An Approach to the Existence of Unique Invariant Probabilities for Markov Processes *

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Abstract

A notion of *localized splitting* is introduced as a further extension of the splitting notions for iterated monotone maps introduced earlier by Dubins and Freedman (1966) and more generally by Bhattacharya and Majumdar (1999). We will see that under quite general conditions, *localized splitting theory* is a natural extension of the Doeblin (1937) minorization theory, Harris (1956) recurrence theory, splitting theory of Nummelin (1978) and regeneration theory of Athreya and Ney (1978), under which we can prove the existence of a unique invariant probability. By way of introduction we also provide a natural coupling proof of the ergodic problem for Markov processes on general state spaces under Doeblin's minorization condition which seems to have heretofore gone unnoticed. The paper is concluded with some new applications of splitting theory to random iterated quadratic maps.

1 Introduction and Some Background Results

The focus of this paper is the ergodic theory of Markov processes on a general state space viewed as actions of iterated random maps. There are a couple of ways in which Markov processes $\{X_n\}_{n=0}^{\infty}$ on a measurable state space $(S, \mathcal{S})$ may be viewed as naturally arising from iterations of *i.i.d.* random maps. In the case that S is a Borel subset of a Polish space a Markov process with arbitrarily prescribed transition probability $p(x, dy)$ and initial state $x \in S$ may be represented means of a sequence of *i.i.d.* random maps $\alpha_n, n \geq 1$, on S into S, defined on some probability space $(\Omega, \mathcal{F}, P)$, as:

$$X_0 = x, \quad X_1 = \alpha_1 x, \ldots, X_n = \alpha_n \cdots \alpha_1 x, n \geq 1. \tag{1}$$

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Here $\alpha_n(\omega)$ is a map on S for $\omega \in \Omega$ whose value at $x \in S$ is denoted $\alpha_n x$ under the usual probability convention of suppressing ω. Also $\alpha_n \cdots \alpha_1$ denotes the n-fold composition of the maps $\alpha_n, \ldots, \alpha_1$.

Although the representation is not unique, the familiar "inverse distribution function method" used to generate simulations from a given distribution on the real number line lies at the heart of this representation; see Kifer (1986) or Bhattacharya and Waymire (p. 228,1990). However another way in which Markov processes on a general state space with such representation occur, and which need not involve topological conditions on the state space, is simply as a *model* in some specific context, eg. linear/nonlinear autoregressive and ARMA models, Markov Chain Monte Carlo, Bayesian statistics, economics, temporal discretization of diffusion, fractal image compression, etc. For an expository orientation to the breadth of applications accommodated by this viewpoint see the excellent recent article by Diaconis and Freedman (1999), as well as Bhattacharya and Waymire (1990), Bhattacharya and Waymire (1999). The article by Diaconis and Freedman (1999) differs from the present approach in exploiting *contractive properties* of the maps on average, whereas our focus is on certain *splitting properties* of the maps which occur with positive probability.

So one may either assume at the outset that S is a Borel subset of a Polish space, or assume that one is given a Markov process on a measurable state space $(S, \mathcal{S})$ with the representation (1). It is most convenient to exposition to simply make the former assumption, which we will set as the framework for this paper.

Let us now attempt to give some background from the general ergodic theory for Markov processes for perspective on the framework being developed here. The reader is referred to Meyn and Tweedie (1993) for a thorough and state of the art account of general minorization and small set theory, to Diaconis and Freedman (1999) for the contractive mapping theory, and to Bhattacharya and Majumdar (1999) for the recent splitting theory which spawned the present work.

When applicable *Doeblin's minorization* condition provides a powerful approach to check for the existence of a unique invariant probability, which also gives uniform exponential rates of convergence in total variation distance. In fact, Doeblin's minorization is also necessary for uniform exponential rates in total variation distance; see Nummelin (Theorem 6.15, 1984), Tierney (Proposition 2, 1994), Meyn and Tweedie (Theorem 16.2.3,1993). Doeblin's minorization requires a probability measure ν on $(S, \mathcal{S})$, a positive integer N, and a positive real number δ such that

$$p^{(N)}(x, B) \geq \delta\nu(B), x \in S, B \in \mathcal{S}, \tag{2}$$

where $p^{(m)}(x, dy)$ denotes the m-step transition probability defined inductively by

$$p^{(0)}(x, dy) = \delta_x(dy), \quad p^{(m+1)}(x, B) = \int_X p(y, B)p^{(m)}(x, dy), m \geq 0. \tag{3}$$

We give a proof below using another powerful idea introduced by Doeblin (1938) for the ergodic problem of *finite state* Markov chains, namely *coupling.* The proof also illustrates the *backward iteration* method of Furstenberg (1963) which has found wide ranging applications in the study of random iterated maps, e.g. see Diaconis and Freedman (1999) and references therein, and Bhattacharya and Lee (1988), Bhattacharya and Rao (1993), Bhattacharya and Waymire (1990), Bhattacharya and Waymire (1999).

Theorem 1.1 *(Doeblin's Minorization). Under Doeblin's Minorization condition there is a unique invariant probability* π *on* $(S, \mathcal{S})$*. Moreover*

$$|p^{(n)}(x,B) - \pi(B)| \leq (1-\delta)^{[\frac{n}{N}]} \quad n \geq 1.$$

Proof. By viewing the process $\{X_n\}_{n=0}^{\infty}$at times $0, N, 2N, \ldots$, one may regard $p^{(N)}(x,dy)$ as a one-step transition probability. By this device one may restrict attention to the case $N = 1$. First observe that Doeblin's minorization is equivalent to the existence of a representation by *i.i.d.* maps $\alpha_1, \alpha_2, \ldots,$ of the form (1) such that α_n is constant on S with probability p, for any $0 < p < \delta$. For clearly if such a map exists then

$$p(x,B) = P(\alpha_1 x \in B) \geq P(\alpha_1 \in \Gamma_c, \alpha_1 x \in B) \geq pP(\alpha_1 x \in B | \alpha_1 \in \Gamma_c),$$

where Γ_c denotes the collection of constant maps on S. Conversely, if Doeblin's minorization holds then define *i.i.d.* random maps as follows: At time step n, with probability p and independently of $\alpha_k, k < n$, select a random point distributed as ν and define α_n as the constant map with this value, or with probability $1-p$ select α_n as the map such that $\alpha_n x$ is distributed as $(1-p)^{-1}(p(x,dy) - p\nu(dy))$. Then by construction

$$p(x,dy) = P(\alpha_n x \in dy), x \in S.$$

Now, since a constant map will occur among the *i.i.d.* sequence $\alpha_1, \alpha_2, \ldots$ with probability one, and since, for each n, the backward and forward iterations have the same distribution, the a.s. limit of the *backward iteration* exists and is given by

$$\lim_{n\to\infty} \alpha_1 \cdots \alpha_n x = \alpha_T \cdots \alpha_1 x,$$

where

$$T := \inf\{n \geq 1 : \alpha_n \in \Gamma_c\}.$$

One may readily check that the distribution π of this limit is an invariant probability. Now, with the minorized representation defined above one has a natural coupling of the processes with transition probability $p(x,dy)$ started in state $x \in S$ and initial distribution π, respectively, given by $\{(X_n, Y_n)\}_{n=0}^{\infty}$, where $X_n = \alpha_n \cdots \alpha_1 x, Y_n = \alpha_n \cdots \alpha_1 Y_0, n \geq 1$, with Y_0 distributed as π independently of the $\alpha_n' s$. In particular it follows that

$$|p^{(n)}(x,B) - \pi(B)| \leq P(T > n) \leq (1-p)^n, \quad x \in S, B \in \mathcal{S}.$$

