-5/1. The author would like to thank the Mathematical Research Center of the University of Warwick and especially David Elworthy for their hospitality, Guido Schneider for pointing out this problem, and Martin Hairer for many discussions on fractional Brownian motions.

References

[Ar98] **L. Arnold.** Random dynamical systems. Springer Monographs in Mathematics. Berlin, Springer, 1998.

[B02] **D. Blömker.** Non-homogeneous noise and Q-Wiener processes on bounded domains. Submitted for publication, 2002.

[B03] **D. Blömker.** Amplitude equations for locally cubic non-autonomous nonlinearities. Submitted for publication, 2003.

[BG03] **D. Blömker, C.Gugg.** Thin-Film-Growth-Models: On local solutions. To appear in *Proceedings of "The First Sino-German Conference on Stochastic Analysis", Beijing 2002* Eds. S. Albeverio, Z.M. Ma, M. Röckner.

[BGR02] **D. Blömker, C.Gugg, M. Raible.** Thin-film-growth models: roughness and correlation functions. *European J. Appl. Math.* 13(4):385–402, (2002).

[BMS01] **D. Blömker, S. Maier-Paape, G. Schneider.** The stochastic Landau equation as an amplitude equation. *Discrete and Continuous Dynamical Systems, Series B*, 1(4):527–541, (2001).

[BP99] **Z. Brzeźniak, S. Peszat.** Space-time continuous solutions to SPDE's driven by a homogeneous Wiener process. *Studia Math.* 137(3):261–299, (1999).

[BP00] **Z. Brzeźniak, S. Peszat.** Strong local and global solutions for stochastic Navier-Stokes equations. *Infinite Dimensional Stochastic Analysis* (Amsterdam, 1999), 85–98, Verh. Afd. Natuurkd. 1. Reeks. K. Ned. Akad. Wet., 52, R. Neth. Acad. Arts Sci., Amsterdam, (2000).

[CB95] **Z. R. Cuerno and A.-L. Barabási.** Dynamic Scaling of Ion-Sputtered Surfaces *Phys. Rev. Lett.* 74:4746–4749, (1995).

[CF98] **H. Crauel, F. Flandoli.** Additive noise destroys a pitchfork bifurcation. *J. Dynam. Differential Equations* 10(2):259–274, (1998).

[dPZ92] **G. Da Prato, J. Zabczyk.** *Stochastic Equations in Infinite Dimensions.* Cambridge University Press, 1992.

[DÜ99] **L. Decreusefond, A. S. Üstünel.** Stochastic analysis of the fractional Brownian motion. *Potential Anal.* 10(2):177–214, (1999).

[DJ01] **J. Duan, V.J. Ervin.** On the stochastic Kuramoto-Sivashinsky equation. *Nonlinear Anal., Ser. A: Theory Methods*, 44(2):205–216, (2001).

[DPM02] **T. E. Duncan, B. Pasik-Duncan, B. Maslowski.** Fractional Brownian motion and stochastic equations in Hilbert spaces. *Stoch. Dyn.*, 2(2):225–250, (2002).

[FB+02] **S. Facsko, T. Bobek, H. Kurz, T. Dekorsy, S. Kyrsta, R. Cremer** Ion-Induced Formation Of Regular Nanostructures On Amorphous GaSb Surfaces. *Applied Physics Letters*, 80(1):130–132, (2002).

[FvE03] **I. Fatkullin, E. vanden-Eijnden.** Asymptotic dynamics of stochastically perturbed gradient systems. *Preprint*, (2003).

[Fr96] **M. Freidlin.** *Markov processes and differential equations: asymptotic problems.* Lectures in Mathematics ETH Zrich. Birkhäuser Verlag, Basel, 1996.

[Ge98] **A.V. Getling** *Rayleigh-Bénard Convection - Structures and Dynamics.* World Scientific Press, 1998.

[H83] **H. Haken.** *Synergetics. An introduction. Nonequilibrium phase transitions and self-organization in physics, chemistry, and biology.* Springer Series in Synergetics, Vol. 1. Berlin etc.: Springer, 1983.

[KK01] **R. Khasminskii, N. Krylov.** On averaging principle for diffusion processes with null-recurrent fast component. *Stochastic Process. Appl.* 93(2):229–240, (2001).

[KSM92] **P. Kirrmann, G. Schneider, A. Mielke.** The validity of modulation equations for extended systems with cubic nonlinearities. *Proceedings of the Royal Society of Edinburgh* 122A:85–91, (1992).

[K01] **R. Kuske.** Asymptotic analysis of noise-amplified oscillations for subcritical delays. *Differential Equations Dynam. Systems* 9(3-4):221–241, (2001).

[L94] **A. Lunardi.** *Analytic Semigroups and Optimal Regularity in Parabolic Problems.* Progress in Nonlinear Differential Equations and their Applications, 16. Birkhäuser Verlag, Basel, 1995.

[MN02] **B. Maslowski, D. Nualart.** Evolution equations driven by a fractional Brownian motion. *Preprint*, (2002).

[PS03a] **G. Pavliotis, A. Stuart.** White noise limits for inertial particles in a random field. Preprint, (2003).

[PS03b] **G. Pavliotis and A. Stuart.** Ito versus Stratonovich white noise limits. Preprint, (2003).

[RLH00] **M. Raible, S. J. Linz, and P. Hänggi.** Amorphous thin film growth: Minimal deposition equation *Physical Review E*, 62:1691–1705, (2000).

[RLH01] **M. Raible, S. J. Linz, and P. Hänggi.** Amorphous thin film growth: Effects of density inhomogeneities. *Physical Review E*, 64:31506, (2001).

[RM+00] **M. Raible, S. G. Mayr, S. J. Linz, M. Moske, P. Hänggi, and K. Samwer.** Amorphous thin film growth: Theory compared with experiment. *Europhysics Letters*, 50:61–67, (2000).

[S93] **G. Schneider.** Analyticity of Ginzburg-Landau modes. *Journal of Differential Equations,* 121:233–257, (1995).

[SP94] **M. Siegert and M. Plischke.** Solid-on-solid models of molecular-beam epitaxy. *Physical Review E,* 50:917–931, (1994).

[W97] **D. Walgraef** *Spatio-Temporal Pattern Formation.* Partially ordered systems, Springer, 1997.

[W80] **J. Weidmann.** *Linear operators in Hilbert spaces.* Springer, 1980.

[Zä98] **M. Zähle.** Integration with respect to fractal functions and stochastic calculus. I. *Probab. Theory Related Fields* 111(3):333–374 (1998).