Since $0 < p < \delta$ is arbitrary the asserted rate follows. □

The notion of *small set* A_0 provides a *localized minorization* condition defined by a subset $A_0 \in \mathcal{S}$ of the state space S, a probability measure ν on $(A_0, A_0 \cap \mathcal{S})$, a positive integer N, and a positive real number δ such that

$$p^{(N)}(x, B) \geq \delta\nu(B), x \in A_0, B \in \mathcal{S}; \tag{4}$$

see Meyn and Tweedie (1993) for a treatment of small sets. These are also the so-called C-sets in Orey (1971). We simply refer to (4) as *local minorization on* A_0, where we will assume further that this occurs on a *recurrent set* A_0 in the sense that

$$P_x \cup_{n=0}^{\infty} [X_n \in A_0] = 1, \; x \in S. \tag{5}$$

It was pointed out to the authors by K. B. Athreya (personal communication) that local minorization on a recurrent set is an equivalent condition for Harris' familiar notion of *φ-recurrence* whenever $\mathcal{S}$ is countably generated; neccessity is proved in Orey (1971) and sufficiency is straightforward to check.

A formulation in terms of iterated maps may be obtained along the lines introduced by Athreya and Ney (1978) to identify regeneration structure in locally minorized Markov processes on a recurrent set A_0.

Proposition 1.1 *(Local Minorization). The local minorization condition (4) on a recurrent set A_0 is equivalent to the existence of a representation by i.i.d. maps $\alpha_1, \alpha_2, \ldots,$ of the form (1) such that α_n is a constant on A_0 into A_0 with probability $p = \delta$.*

Proof. By the device noted in the proof of Theorem 1.1 one may restrict attention to the case $N = 1$. First observe that if such maps exist then for $x \in A_0, B \in \mathcal{S}$, one has

$$p(x, B) = P(\alpha_1 x \in B) \geq P(\alpha_1 \in \Gamma_c, \alpha_1 x \in B) \geq pP(\alpha_1 x \in B | \alpha_1 \in \Gamma_c),$$

where Γ_c denotes the collection of constant maps on A_0 into A_0. Conversely, suppose that local minorization holds on a recurrent set A_0. Let α_n be a representation by *i.i.d.* maps and define an alternative representation by *i.i.d.* maps $\beta_n, n \geq 1$ constructed as follows: Toss a coin with probability $p = \delta$ of heads. If head occurs then select a random point in A_0 distributed as ν and define $\beta_1 x$ as this constant value for all $x \in A_0$, but if tail occurs then for each $x \in A_0$ let $\beta_1 x$ be distributed as $(1-p)^{-1}(p(x, dy) - p\nu(dy))$. If $x \in S - A_0$, then define $\beta_1 x = \alpha_1 x$. Now let β_n be an *i.i.d.* sequence of maps distributed Then by such a construction

$$p(x, dy) = P(\beta_n x \in dy), x \in S.$$

We leave the detailed construction of the probability space etc to the reader; or see Bhattacharya and Waymire (2001). □

The following theorem is a centerpiece of general Markov process theory based on various notions of Harris recurrence by Orey (1971), regeneration by

Athreya and Ney (1978), or so-called Nummelin splitting by Nummelin (1978); see Meyn and Tweedie (1993) for a proof and comprehensive treatment.

Theorem 1.2 *Assume the local minorization condition (4) on a recurrent set A_0. In addition assume that*

$$\sup_{x \in A_0} \mathbf{E}_x \tau_{A_0} < \infty,$$

where $\mathbf{E}_x$ denotes expectation when $X_0 = x$, and

$$\tau_{A_0} = \inf\{n \geq 1 : X_n \in A_0\}.$$

Then there is a unique invariant probability π on $(S, \mathcal{S})$. Moreover for $x \in S$,

$$\sup_{B \in \mathcal{S}} |\frac{1}{n} \sum m = 1^n p^{(m)}(x, B) - \pi(B)| \to 0 \text{ as } n \to \infty.$$

Remark. The finiteness of the expected time to renew a visit to A_0 may often be checked by the extensions of Foster's *drift conditions,* or equivalently, *stochastic Liapounov conditions;* see Meyn and Tweedie (1993).

Finally let us record a generalization of the Dubins and Freedman (1966) splitting condition for monotone maps given by Bhattacharya and Majumdar (1999). This condition is defined by a sub-collection $\mathcal{A}$ of $\mathcal{S}$, a positive integer N and a positive real number δ such that

$$P(\alpha_N \cdots \alpha_1^{-1} A = S \text{ or } \emptyset) \geq \delta \quad \text{for all } A \in \mathcal{A}. \tag{6}$$

We will refer to this condition as *full splitting* and to the parameter N as a *splitting scale.* The sub-collection $\mathcal{A}$ of $\mathcal{S}$ will be called the *splitting class* of sets.

With a remarkably simple proof, Bhattacharya and Majumdar (1999) obtained the following theorem. Denote by Q the distribution of α_1 on an appropriate space Γ of measurable maps γ of S into S.

Theorem 1.3 *Assume the full splitting condition (6). Assume that $(\mathcal{P}(S), d)$ is a complete metric space under*

$$(i.) \quad d(\mu, \nu) = \sup_{B \in \mathcal{A}} |\mu(B) - \nu(B)|, \mu, \nu \in \mathcal{P}(S),$$

where $\mathcal{P}(S)$ denotes the space of probability measures on $(S, \mathcal{S})$. Also with N as the splitting scale, assume that for each measurable map $\gamma : S \to S$

$$(ii.) \quad d(\mu \circ \gamma_N \cdots \gamma_1^{-1}, \nu \circ \gamma_N \cdots \gamma_1^{-1}) \leq d(\mu, \nu), \mu, \nu \in \mathcal{P}(S).$$

Then there is a unique invariant probability π on $(S, \mathcal{S})$. Moreover

$$\sup_{x \in S, B \in \mathcal{A}} |p^{(n)}(x, B) - \pi(B)| \leq (1 - \delta)^{[\frac{n}{N}]}.$$

Remark. It is known that the sufficient condition in Theorem1.2 is necessary if the process is aperiodic (Meyn and Tweedie 1993, p. 384). Also, in the aperiodic case one may replace the Caesaro mean $\frac{1}{n}\sum_{m=1}^{n} p^{(m)}(x,B)$ by $p^{(n)}(x,B)$ by making use of the renewal theorem.

This is the starting point for the present paper. The remainder is organized as follows. In section 2 we introduce a localized version of splitting and state our theorem asserting the existence and uniqueness of an invariant probability under localized splitting conditions. This is followed by the proof of existence. The proof of uniqueness is taken up in section 3. In the end we have a generalization of Theorem 1.3. Also, Theorem 1.3 and/or the Dubins and Freedman theory have found interesting applications to random iterations of (nonmonotone) quadratic maps, eg. see Bhattacharya and Rao (1993), Athreya and Dai (1999), Dai (1999). These results are essentially obtained by finding an invariant set on which the maps are monotone, but may involve some delicate considerations of a splitting class of sets. This is illustrated with the introduction of a notion of *strict splitting* in the context of quadratic maps to extend previously known results there.

2 A Localized Splitting and Existence

Let us begin by introducing a localized version of the splitting condition (6).
Definition 2.1 The Markov process $\{X_n\}_{n=0}^{\infty}$ on $(S,\mathcal{S})$ is said to have a *locally splitting* representation by *i.i.d.* maps $\alpha_1, \alpha_2, \ldots$ if there is a recurrent set $A_0 \in \mathcal{S}$, a sub-collection $\mathcal{A}$ of $\mathcal{S}$, a positive integer N, and a positive real number δ such that for each $A \in \mathcal{A}_0 \equiv A_0 \cap \mathcal{A}$,

$$P(A_0 \cap \alpha_N \cdots \alpha_1^{-1} A_0 = A_0, A_0 \cap \alpha_N \cdots \alpha_1^{-1} A = A_0 \text{ or } \emptyset) \geq \delta. \tag{7}$$

The parameter N is referred to as the *local splitting scale,* and the collection $\mathcal{A}_0$ is the *local splitting class.*

One may easily check that from Proposition 1.1 it follows that local minorization on a recurrent set A_0 implies local splitting on A_0 with $\mathcal{A} = \mathcal{S}$, i.e. with local splitting class $\mathcal{A}_0 = A_0 \cap \mathcal{S}$. We will consider a localization of splitting which generalizes this framework. Nonetheless, we will require the local splitting class to

$$\mathcal{A}_0 = A_0 \cap \mathcal{S}$$

for this generalization. Under this condition, then, the space $\mathcal{P}(A_0)$ of probability measures on $(A_0, A_0 \cap \mathcal{S})$ is a complete metric space under

$$d_0(\mu,\nu) = \sup_{B \in \mathcal{A}_0} |\mu(B) - \nu(B)|, \mu, \nu \in \mathcal{P}(A_0). \tag{8}$$

Theorem 2.1 . *Assume the local splitting condition (7) with local splitting class $\mathcal{A}_0 = A_0 \cap \mathcal{S}$. If for all $x \in S$*

$$\sup_{x \in A_0} \mathbf{E}_x \tau_{A_0} < \infty, \quad P_x(\tau_{A_0} < \infty) = 1,$$

then there is a unique invariant probability π on $(S, \mathcal{S})$ and

$$\lim_{n\to\infty} |\frac{1}{n}\sum_{j=0}^{n-1} p^{(j)}(x,B) - \pi(B)| = 0, x \in S, B \in \mathcal{A}.$$

As has been our practice in the previous section, we will continue to restrict attention to the case $N = 1$ in the proofs. To prepare for the proof define

$$\tau_{A_0}^{(0)} := 0, \quad \tau_{A_0}^{(n+1)} := \inf\{n \geq \tau_{A_0}^{(n)} + 1 : X_n \in A_0\}, \ n = 0, 1, 2, \ldots. \tag{9}$$

Also we write $\tau_{A_0} \equiv \tau_{A_0}^{(1)}$. The process viewed only on its returns to A_0 will be denoted

$$\tilde{X}_n = X_{\tau_{A_0}^{(n)}}, \quad n = 1, 2, \ldots. \tag{10}$$

Define the kernel $p_{A_0}(x, B), x \in S, B \in \mathcal{S}$, by

$$p_{A_0}(x,B) = \sum_{n=1}^{\infty} P_x(X_n \in B, X_k \in A_0^c \ 1 \leq k < n). \tag{11}$$

It is well-known that the process $\{\tilde{X}_n\}_{n=0}^{\infty}$ is a Markov process with transition probabilities obtained from $p_{A_0}(x,B)$ by restricting x to A_0, and B to $\mathcal{S} \cap A_0$ (e.g. see Orey (1971)).

It will be important to observe that for each $x \in S$, the kernel $B \to p_{A_0}(x,B)$ defines a measure on the sigma-field $\mathcal{S}$. The probabilistic interpretation is the expected number of visits to $B \in \mathcal{S}$ prior to revisiting A_0. In particular under the assumption

$$\sup_{x \in A_0} \mathbf{E}_x \tau_{A_0} < \infty, \tag{12}$$

one sees that

$$p_{A_0}(x, S) = \mathbf{E}_x \tau_{A_0} < \infty, x \in A_0. \tag{13}$$

Remark. One may check that if $p(x, dy)$ satisfies local minorization (4) on A_0 then $p_{A_0}(x, dy)$ $x \in A_0$, will satisfy Doeblin minorization (2) on A_0 as a warm-up exercise to Lemma 2.1.

Lemma 2.1 . *Under the conditions of Theorem 2.1, the process $\{\tilde{X}_n\}_{n=0}^{\infty}$, started at $\tilde{X}_0 = x \in A_0$, has a unique invariant probability π_{A_0} on $(A_0, A_0 \cap \mathcal{S})$. Moreover,*

$$\sup_{x \in A_0, B \in \mathcal{A}_0} |p_{A_0}^{(n)}(x,B) - \pi_{A_0}(B)| \leq (1-\delta)^{[\frac{n}{N}]}, n \geq 1.$$

Proof. In view of Theorem 1.3 it suffices to show that the process $\{\tilde{X}_n\}_{n=0}^{\infty}$, started at $\tilde{X}_0 = x \in A_0$, has a full splitting representation on A_0. For this first let $\{\alpha_n\}_{n=1}^{\infty}$ denote a localized splitting representation of $\{X_n\}_{n=0}^{\infty}$. Define a random map β_1 on A_0 as follows: For $\omega \in \Omega$, $\beta_1(\omega) : A_0 \to A_0$ by

$$\beta_1(\omega)y = \alpha_{\tau_{A_0}(y)(\omega)}(\omega) \cdots \alpha_1(\omega)y, y \in A_0.$$

Now let $\beta_1, \beta_2, \ldots$ be an *i.i.d.* sequence of maps. This provides a representation since both $\{\tilde{X}_n\}_{n=0}^{\infty}$ and the process generated by the *i.i.d.* maps $\beta_1, \beta_2, \ldots$ are Markov processes with the same transition probabilities. To see that this is a fully splitting representation on A_0 simply note that

$$P(\beta_1^{-1}A = A_0 \text{ or } \emptyset) \geq P(A_0 \cap \alpha_1^{-1}A_0 = A_0,\ A_0 \cap \alpha_1^{-1}A = A_0 \text{ or } \emptyset) \geq \delta.$$

The other required conditions for Theorem 1.3 follow immediately from the conditions of Theorem 2.1 since $\mathcal{A} = \mathcal{S}$. □

Remark. One may notice from the proof of Lemma 2.1 that it is the condition (ii) of Theorem 1.3 that is the reason for our restriction on the local splitting class. Nonetheless, as will be seen, this localization significantly extends the applicability of Theorem 1.3. An often quoted specific example from queuing theory which illustrates the nature of this extension in concrete terms is the random walk on the half-line $[0, \infty)$, defined by

$$X_{n+1} = (Y_{n+1} + X_n)^+, \quad n \geq 0, X_0 = x \geq 0, \tag{14}$$

where $\{Y_n\}_{n=1}^{\infty}$ is an *i.i.d.* sequence with $\mathbf{E}Y_1 < 0$. Of course, this is more generally illustrated by the broad class of locally minorized Markov processes.

With Lemma 2.1 the proof of the existence part of Theorem 2.1 is now easily completed by spreading π_{A_0} to $\mathcal{S}$ by defining

$$\pi(B) = c \int_{A_0} p_{A_0}(x, B)\pi_{A_0}(dx), B \in \mathcal{S}, \tag{15}$$

where

$$c^{-1} = \mathbf{E}_{\pi_{A_0}} \tau_{A_0} > 0, \tag{16}$$

and $\mathbf{E}_{\pi_{A_0}}$ denotes expectation when X_0 has distribution π_{A_0}. Note that on $A_0, \pi = c\pi_{A_0}$. Now the existence is completed by virtue of the following straightforward lemma. For $x \in A_0, B \in \mathcal{S} \cap A_0$, $p_{A_0}^{(n)}(x, B)$ denotes the n-step transition probability for the process on A_0 (with one-step transition probability p_{A_0}). For general $B \in \mathcal{S}$, we will write $p_{A_0}^{(n+1)}(x, B)$ for $\int_{A_0} p_{A_0}(y, B) p_{A_0}^{(n)}(x, dy) (n \geq 1)$, with $p_{A_0}(y, B)$ as defined in (11).

Lemma 2.2 . *Under the conditions of Theorem 2.1 (15) defines an invariant probability on $(S, \mathcal{S})$ for $p(x, dy)$. Moreover for $x \in A_0, B \in \mathcal{S}$,*

$$|cp_{A_0}^{(n+1)}(x, B) - \pi(B)| \leq (1 - \delta)^{[\frac{n}{N}]}, n \geq 1.$$

Proof. Let $B \in \mathcal{S}$. Then using Lemma 2.1,

$$\begin{aligned}
&\int_S p(y,B)\pi(dy) \\
&= \int_S p(y,B)c\int_{A_0} p_{A_0}(x,dy)\pi_{A_0}(dx) \\
&= c\int_S\int_{A_0}\sum_{n=1}^{\infty} p(y,B)P_x(X_n \in dy, X_k \in A_0^c 1 \le k < n)\pi_{A_0}(dx) \\
&= c\int_{A_0}\sum_{n=1}^{\infty} P_x(X_{n+1} \in B, X_k \in A_0^c 1 \le k < n)\pi_{A_0}(dx) \\
&= c\int_{A_0}\{p_{A_0}(x,B) - p(x,B) + \sum_{n=1}^{\infty} P_x(X_{n+1} \in B, X_n \in A_0, X_k \in A_0^c 1 \le k < n)\}\pi_{A_0}(dx) \\
&= \pi(B) - c\int_{A_0} p(x,B)\pi_{A_0}(dx) + c\int_{A_0}\int_{A_0} p(y,B)p_{A_0}(x,dy)\pi_{A_0}(dx) = \pi(B). \qquad (17)
\end{aligned}$$

In addition note that for $x \in A_0, B \in \mathcal{S}$,

$$\begin{aligned}
|cp_{A_0}(x,B) - \pi_{A_0}(B)| &= |c\int_{A_0} p(y,B)\{p_{A_0}^{(n)}(x,dy) - \pi(dy)\}| \\
&\le \int_{A_0} |p_{A_0}^{(n)}(x,dy) - \pi_{A_0}(dy)|.
\end{aligned}$$

So the convergence follows from Theorem 1.3. □

3 Localized Splitting and Uniqueness

Let us suppose that π is an arbitrary invariant probability. The uniqueness problem is a bit of an issue because of the nonstationarity of the process viewed as it revisits A_0 under P_π; nonstationarity is made transparent by considering the simple 3-state Markov chain on $\{1,2,3\}$ with $p = p_{2,1} = 1 - p_{2,3}, r = p_{1,2} = 1 - p_{1,3}, p_{3,2} = 1, A_0 = \{1,3\}$. However the following lemma will be useful for extrapolating from stationarity on $P_{\pi_{A_0}}$.

Lemma 3.1 . *Suppose π_{A_0} is the unique invariant probability for the process $\{\tilde{X}_n\}_{n=0}^{\infty}$ on A_0. Also assume (12) and (13) hold. If π is any invariant probability for $p(x,dy)$ then $\pi = c\pi_{A_0}, c = \pi(A_0)$, on $A_0 \cap \mathcal{S}$. In particular,*

$$P_\pi(E) \ge cP_{\pi_{A_0}}(E), E \in \mathcal{F}.$$

Proof. In view of recurrence it follows from Meyn and Tweedie (Theorem 10.4.7,p. 243,1993) that π restricted to A_0 is invariant under $p_{A_0}(x,dy)$. Thus the first assertion follows from the uniqueness of π_{A_0}, and the second by

$$P_\pi(E) \geq \int_{A_0} P_y(E)\pi(dy) = c\int_{A_0} P_y(E)\pi_{A_0}(dy) = P_{\pi_{A_0}}(E).$$

□

Proof of Uniqueness in Theorem 2.1. Without loss of generality assume that π is an ergodic invariant probability; else, in view of the topological structure of S, one may take an ergodic component in the ergodic decomposition. The idea for proving uniqueness is to use the ergodic theorem to show that for bounded measurable functions $f : S \to \mathbf{R}$, $\int_S f d\pi$ is determined by f and expected values with respect to π_{A_0}. In particular, first note that by ergodicity of the process $\{X_n\}_{n=0}^\infty$ under P_π one has P_π−a.s. that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^n f(X_j) \to \int_S f d\pi. \tag{18}$$

In particular, taking $f = \mathbf{1}[A_0]$, one has P_π−a.s.

$$\lim_{n\to\infty} \frac{N_n}{n} = \pi(A_0), \tag{19}$$

where

$$N_n = \sum_{j=1}^n \mathbf{1}[X_j \in A_0] \tag{20}$$

denotes the number of visits to A_0 during $[1, n]$. Now, for arbitrary bounded measurable functions $f : S \to \mathbf{R}$ we have P_π−a.s.

$$\frac{1}{n}\sum_{j=1}^n f(X_j) - \frac{N_n}{n}\cdot\frac{1}{N_n}\sum_{m=1}^{N_n} Z_m \to 0 \tag{21}$$

as $n \to \infty$, where, for times $\tau^{(m)} := \tau_{A_0}^{(m)}$ defined by (9),

$$Z_m := \sum_{j=\tau^{(m-1)}+1}^{\tau^{(m)}} f(X_j), m \geq 1. \tag{22}$$

It follows from (18) and (21) that the limit $\lim_{n\to\infty} \frac{1}{N_n}\sum_{m=1}^{N_n} Z_m$ exists P_π−a.s. and is given by $\frac{1}{\pi(A_0)}\int_S f d\pi$. But $\{Z_m\}_{m=1}^\infty$ is a stationary process under $P_{\pi_{A_0}}$, and therefore the sequence $\{\frac{1}{N}\sum_{m=1}^N Z_m\}_{N=1}^\infty$ will converge $P_{\pi_{A_0}}$−a.s. and in L^1. On the other hand, in view of Lemma 3.1, these imply that $P_{\pi_{A_0}}$−a.s.

$$\lim_{N\to\infty} \frac{1}{N}\sum_{m=1}^N Z_m = \frac{1}{\pi(A_0)}\int_S f d\pi. \tag{23}$$

Hence one may take expectations in (23) to get

$$\frac{1}{\pi(A_0)}\int_S f d\pi = \lim_{N\to\infty} \mathbf{E}_{\pi_{A_0}} \frac{1}{N}\sum_{m=1}^{N} Z_m = \mathbf{E}_{\pi_{A_0}} Z_1. \tag{24}$$

Taking $f = 1$ in (24) identifies $\pi(A_0)$ as

$$\pi(A_0) = c = \frac{1}{\mathbf{E}_{\pi_{A_0}}\tau_{A_0}}. \tag{25}$$

Thus we finally arrive at the unique determination of π via the formula

$$\int_S f d\pi = c\mathbf{E}_{\pi_{A_0}} Z_1 \tag{26}$$

for all bounded measurable functions f on S. □

4 Applications to Quadratic Maps.

In this section Theorem 1.3 is applied to iterations of *i.i.d.* quadratic maps $\alpha_n = F_{\theta_n}(n \geq 1)$, where θ_n $(n \geq 1)$ are *i.i.d.* random variables taking values in the parameter space [0,4] and, for each $\theta \in [0,4]$,

$$F_\theta(x) = \theta x(1-x), 0 \leq x \leq 1. \tag{27}$$

Since 0 is a common fixed point of all F_θ, the Dirac measure δ_0 is always an invariant probability for the Markov process $X_n(x) := \alpha_n \cdots \alpha_1 x (n \geq 1), X_0(x) = x$, on the state space [0,1]. We focus on the existence and uniqueness of an invariant probability other than δ_0. The appropriate state space is then $S = (0,1)$ left invariant by all F_θ.

We begin by recalling a few basic facts about the quadratic family $\{F_\theta : \theta \in [0,4]\}$, shared by other unimodal families as well; see Collet and Eckman (1980) and Devaney (1989) for proofs and further properties. It is easily checked that for $0 \leq \theta \leq 1$ the map F_θ has the unique attracting fixed point 0. For $\theta > 1$, 0 is repelling for F_θ and a new fixed point $p_\theta = 1 - \frac{1}{\theta}$ appears, which is attractive for $1 < \theta \leq 3$, and repelling for $\theta > 3$. A period two orbit for F_θ appears for $\theta > 3$, which remains attractive for $3 < \theta \leq 1 + \sqrt{6}$, becomming repelling for $\theta > 1 + \sqrt{6}$, at which point a period-four orbit appears. In this manner period-doubling bifurcations take place for all periods $2^n (n \geq 0)$. Beyond this, other periods appear each with a period doubling sequence of its own.The last to appear is a period-three orbit. A well-known theorem of Sarkovskii says that a continuous map with a period-three orbit has periodic orbits of all periods. Beyond the period-three regime, there are θ values which have no attractive periodic orbits, and chaos sets in. Although the set of θ's for which F_θ has an attractive periodic orbit is dense in [0,4], the set of θ's for which F_θ is chaotic or even has an absolutely continuous invariant probability, has positive Lebesgue measure.

Turning to the Markov process $X_n := \alpha_n \cdots \alpha_1 X_0 (n \geq 1)$, with X_0 independent of $\{\alpha_n = F_{\theta_n} : n \geq 1\}$, the following lemma allows one to extend earlier results of Bhattacharya and Rao (1993) and Bhattacharya and Majumdar (1999). In order to state it, we will recast the splitting class as $\mathcal{A} = \{[c, x] : c \leq x \leq d\}$ for the case of *i.i.d.* monotone maps $\{\alpha_n : n \geq 1\}$ on an interval$[c, d]$ as follows. The Markov process $X_n(x), (n \geq 1), X_0(x) = x, x \in [c, d]$ is said to have the splitting property if there exists $\delta > 0, x_0 \in [c, d]$, and an integer N such that

$$P(X_N(x) \leq x_0 \forall x \in [c, d] \text{ or } X_N(x) \geq x_0 \forall x \in [c, d]) \geq \delta. \tag{28}$$

It is shown in Dubins and Freedman (1966) that (28) implies the existence and uniqueness of an invariant probability π on $[c, d]$; i.e. Theorem 1.3 holds. If the inequalities '$\leq x_0$'and '$\geq x_0$'appearing within parenthesis in (28) are replaced by strict inequalities '$< x_0$'and '$> x_0$', respectively, then the above property will be referred to as a *strict splitting property*. Note that (28), or its strict version, is a property of the distribution Q of θ_n. Denote by Q^N the (product) probability distribution of $(\theta_1, \ldots, \theta_N)$.

Lemma 4.1 . *Let $\theta_1 < \theta_2$ and $m \geq 1$ be given. (a). If, for all $\theta^i \in \{\theta_1, \theta_2\}$, $1 \leq i \leq m$, the range of $F_{\theta^1} \cdots F_{\theta^m}$ on an interval $I_1 = [u_1, v_1]$ is contained in $I_2 = [u_2, v_2]$, then the same is true for all $\theta^i \in [\theta_1, \theta_2]$. In particular, if $F_{\theta^1} \cdots F_{\theta^m}$ leaves an interval $[c, d]$ invariant for all $\theta^i \in \{\theta_1, \theta_2\}, 1 \leq i \leq m$, then the same is true for all $\theta^i \in [\theta_1, \theta_2], 1 \leq i \leq m$. (b). Suppose, for all $\theta^i \in \{\theta_1, \theta_2\}$, $1 \leq i \leq m$, $F_{\theta^1} \cdots F_{\theta^m}$ leaves invariant an interval $[c, d]$. Assume also that the strict splitting property above holds, with $N = km$ a multiple of m, for a distribution $Q = Q_0$ whose support is $\{\theta_1, \theta_2\}$. Then (i) $F_{\theta^1} \cdots F_{\theta^m}$ leaves $[c, d]$ invariant for all $\theta^i \in [\theta_1, \theta_2]$, and (ii) the strict splitting property holds for an arbitrary $Q = \hat{Q}$ whose support has θ_1 as the smallest point and θ_2 as its largest.*

Proof (a.) The proof is by induction on m. The assertion is true for $m = 1$, since in this case $u_2 \leq \min\{F_{\theta_1}(y) : y \in [u_1, v_1]\} \leq F_\theta(x) \leq \max\{F_{\theta_2}(y) : y \in [u_1, v_1]\} \leq v_2$, for all $x \in [u_1, v_1]$. Assume the assertion is true for some integer $m \geq 1$. Let $a = \min\{F_{\theta^2} \cdots F_{\theta^{m+1}}(x) : \theta^i \in \{\theta_1, \theta_2\}, 2 \leq i \leq m+1, x \in [u_1, v_1]\}$, and $b = \max\{F_{\theta^2} \cdots F_{\theta^{m+1}}(x) : \theta^i \in \{\theta_1, \theta_2\}, 2 \leq i \leq m+1, x \in [u_1, v_1]\}$. By the induction hypothesis, for arbitrary $\theta^2, \ldots, \theta^{m+1} \in [\theta_1, \theta_2]$, the range of $F_{\theta^2} \cdots F_{\theta^{m+1}}$ on $[u_1, v_1]$ is contained in $[a, b]$. On the other hand, $u_2 \leq \min\{F_{\theta_1}(y) : y \in [a, b]\}, v_2 \geq \max\{F_{\theta_2}(y) : y \in [a, b]\}$. Hence, for arbitrary $\theta^1, \ldots, \theta^{m+1} \in [\theta_1, \theta_2]$, the range of $F_{\theta^1} \cdots F_{\theta^{m+1}}$ on $[u_1, v_1]$ is contained in $[u_2, v_2]$, thus completing the induction argument.

(b)(ii.) Suppose a strict splitting property holds with $\delta > 0, x_0, N$. Define the continuous functions L, l on $[\theta_1, \theta_2]^N$ by

$$L(\theta^1, \dots, \theta^N) := \max_{c \le x \le d} F_{\theta^1} \cdots F_{\theta^N}(x), \quad l(\theta^1, \dots, \theta^N) := \min_{c \le x \le d} F_{\theta^1} \cdots F_{\theta^N}(x) \tag{29}$$

By hypothesis, the open subset U of $[\theta_1, \theta_2]^N$ defined by $U := \{(\theta^1, \dots, \theta^N) \in [\theta_1, \theta_2]^N : L(\theta^1, \dots, \theta^N) < x_0\}$, includes a point $(\theta_0^1, \dots, \theta_0^N) \in \{\theta_1, \theta_2\}^N$. Therefore U contains a rectangle $R = R_1 \times \cdots \times R_N$ where R_i is of the form $R_i = [\theta_i, \theta_i + h_i)$ or $R_i = (\theta_2 - h_i, \theta_2]$ for some $h_i > 0 (1 \le i \le N)$, depending on whether $\theta_0^i = \theta_1$ or $\theta_0^i = \theta_2$. By the hypothesis on $\hat{Q}$, $\delta_1 := \hat{Q}^N(R) = \hat{Q}(R_1) \cdots \hat{Q}(R_N) > 0$. Hence the first inequality in (28) (with '$< x_0$') holds with $\delta_1 > 0$ in place of δ. Similarly, the second inequality in (28) holds (with '$> x_0$') with some $\delta_2 > 0$ in place of δ, since $V := \{(\theta^1, \dots, \theta^N) \in [\theta_1, \theta_2]^N : l(\theta^1, \dots, \theta^N) > x_0\}$, is an open subset of $[\theta_1, \theta_2]^N$, which includes a point $(\theta_0^1, \dots, \theta_0^N) \in \{\theta_1, \theta_2\}^N$. Now take the minimum of δ_1, δ_2 for δ in (28). Finally, the invariance of $[c, d]$ under $F_{\theta^1} \cdots F_{\theta^N}$ for all $\theta^i \in [\theta_1, \theta_2], 1 \le i \le N$, follows from (a). □

In Examples 1 - 4 below the invariant interval $[c, d]$ is either contained in $(0, \frac{1}{2}]$ or in $[\frac{1}{2}, 1)$ to insure monotonicity of $\alpha_n = F_{\theta_n}$, so that the theorem of Dubins and Freedman (1966), i.e. Theorem 1.3 with $\mathcal{A} = \{[c, x] : c \le x \le d\}$, can be applied to the process $\{Y_k := X_{km}\}_{k=0}^{\infty}$. The distribution Q of θ_n in the case $m = 1$ in Lemma 4.1, or Q^m in case $m > 1$, is assumed to have support with θ_1, θ_2 its smallest and largest points, respectively. This generalizes the case of support precisely $\{\theta_1, \theta_2\}$ considered in Bhattacharya and Rao (1993) and Bhattacharya and Majumdar (1999). It is also true in these examples that the probability of reaching $[c, d]$ in finite time, starting from any $x \in (0, 1)$ is one. One may then show, in a manner similar to (but simpler than) the proof of Theorem 2.1, that there is a unique invariant probability on $S = (0, 1)$.

Example 1. Take $1 < \theta_1 < \theta_2 \le 2, m = 1$. Then F_{θ_i} has an attractive fixed point $p_{\theta_i} = 1 - \frac{1}{\theta_i} (i = 1, 2)$. Here $[c, d] = [p_{\theta_1}, p_{\theta_2}] \subset (0, \frac{1}{2}], x_0 \in (c, d)$, and N is a sufficiently large integer such that $F_{\theta_1}^N p_{\theta_2} < x_0$, and $F_{\theta_2}^N p_{\theta_1} > x_0$. The Markov process on $S = (0, 1)$ then has a unique invariant probability π and $X_n(x)$ converges in distribution to π geometrically fast in the Kolmogorov distance, as $n \to \infty$, for every $x \in (0, 1)$.

Example 2. Take $2 < \theta_1 < \theta_2 \le 3, m = 1$. Then F_{θ_i} has an attractive fixed point $p_{\theta_i} = 1 - \frac{1}{\theta_i} (i = 1, 2)$ and $[c, d] = [p_{\theta_1}, p_{\theta_2}] \subset [\frac{1}{2}, 1)$. The same conclusion as in Example 1 holds in this case as well.

Example 3. Take $2 < \theta_1 \le 3 < \theta_2 \le 1 + \sqrt{5}, \theta_1 \in [\frac{8}{\theta_2(4 - \theta_2)}, \theta_2), m = 1$. Then F_{θ_1} has an attractive fixed point $p_{\theta_1} = 1 - \frac{1}{\theta_1}$, and F_{θ_2} has an attractive period-two orbit $\{q_1, q_2\}, q_1 < q_2$. The interval $[c, d] = [\frac{1}{2}, \frac{\theta_2}{4}]$ is invariant under $F_{\theta_i} (i = 1, 2)$, and strict splitting occurs with $x_0 \in (p_{\theta_2}, q_2)$ and N a sufficiently large even integer. Also with probability one, the Markov process $\{X_n(x)\}_{n=0}^{\infty}$ reaches $[\frac{1}{2}, \frac{\theta_2}{4}]$ in finite time, whatever be $x \in (0, 1)$. Thus there is a unique invariant probability π on $S = (0, 1)$ and $X_n(x)$ converges in distribution to π

as $n \to \infty$, for every $x \in (0,1)$.

Example 4. Take $\theta_1 = 3.18, \theta_2 = 3.20, m = 2$. Then F_{θ_i} has an attractive periodic orbit $\{q_{1i}, q_{2i}\}, q_{1i} < q_{2i} (i = 1,2)$. One may choose $[c,d] = [q_{21} - \epsilon, q_{22} + \epsilon]$ with $\epsilon \geq 0$ sufficiently small and show that $F_{\theta^1} F_{\theta^2} (\theta^i \in \{\theta_1, \theta_2\})$ leaves $[c,d]$ invariant, and a (strict) splitting occurs for $x_0 \in (q_{21}, q_{22})$ and N a sufficiently large even integer. Also with probability one, the Markov process $\{X_n(x)\}_{n=0}^{\infty}$ reaches $[c,d]$ in finite time, whatever be $x \in (0,1)$. Thus there is a unique invariant probability π on $S = (0,1)$. Examples 1-4 may be derived as special cases of the following result. As before, Q denotes the common distribution of the *i.i.d.* sequence $\theta_n (n \geq 1)$ and $X_n(x) = F_{\theta_n} \cdots F_{\theta_1} x (n \geq 1)$.

Theorem 4.1 *Let θ_1, θ_2 denote the smallest and largest points of the support of Q, $1 < \theta_1 < \theta_2 < 4$. Assume that F_{θ_i} has an attractive periodic orbit of period $m_i \geq 1 (i = 1,2)$. Assume that $\{q_1, q_2\}$ are points of the attractive orbits of $F_{\theta_1}, F_{\theta_2}$ with the following properties: (i) There is an interval I containing $\{q_1, q_2\}$ which is contained either in $(0, \frac{1}{2}]$ or in $[\frac{1}{2}, 1)$ such that $F_{\theta_i}^m$ leaves I invariant for some multiple m of both m_1 and $m_2 (i = 1,2)$, (ii) $F_{\theta_i}^{km} I \to \{q_i\}$ as $k \to \infty (i = 1,2)$, and (iii) $P_x(\tau_I < \infty) = 1$ for all $x \in (0,1)$. Then the Markov process $\{X_n\}_{n=0}^{\infty}$ has a unique invariant probability on $S = (0,1)$.*

Proof. By Lemma 4.1(a), the Markov process $\{Y_k := X_{km}\}_{k=0}^{\infty}$ may be defined on the state space I, on which it is generated by *i.i.d.* monotone maps $\beta_k := \alpha_{km} \cdots \alpha_{(k-1)m+1} (k \geq 1)$. Let x_0 belong to the interior of the line segment joining q_1, q_2. Then the (strict) splitting condition (29) holds, with Y_N in place of X_N, if N is sufficiently large, by virtue of assumption (ii). Hence there exists a unique invariant probability π_0, say, of $\{Y_k\}_{k=0}^{\infty}$ on I. Under P_{π_0}, by considering the proportion of times spent in a set by $\{X_n\}_{n=0}^{\infty}$ in the first M stationary blocks $(X_{(k-1)m}, X_{(k-1)m+1}, \ldots, X_{km-1}), 1 \leq k \leq M$, as $M \to \infty$, one obtains an invariant probability π for $\{X_n\}_{n=0}^{\infty}$ on S given by $\pi(B) = \frac{1}{m} \sum_{n=0}^{m-1} P_{\pi_0}(X_n \in B)$. In view of (iii), the invariant probability π is unique. On may also mimick the proof of Theorem 2.1, however the steps are much simpler here. □

Remark. Given any integer $n \geq 0$, there exists $\theta_1 < \theta_2$ so that $F_{\theta_1}, F_{\theta_2}$ have periodic orbits of period 2^n. One may choose θ_1, θ_2 sufficiently close so that the largest points in their orbits, q_1, q_2, say, have no other periodic (or fixed) point of F_{θ_i} between them $(i = 1,2)$, and so that q_1, q_2 both lie in $(\frac{1}{2}, 1)$ (or, $(0, \frac{1}{2})$, and in some cases with $n = 0$). The hypothesis of Theorem 4.1 hold in this case.

Suppose next that Q has a density component. Then under broad conditions one can show the existence of a unique invariant probability on $S = (0,1)$, as the following theorem shows.

Theorem 4.2 *Let $1 < \mu < \lambda < 4$. Suppose $Q([\mu, \lambda]) = 1$ and Q has a nonzero absolutely continuous component with a density which is strictly positive on some* *open interval $(\mu_1, \mu_2)(\mu_1 < \mu_2)$ containing a parameter value θ for which F_θ has an attractive periodic orbit of some period $2^n (n \geq 0)$. Then the Markov process has a unique invariant probability on $S = (0, 1)$.*

We first need a preliminary lemma.

Lemma 4.2 . *Let $1 < \mu < \lambda < 4$. Let $u = \min\{1 - \frac{1}{\mu}, F_\mu(\frac{\lambda}{4})\}, v = \frac{\lambda}{4}$. Then for every $\theta \in [\mu, \lambda]$, $[u, v]$ is invariant under F_θ.*

Proof We need to prove that (i) $\max\{F_\theta(x) : u \leq x \leq v\} \leq v$ for $\theta \in [\mu, \lambda]$ and (ii) $\min\{F_\theta(x) : u \leq x \leq v\} \geq u$ for $\theta \in [\mu, \lambda]$. The first of these follows from the relations $\max\{F_\theta(x) : u \leq x \leq v\} \leq \max\{F_\lambda(x) : u \leq x \leq v\} \leq \frac{\lambda}{4}$. For (ii) note that by unimodality, $\min\{F_\theta(x) : u \leq x \leq v\} = \min\{F_\theta(u), F_\theta(v)\} \geq \min\{F_\mu(u), F_\mu(v).\}$ If $u = 1 - \frac{1}{\mu} \leq F_\mu(\frac{\lambda}{4})$, then the last minimum is $F_\mu(u) = 1 - \frac{1}{\mu} = u$. If $u = F_\mu(\frac{\lambda}{4}) < 1 - \frac{1}{\mu}$, then $\min\{F_\mu(u), F_\mu(\frac{\lambda}{4})\} = F_\mu(\frac{\lambda}{4}) = u$, since on $(0, 1 - \frac{1}{\mu}), F_\mu(x) > x$. □

Proof of Theorem 4.2. We will sketch only the main ideas behind the proof. Under the hypothesis there exist $n \geq 0$ and an interval $[\delta_1, \delta_2] \subset (\mu_1, \mu_2)$, $\gamma_1 < \gamma_2$, such that for every $\theta \in [\gamma_1, \gamma_2]$ F_θ has an attractive periodic orbit of period $m = 2^n$. For simplicity assume that Q is absolutely continuous with a continuous density h which is strictly positive on $[\delta_1, \delta_2]$. Let q_1, q_2 be the largest points on the attractive orbits of F_{γ_1} and F_{γ_2}, respectively. One may choose γ_1, γ_2 sufficiently close to each other so that (1) there is no other periodic (or fixed) point of F_{γ_i} in the line segment joining q_1, q_2, and (2) there is an interval I containing $\{q_1, q_2\}$ which is left invariant by $F^m_{\gamma_i}$ and $F^{km}_{\gamma_i} I \to \{q_i\}$ as $k \to \infty (i = 1, 2)$. It follows from Lemma 4.1(a) that I is left invariant by $F_{\theta^1} \cdots F_{\theta^m}$ for all $\theta^i \in [\gamma_1, \gamma_2], 1 \leq i \leq m$.

The m-step transition probability density $p^{(m)}(x, y)$ of the Markov process $\{X_n\}_{n=0}^\infty$ is easily shown to be given by

$$p(x, y) = \frac{1}{x(1-x)} h(\frac{y}{x(1-x)}),$$

$$p^{(2)}(x, y) = \frac{1}{x(1-x)} \int_{[u,v]} \frac{1}{z(1-z)} h(\frac{y}{z(1-z)}) h(\frac{z}{x(1-x)}) dz,$$

$$p^{(n+1)}(x, y) = \int_{[u,v]} \frac{1}{z(1-z)} h(\frac{y}{z(1-z)}) p^{(n)}(x, z) dz \quad (n \geq 1), \quad (u \leq x, y \leq v).$$

Here u, v are as in Lemma 4.2. Using the fact that $F^m_\theta q = q$ for a point q on the attractive 2^n- periodic orbit of F_θ, with $\theta \in [\gamma_1, \gamma_2]$, one may show that $p^{(m)}(x, y) > 0$ for all $x, y \in I$ (if γ_1, γ_2 are sufficiently close). This implies that the restriction $\{\tilde{X}_n\}_{n=0}^\infty$ of the process $\{X_n\}_{n=0}^\infty$ to I at times of visits to I

satisfies Doeblin's local minorization. Also, using the compactness of $[u, v]$ and the fact that every point in $[u, v]$ belongs to an interval of attraction to a point in the attractive orbit of some F_θ with $\theta \in (\gamma_1, \gamma_2)$, one can show $\sup\{\mathbf{E}_x \tau_I : x \in I\} < \infty$. Similarly one proves $P_x(\tau_I < \infty) = 1$ for all $x \in S = (0, 1)$. □

Remarks.

(1.)It has recently been shown by Dai (1999), generalizing an earlier result of Bhattacharya and Rao (1993) (Remark 4.1.2), that if Q has an absolutely continuous component with a positive density on a subinterval of $(1, 3)$ then the Markov process is Harris recurrent and, therefore, the invariant probability is unique. One may arrive at this by an application of Theorem 1.2.

(2.) Lemma 4.2 corrects an oversight in Bhattacharya and Rao (1993) and in Dai (1999), where u is taken to be $1 - \frac{1}{\mu}$.

(3.) Finally, in a recent article Athreya and Dai (1999) has proved the existence (but not uniqueness) of an invariant probability on $S = (0, 1)$ under the assumption that $\mathbf{E} \ln \theta_1 > 0, \mathbf{E}|\ln(4 - \theta_1)| < \infty$. Athreya and Dai (1999) also have shown that '$\mathbf{E} \ln \theta_1 > 0$'is a necessary condition for the existence of an invariant probability π on $(0, 1)$, by using the functional equation $X_{n+1} = \theta_{n+1} X_n (1 - X_n)$ to point out that an invariant π must satisfy the equation $-\int \ln(1 - x)\pi(dx) = \mathbf{E} \ln \theta_1$.

(4.) In view of Theorem 4.1, and the density of the set of θ's such that F_θ has an attractive periodic orbit, one may conjecture that if the Athreya-Dai sufficiency condition holds and Q has a nonzero absolutely continuous component with a density which is positive on some interval $(1, 4)$, then there exists a unique invariant probability on $S = (0, 1)$.

(5.) Although we have no example for which there exists more than one invariant probability on $S = (0, 1)$, we believe that there are lots of Q's for which there are more than one invariant probability.

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